Su-Yee Chan

St George Girls High School

Year 12

Mid-HSC Course Examination

2004

Mathematics Extension 1

General Instructions
- Working time – 1½ hours
- Write using blue or black pen
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.
- Write on one side of the page only.
- Start each question on a NEW page.

Total marks – 78
- Attempt Questions 1 – 6
- All questions are of equal value

Question 1 – (13 marks) – Start a new page **Marks**

a) Write down an expression for $\tan(x+y)$ and hence calculate $\tan 75^\circ$ in simplest irrational form. 3

b) State the domain and range of 2

$$y = e^{2x-1}$$

c) If $y = e^{3x}$ show $\dfrac{d^2y}{dx^2} - 5\dfrac{dy}{dx} + 6y = 0$ 3

d) (i) Differentiate $\log_e(1+\tan x)$, 1

(ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{1+\tan x}\,dx$ 1

e) Given $f(x) = \ln\left[x + \sqrt{x^2-1}\right]$ calculate the largest possible domain of the function. Give reasons. 3

Question 2 – (13 marks) – Start a new page **Marks**

a) $\frac{d}{dx}(x\sin x + \cos x)$ 2

b) What is the primitive function of $\frac{1}{x+7}$? 1

c) (i) Sketch the curve $y = e^{-x}$ 1

(ii) The arc joining the points on the curve where $x = 0,\ x = 1$ is rotated around the x-axis. Calculate the volume of the solid so formed. (Leave the answer in terms of e). 3

d) Evaluate $\lim\limits_{x \to 0} \frac{2\sin\frac{1}{2}x}{x}$ 2

e) Evaluate (in exact form)

(i) $\int_0^1 e^{-x} + e^x \, dx$ 2

(ii) $\int_0^{\frac{1}{3}} \frac{dx}{3x+1}$ 2

Question 3 – (13 marks) – Start a new page **Marks**

a) Differentiate $\sqrt{e^{2x+1}}$ 2

b) Find the derivative of

(i) $\log_e\left(\frac{x+1}{x-1}\right)$ 2

(ii) $\log_e\sqrt{x^2+1}$ 2

c) ABC is an isosceles triangle with a right angle at B. The sides $AB,\ BC$ are each of length 2cm. An arc, centre A, radius 2cm cuts the side AC at D.

(i) Draw a diagram to represent all this information. 1

(ii) If BDC is the part of ΔABC outside the circle show that the area of BDC is $\left(2 - \frac{\pi}{2}\right)$ cm^2. 3

d) Evaluate $\int_0^2 \frac{e^x - 1}{e^x} dx$, giving the answer to two decimal places. 3

Question 4 – (13 marks) – Start a new page **Marks**

a) (i) On the same diagram, sketch the curves $y = \sin x$, $y = \cos x$ for $0 \le x \le \frac{\pi}{2}$ **2**

(ii) Determine the x-coordinate of the point of intersection, P, of the two curves, in this interval. **1**

(iii) Calculate the area bounded by the curves $y = \sin x$ and $y = \cos x$, and the x-axis between $x = 0$, $x = \frac{\pi}{2}$ **3**

(iv) If the acute angle between the tangents at P to the curves is ϕ, prove that $\tan \phi = 2\sqrt{2}$ **4**

b) The diameter of a planet is 73600 km and the distance between the centre of the Earth and the centre of the planet is 7.32×10^8 km. Determine in seconds, to two significant figures, the angle subtended, at that time by the planet, at the centre of the Earth. **3**

Question 5 – (13 marks) – Start a new page **Marks**

a) (i) Sketch the curve $y = \log_e x$ **1**

(ii) Using Simpson's Rule with five function values, calculate an approximation to $\int_1^5 \log_e x \, dx$ (correct to 2 decimal places). **4**

(iii) Find the derivative of $x \log_e x - x$ and hence evaluate the exact answer to $\int_1^5 \log_e x \, dx$ **3**

b) The second derivative of a curve is given by $\frac{d^2y}{dx^2} = \frac{1}{x^2}$ and the curve passes through $(1, -1)$ with a slope of 2. Determine the equation of the curve. **4**

c) Evaluate $\log_5 7$ to 3 significant figures **1**

Question 6 – (13 marks) – Start a new page **Marks**

a) Differentiate $xe^x - e^x$ and hence evaluate $\int_0^1 xe^x \, dx$ 3

b) (i) Sketch the curve $y = \sin^2 x$ in the interval $0 \le x \le 2\pi$ 2

(ii) What is the period of $y = \sin^2 x$? 1

c) The rate of decay of radio-active material is proportional to the mass present, ie $\frac{dm}{dt} = -km$ where m is the mass at time t and k is a constant.

