St George Girls High School

Year 12

Mid-HSC Course Examination

2006

Mathematics

General Instructions

- Working time: 1½ hours
- Reading time: 5 minutes
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.

Total marks – 78

- Attempt Questions 1 – 6
- All questions are of equal value
- Begin each question in a new booklet
- All necessary working should be shown in every question.

Question 1 – (13 marks) – (Start a new page) Mar

a) Find the primitive of:

(i) $\int (1-2x)^3$ $= \frac{(1-2x)^4}{-2\cdot 4} + C$ $= -\frac{(1-2x)^4}{8} + C$ 1

(ii) $\int \frac{1}{3x^2}$ 2

(iii) $\int \frac{3x^2 - x^3}{x^2}$ $= 3 - x$ 2

$= 3x - \frac{x^2}{2} + C$

b) (i) Using the table of standard integrals show that $\int_{-3}^{3} \frac{1}{\sqrt{x^2+16}}\, dx = \ln 4$ 2

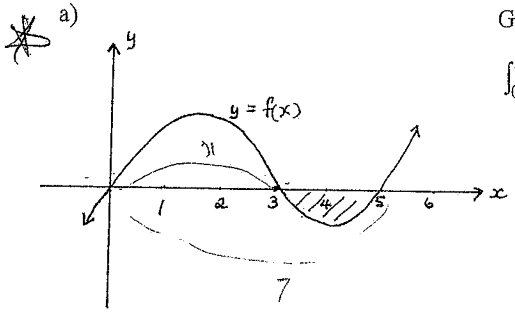
(ii) Use Simpson's rule with 3 function values to approximate $\int_{-3}^{3} \frac{1}{\sqrt{x^2+16}}\, dx$. Hence, show that $\ln 4 \doteqdot 1\cdot 4$ 2

c) (i) Without using calculus, sketch $y = x^2 - x$ 1

(ii) Calculate the area between the curve $y = x^2 - x$, the x axis and the line $x = -1$ 3

Question 2 – (13 marks) – (Start a new page) **Marks**

a)

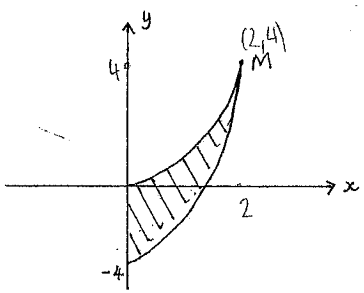


Given that $\int_0^3 f(x)\,dx = 11$ and that $\int_0^5 f(x)\,dx = 7$

(i) What is the area of the shaded region? 2

(ii) What is the value of $\int_3^5 f(x)\,dx$?

b)



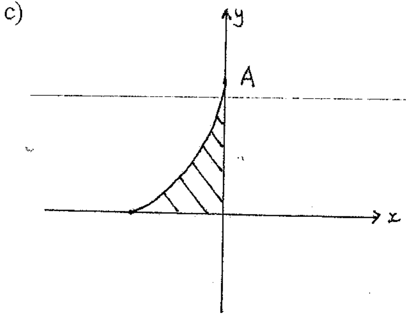
M is a point of intersection of $y = x^2$ and $y = 2x^2 - 4$ as shown.

(i) Find the x coordinate of M. 1

(ii) Find the area of the shaded region. 3

Question 2 (cont'd) **Mark**

c)



The shaded region is bounded by the coordinate axes and part of the curve $y = (x+2)^2$

(i) What are the coordinates of point A?

(ii) If the shaded region is revolved about the y axis show that the volume V of the resulting solid is given by $V = \pi \int_0^4 \left(\sqrt{y} - 2\right)^2 dy$ 2

(iii) Calculate the volume. 4

Question 3 – (13 marks) – (Start a new page) **Marks**

a) Differentiate

(i) e^{1-3x} 1

(ii) $2xe^{x}$ 1

b) Find $\int e^{\frac{x}{3}}\,dx$ 1

c) Evaluate $\int_0^{\ln 3} e^{2x}\,dx$ 3

d) (i) Find the equation of the normal to the curve $y = e^{x}$ at the point where $x = 1$. 3

$y' = e^x$ normal $= \frac{-1}{e^x}$ $x=1$ $\frac{-1}{e}$

(ii) Where does this normal cut the x axis? 1

(iii) Find the area between this normal, the curve $y = e^{x}$ and the coordinate axes. 3

Question 4 – (13 marks) – (Start a new page) **Marks**

a) Evaluate $e^{-3.5}$ correct to 3 significant figures. 1

b) Expand $\left(1+e^{x}\right)^{2}$ 1

c) Consider the function $f(x) = e^{2x}(1-x)$ where $-3 \le x \le 1$

(i) Copy and complete this table of values. Give values correct to 2 decimal places where necessary. 1

x	-3	-2	-1	0	1
$f(x)$	0.01	0.05	0.28	1	

(ii) Differentiate $f(x)$ and hence show that the function has only one stationary point. Find the coordinates of this point. 4

(iii) Determine the nature of this stationary point. 1

(iv) Sketch the curve $y = f(x)$ for $-3 \le x \le 1$ 2

(v) Using the trapezoidal rule with 5 function values approximate the area under the curve $y = f(x)$ for $-3 \le x \le 1$ 2

(vi) From your diagram, decide whether this approximation is an over estimate or an under estimate of the true value of the area. Give a brief reason. 1

Question 5 – (13 marks) – (Start a new page) **Marks**

a) Evaluate $\log_e 11$ to 2 significant figures. 1

b) Given that $\log a = 0.86$ and $\log b = 0.42$ find the value of:

(i) $\log \frac{a}{b}$ 2

(ii) $\log \sqrt{ab}$ 2

c) Solve $\log_2(x+1) + \log_2(x-3) = 5$ 4

d) (i) State the domain and range of $y = \log_e(2x+1)$ 2

(ii) Sketch $y = \log_e(2x+1)$ showing all essential features. 2

Question 6 – (13 marks) – (Start a new page) **Marks**

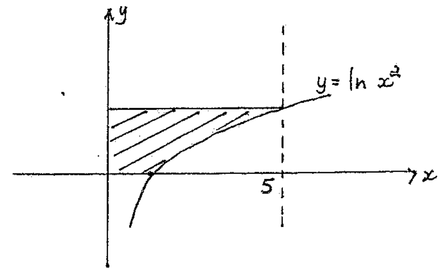
a) Differentiate

(i) $y = \ln(3x+2)$ 1

(ii) $y = \ln e^{3x}$ 1

(iii) $y = \ln\sqrt{1-4x}$ 2

b) The graph of $y = \ln x^2$ is drawn.



Find the exact value of the shaded area, in simplified form. 6

c) Given that $a^2 + b^2 = 23ab$ express $\left(\frac{a+b}{5}\right)^2$ in terms of a and b. Hence show that $\log\left(\frac{a+b}{5}\right) = \frac{1}{2}(\log a + \log b)$ 3

$\log_2 100 = 10$

End of Paper

Year 12

HSC Mid-Course Exam

Mathematics

Question 1

a.) (i) $\frac{(1-2x)^4}{4\times -2}+c = -\frac{(1-2x)^4}{8}+c$

(ii) $\frac{x^{-2}}{3} = f'(x)$ $\boxed{\frac{1}{3x^2} = \frac{x^{-2}}{3}}$

$f(x) = \frac{x^{-1}}{3\times -1}+c$

$= \frac{-1}{3x}+c$

(iii) $f'(x) = 3-x$

$f(x) = 3x - \frac{x^2}{2}+c$

b.) (i) $\int_{-3}^{3} \frac{dx}{\sqrt{x^2+16}} = \left[\ln(x+\sqrt{x^2+16})\right]_{-3}^{3}$

$= \ln(3+\sqrt{9+16}) - \ln(-3+\sqrt{9+16})$

$= \ln 8 - \ln 2$

$= \ln\left(\frac{8}{2}\right)$

$= \ln 4$

(ii)

x	-3	0	3
$f(x)$	$\frac{1}{5}$	$\frac{1}{4}$	$\frac{1}{5}$

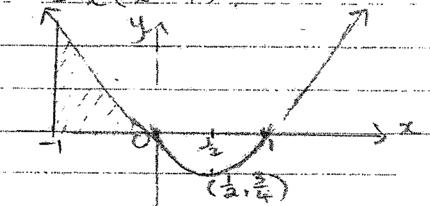
$\int_{-3}^{3} \frac{dx}{\sqrt{x^2+16}} \doteqdot \frac{6}{6}\left[\frac{1}{5}+4\left(\frac{1}{4}\right)+\frac{1}{5}\right]$

$= 1.4$

$\therefore \ln 4 \doteqdot 1.4$

c.) (i) $y = x^2 - x$

$= x(x-1)$



(ii) $A = \int_{-1}^{0} (x^2-x) - \int_{0}^{1}(x^2-x)\,dx$

$= \left[\frac{x^3}{3}-\frac{x^2}{2}\right]_{-1}^{0} - \left[\frac{x^3}{3}-\frac{x^2}{2}\right]_{0}^{1}$

$= \left(0-\left(-\frac{1}{3}-\frac{1}{2}\right)\right) - \left(\frac{1}{3}-\frac{1}{2}-0\right)$

$= 1$ unit^2

Question 2

a.) (i) shaded region $= 4$ units^2

(ii) $\int_3^5 f(x)\,dx = -4$

b.) (i) $y = x^2$, $y = 2x^2-4$.