(i) Show that $m = m_0 e^{-kt}$ is a solution to $\frac{dm}{dt} = -km$. 1

(ii) If the half life of the mass is 10 years, calculate the exact value of k. 2

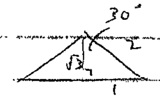
(iii) Calculate the fraction of the original mass of the material left after another 20 years. 2

(iv) Calculate the time to the nearest year for the mass to be one-fifth of its original mass. 2

Q2: Solutions to
Mid-HSC Course Exam
2004 Ext 1 (yr 12)

Q1

a) $\tan(x+y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$



$\therefore \tan 75° = \tan(45+30)°$

$= \frac{\tan 45° + \tan 30°}{1 - \tan 45° \tan 30°}$

$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}+1}{\sqrt{3}+1}$

$= \frac{(\sqrt{3}+1)^2}{2}$

$= \frac{4+2\sqrt{3}}{2}$

$= 2+\sqrt{3}$

b) $y = e^{2x-1}$
D: $x \in \mathbb{R}$
R: $y > 0$

c) ~~(i)~~ $f(x) = \ln(x + \sqrt{x^2-1})$
From $\sqrt{x^2-1}$, $x \ge 1, x \le -1$
~~D~~: We also need $x + \sqrt{x^2-1} > 0$
$\therefore x \ne -1$
D: $x \ge 1$

c) $y = e^{3x}$

$\frac{dy}{dx} = 3e^{3x}$

$\frac{d^2y}{dx^2} = 9e^{3x}$

$\therefore \frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y =$

$9e^{3x} - 15e^{3x} + 6e^{3x} = 0$ ✓

d) (i) $\frac{d}{dx} \ln(1+\tan x) = \frac{\sec^2 x}{1+\tan x}$.

(ii) $\int_0^{\frac{\pi}{4}} \frac{\sec^2 x}{1+\tan x} dx = [\ln(1+\tan x)]_0^{\frac{\pi}{4}}$

$= \ln(1+\tan\frac{\pi}{4}) - \ln 1$

e) (i) $y = \ln(2x-1)$ ~~$= \ln 2$~~

$D: x > \frac{1}{2}$ $\qquad 2x-1>0$

(ii) ~~$D: x \in \mathbb{R}$~~

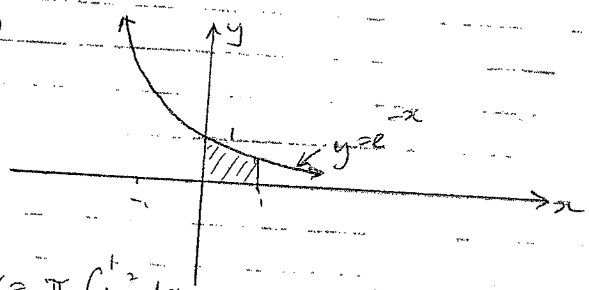
Q2

2 a) $\frac{d}{dx}(x\sin x + \cos x) = x\cdot\cos x + \sin x - \sin x$

$= x\cos x$

1 b) $\int \frac{1}{x+7}\,dx = \ln(x+7) + C$ $\frac{1}{2}$

c)



1 $\frac{1}{2}$ for shape.

$V = \pi\int_0^1 y^2\,dx$ 1

$= \pi\int_0^1 e^{-2x}\,dx$

3 $= \pi\left[-\frac{1}{2}e^{-2x}\right]_0^1$ 1

$= \pi\left(-\frac{1}{2}e^{-2} + \frac{1}{2}\right)$ $\quad 2\pi$ (2)

$= \frac{\pi}{2}\left(1 - e^{-2}\right)$ units^3 1

$\frac{1}{2}$ each error.

d) $\lim_{x\to 0} \frac{2\sin\frac{1}{2}x}{x}$

2 $= \lim_{x\to 0} \frac{\sin\frac{x}{2}}{\frac{x}{2}}$ $\quad \left(\lim_{x\to 0}\frac{\sin ax}{ax} = 1\right)$

$= 1$

$\frac{1}{2}$ ...

e)(ii) $\int_0^{\frac{1}{3}} \frac{dx}{3x+1} = \frac{1}{3}\int_0^{\frac{1}{3}} \frac{3}{3x+1}\,dx$

$= \frac{1}{3}\left[\ln(3x+1)\right]_0^{\frac{1}{3}}$ 1 $\quad -\frac{1}{2}$

$= \frac{1}{3}(\ln 2 - \ln 1)$ 2.