$2x^2-4 = x^2$

$x^2-4=0$.

$(x-2)(x+2)=0$

$x = \pm 2$.

when $x=2$, $y=4$

$\therefore$ m is $(2,4)$ since it's in the first quadrant.

(ii) $A = \int_0^2 (x^2-(2x^2-4))\,dx$

$= \int_0^2 (-x^2+4)\,dx$.

$= \left[4x-\frac{x^3}{3}\right]_0^2$

$= \left(8-\frac{8}{3}\right)-(0)$

$= \frac{16}{3}$ units^2.

c.) (i) $y = (x+2)^2$

when $x=0$, $y=4$.

$\therefore$ A is $(0,4)$.

(ii) $V = \pi\int_a^b x^2\,dy$.

$\sqrt{y} = x+2$

$\therefore V = \pi\int_0^4 (\sqrt{y}-2)^2\,dy$

(iii) $V = \pi\int_0^4 (y-4y^{1/2}+4)\,d$

$= \pi\left[\frac{y^2}{2}-\frac{8y^{3/2}}{3}+4y\right]_0^4$

$= \pi\left(\left(8-\frac{64}{3}+16\right)-0\right)$

$= \frac{8\pi}{3}$ units^3

Question 3

a.) (i) $y' = -3e^{1-3x}$

(ii) $y' = e^x.(2) + 2x(e^x)$

$= 2e^x(1+x)$.

b.) $\int e^{x/3}\,dx = \dfrac{e^{x/3}}{\frac{1}{3}} + C$

$= 3e^{\frac{x}{3}} + C$

c.) $\int_0^{\ln 3} e^{2x}\,dx = \left[\dfrac{e^{2x}}{2}\right]_0^{\ln 3}$

$= \frac{1}{2}\left(e^{2\ln 3} - e^0\right)$

$= \frac{1}{2}\left(e^{\ln 3^2} - 1\right)$

$= \frac{1}{2}(9-1)$

$= 4$

d.)(i) $y = e^x \quad x = 1$

$\dfrac{dy}{dx} = e^x$

when $x = 1$, $\dfrac{dy}{dx} = e$

$\therefore$ gradient of normal is $-\dfrac{1}{e}$

$y - e = -\dfrac{1}{e}(x-1)$

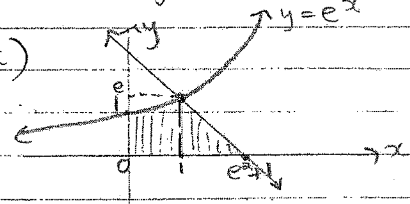
$ey - e^2 = -x + 1$

$\therefore$ equation of normal is

$x + ey - e^2 - 1 = 0$.

(ii) when $y = 0$, $x = e^2 + 1$.

(iii)



$A = \int_0^1 e^x\,dx + \left(\frac{1}{2} \times e^2 \times e\right)$

$= \left[e^x\right]_0^1 + \dfrac{e^3}{2}$

$= (e^1 - e^0) + \dfrac{e^3}{2}$

$= \left(e + \dfrac{e^3}{2} - 1\right)$ units^2

Question 4.

a.) $e^{-3.5} = 0.030197\ldots$

$= 0.0302$ (3 sig. fig.)

b.) $(1+e^x)^2 = 1 + 2e^x + e^{2x}$

c.) (i) $f(x) = e^{2x}(1-x)$

x	-3	-2	-1	0	1
$f(x)$	0.01	0.05	0.27	1	0

(ii) $f'(x) = (1-x).2e^{2x} + e^{2x}.(-1)$

$= e^{2x}(2 - 2x - 1)$

$= e^{2x}(1 - 2x)$.

stat. pt. occurs when $f'(x) = 0$.

$e^{2x}(1-2x) = 0$.

$\therefore e^{2x} = 0$ or $1 - 2x = 0$.

no solution. $\quad x = \frac{1}{2}$

$\therefore$ there is only one stat. pt at $x = \frac{1}{2}$.

when $x = \frac{1}{2}$, $y = \frac{e}{2}$

$\left(\frac{1}{2}, \frac{e}{2}\right)$.

(iii) $f''(x) = (1-2x).2e^{2x} + e^{2x}.(-2)$

$= 2e^{2x}(1 - 2x - 1)$

$= -4xe^{2x}$

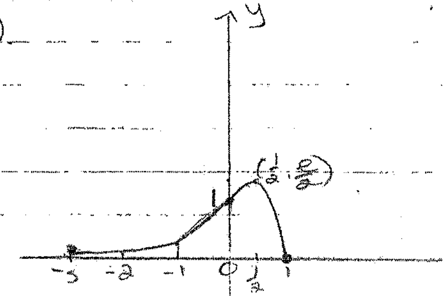
when $x = \frac{1}{2}$, $f''(x) = -4.\frac{1}{2}.e^{2.\frac{1}{2}}$

$= -2e$.

< 0.

$\therefore$ there is a max. stat. pt at $\left(\frac{1}{2}, \frac{e}{2}\right)$.

(iv)



(v) $A \approx \frac{1}{2}\left[f(-3) + 2\left(f(-2) + f(-1) + f(0)\right.\right.$

$= \frac{1}{2}\left[0.01 + 2(0.05 + 0.27 + 1) + 0\right]$

$= 1.325$ units^2.

(vi) The area is an underestima because the trapeziums drai from $x=0$ to $x=-3$ would onl just be over the true area, however from $x=0$ to $x=1$ a large area is neglected.

Question 5

a.) $\log_e 11 \doteqdot 2.4$ (2 sig. fig.)

b.) $\log a = 0.86 \quad \log b = 0.42$

(i) $\log\left(\frac{a}{b}\right) = \log a - \log b$

$= 0.86 - 0.42$

$= 0.44$

(ii) $\log\sqrt{ab} = \frac{1}{2}\log ab$

$= \frac{1}{2}(\log a + \log b)$

$= \frac{1}{2}(0.86 + 0.42)$

$= 0.64$

c.) $\log_2(x+1) + \log_2(x-3) = 5$

$\log_2((x+1)(x-3)) = 5$

$(x+1)(x-3) = 2^5$

$x^2 - 2x - 3 = 32$

$x^2 - 2x - 35 = 0$

$(x+5)(x-7) = 0$

$\therefore x = -5, 7$

But $x \neq -5 \quad \therefore x = 7$

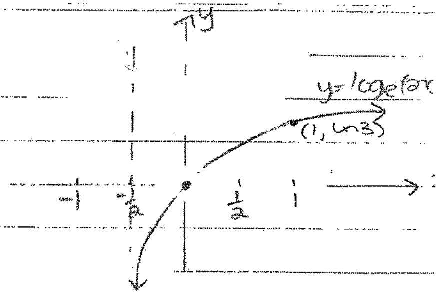
d.) (i) $2x+1 > 0$

$x > -\frac{1}{2}$

$\therefore$ domain: $x > -\frac{1}{2}$

range: all real y

(ii)



when $x = 0, \ y = 0$

Question 6

a.) (i) $y = \ln(3x+2)$

$y' = \dfrac{3}{3x+2}$

(ii) $y = \ln e^{3x}$

$= 3x \ln e$

$= 3x$

$y' = 3.$

(iii) $y = \ln\sqrt{1-4x}$

$= \frac{1}{2}\ln(1-4x).$

$= \frac{1}{2} \times \dfrac{-4}{1-4x}$

$= \dfrac{-2}{1-4x}$

b.) $y = \ln x^2$

$= 2\ln x.$

$\ln x = \frac{y}{2}$

$e^{y/2} = x.$

$A = \int_0^{\ln 25} e^{y/2}\,dy$

$= \left[2e^{y/2}\right]_0^{\ln 25}$

$= 2(e^{\frac{1}{2}\ln 25} - e^0)$

$= 2(e^{\ln 5} - 1)$

$= 2(5-1)$

$= 8 \text{ units}^2.$

c.) $a^2 + b^2 = 23ab.$

$(a+b)^2 = a^2 + 2ab + b^2$

$\therefore \left(\frac{a+b}{5}\right)^2 = \dfrac{25ab}{25}$

$= ab.$

$\therefore \log\left(\frac{a+b}{5}\right)^2 = \log(ab)$

$2\log\left(\frac{a+b}{5}\right) = \log a + \log$

$\therefore \log\left(\frac{a+b}{5}\right) = \frac{1}{2}(\log a + \log$