$= \frac{1}{3}\ln 2$.

(i) $\int_0^1 e^{-x} + e^{x}\,dx$

$= \left[-e^{-x} + e^{x}\right]_0^1$

2 $= -e^{-1} + e - (-1+1)$

$= e - \frac{1}{e}$

$-\frac{1}{2}$ each error.

Q3

a) $\frac{d}{dx}\sqrt{e^{2x+1}} = \frac{d}{dx}\left(e^{2x+1}\right)^{\frac{1}{2}}$

$= \frac{d}{dx} e^{x+\frac{1}{2}}$

$= e^{x+\frac{1}{2}}$

b) $\frac{d}{dx}\ln\left(\frac{x+1}{x-1}\right) = \frac{x-1}{x+1}\cdot\frac{d}{dx}\left(\frac{x+1}{x-1}\right)$

$= \frac{x-1}{x+1}\cdot\frac{(x-1)\cdot 1 - (x+1)\cdot 1}{(x-1)^2}$

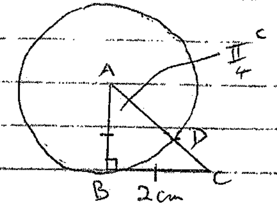
$= \frac{-2}{x^2-1}$

(ii) $\frac{d}{dx}\ln\sqrt{x^2+1} = \frac{1}{2}\frac{d}{dx}\ln(x^2+1)$

$= \frac{1}{2}\cdot\frac{2x}{x^2+1}$

$= \frac{x}{x^2+1}$

c)(i)



(ii) Portion BDC has area

$=$ Area $\triangle ABC$ − area sector ABD

$= \frac{1}{2}\times 2^2 - \frac{1}{2}\cdot 2^2\cdot\frac{\pi}{4}$

$= \left(2 - \frac{\pi}{2}\right)$ cm^2

d) $\int_0^2 \frac{e^x - 1}{e^x}\,dx$

$= \int_0^2 1 - e^{-x}\,dx$

$= \left[x + e^{-x}\right]_0^2$

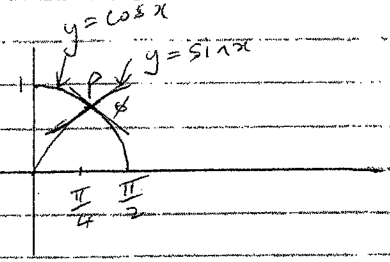
$= 2 + e^{-2} \cancel{+} - e^0$

$= 1 + e^{-2}$

$= 1.14$ (to 2 dec. pl.)

Q4

a)(i)



(ii) $\sin x > \cos x$

$\therefore \tan x = 1$

$x > \frac{\pi}{4}$.

(iii) $A = \int_0^{\frac{\pi}{4}} \sin x\,dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cos x\,dx$

$= \left[-\cos x\right]_0^{\frac{\pi}{4}} + \left[\sin x\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= -\cos\frac{\pi}{4} + \cos 0 + \sin\frac{\pi}{2} - \sin\frac{\pi}{4}$

$= -\frac{1}{\sqrt{2}} + 1 + 1 - \frac{1}{\sqrt{2}}$

$= (2 - \sqrt{2})$ units^2 $\quad$ ($= 2\int_0^{\frac{\pi}{4}} \sin x\,dx$, from symmetry).

(iv) $y = \cos x \Rightarrow y' = -\sin x$

$y = \sin x \Rightarrow y' = \cos x$

$\therefore$ at $x = \frac{\pi}{4}$, slopes of tangents to curves are $-\sin\frac{\pi}{4}$, $\cos\frac{\pi}{4}$

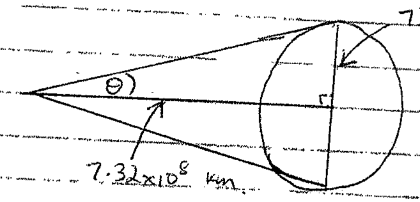
i.e. $m_1 = -\frac{1}{\sqrt{2}}$, $m_2 = \frac{1}{\sqrt{2}}$

$\tan\phi = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{-\frac{2}{\sqrt{2}}}{1 - \frac{1}{2}}\right| = \sqrt{2} \div \frac{1}{2}$

$= 2\sqrt{2}$

b)



$\tan\theta = \dfrac{73600}{2\times 7.32\times10^{8}}$

$\therefore\ \theta = \cancel{0.36''}\ 0.00288\ldots^{\circ}$

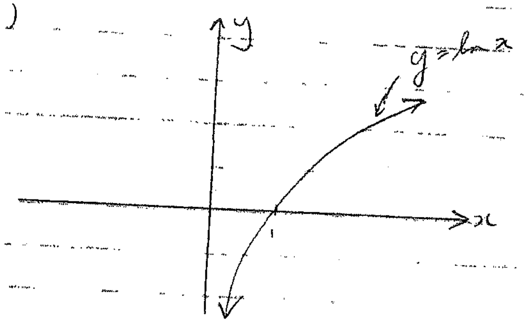
$\therefore\ 2\theta = \cancel{0.72''}\ (5.7608\ldots\times10^{-3})^{\circ}$

ie required angle is ~~0.72 seconds~~ 21 seconds

(to 2 sig fig)

Q5

a) (i)



(ii) $A \doteq \dfrac{h}{3}\left[y_1+y_5+2(y_3)+4(y_2+y_4)\right]$

$\doteq \dfrac{1}{3}\left(\ln 1+\ln 5+2\ln 3+4(\ln 2+\ln 4)\right)$

$\doteq \dfrac{12.124}{3}$ OR $\dfrac{1}{6}(b-a)\left[f(a)+f(b)+4f\left(\dfrac{a+b}{2}\right)\right]$ (twice)

$\doteq 4.04$ (to 2 dec. pl.)

(iii) $\dfrac{d}{dx}(x\ln x - x) = x\cdot\dfrac{1}{x} + \ln x\cdot 1 - 1$

$= \ln x$

$\therefore\ \displaystyle\int_1^5 \ln x\,dx = \left[x\ln x - x\right]_1^5$

$= 5\ln 5 - 5 + 1$

$= 5\ln 5 - 4$

b) $\dfrac{d^2y}{dx^2} = x^{-2}$

$\therefore\ \dfrac{dy}{dx} = -x^{-1} + C$

when $x=1$, $\dfrac{dy}{dx}=2$

$\therefore\ 2 = -1 + C$

$C = 3$

$\therefore\ \dfrac{dy}{dx} = -x^{-1}+3$

$\therefore\ y = -\ln x + 3x + d$

when $x=1, y=-1$

$\therefore\ -1 = -\ln 1 + 3 + d$

$\therefore\ d = -4$

$\therefore\ y = -\ln x + 3x - 4$

c) $\log_5 7 = \dfrac{\log_{10}7}{\log_{10}5}$

$\doteq 1.21$ (to 3 sig figs)

Q6

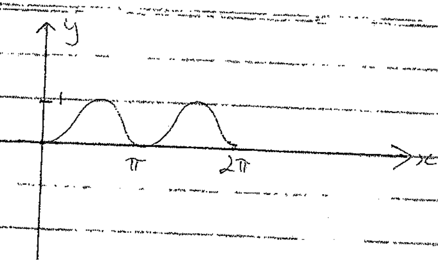
a) $\dfrac{d}{dx}\left(xe^{x}-e^{x}\right) = xe^{x}+e^{x}-e^{x}$

$= xe^{x}$

$\therefore\ \displaystyle\int_0^1 xe^{x}\,dx = \left[xe^{x}-e^{x}\right]_0^1$

$= e - e + 1 = 1$

b) (i)



(ii) $y = \sin^2 x$ has a period of π^c

c) (i) $\frac{dm}{dt} = -km$

If $m = m_0 e^{-kt}$, $LHS = \frac{d}{dt} m_0 e^{-kt}$

$= -m_0 k e^{-kt}$

$RHS = -km$

$= -k m_0 e^{-kt}$

$= LHS$

$\therefore m = m_0 e^{-kt}$ is a soln to $\frac{dm}{dt} = -km$

(ii) $\frac{m_0}{2} = m_0 e^{-10k}$

$\frac{1}{2} = e^{-10k}$

$\ln \frac{1}{2} = -10k$

$k = -\frac{1}{10} \ln \frac{1}{2}$

$= \frac{1}{10} \ln 2$

(iii) $m = m_0 e^{-\frac{1}{10}\ln 2 \times 30}$

$= m_0 e^{-3\ln 2}$

$= m_0 e^{\ln \frac{1}{8}}$

$= \frac{m_0}{8}$

ie $\frac{1}{8}$ of original mass left.

(iv) $\frac{m_0}{5} = m_0 e^{-\frac{1}{10}\ln 2\, t}$

$\frac{1}{5} = e^{-\frac{1}{10}\ln 2\, t}$

$\ln \frac{1}{5} = -\frac{1}{10} \ln 2 . t$

$t = \frac{10 \ln 5}{\ln 2}$

$\doteqdot 23$ years.