St George Girls' High School

Higher School Certificate Course

Assessment Task 1

December 2002

Mathematics Extension 1

General Instructions

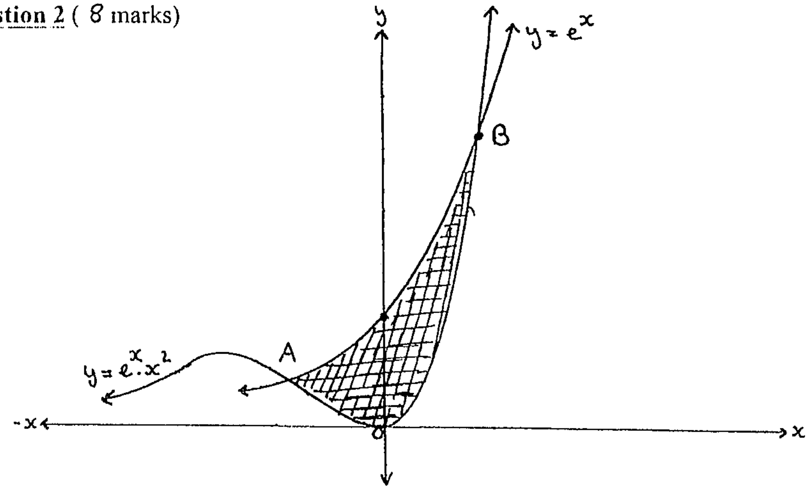
- Working time – 70 minutes
- Write using black or blue pen
- Attempt all questions.
- Start each question on a new page.
- Show ALL working.
- Marks for each question are shown in right column
- Complete cover sheet clearly showing
  - your name
  - mathematics class and teacher

Question 1 (9 marks)

		Marks
a)	Find the equation of the curve which passes through the point (1, 5) and which has its first derivative given by $\frac{dy}{dx} = x^2 - 2x + 1$	2
b)	Evaluate $\int_0^1 x^2(2 - x^3)^3\,dx$	2
c)	(i) Differentiate $\frac{1}{\sqrt{1+x^2}}$	1
	(ii) Hence, evaluate $\int_0^{\sqrt{3}} \frac{x\,dx}{(1+x^2)\sqrt{1+x^2}}$	2
d)	The area enclosed by the parabola $x^2 = 4y$ and the line $y = 1$ is rotated about the x-axis. Find the volume of the solid so formed.	2

Question 2 (8 marks) **Marks**

a)



(i) Show that $A(-1, e^{-1})$ and $B(1, e^1)$ lies on both curves by solving simultaneous equations. [The curves are $y = e^x$ and $y = e^x . x^2$] 2

(ii) Show that the area bounded by $y = e^x$ and the x-axis from $x = -1$ to $x = 1$ is $\left(e - \frac{1}{e}\right)$ sq units. 2

(iii) Use Simpson's Rule with five functional values to estimate the area under the curve $y = e^x . x^2$ [from $x = -1$ to $x = 1$; correct to two decimal places] 3

(iv) Hence, show that an approximation for the shaded area is 1.46 units^2. 1

Question 3 (9 marks) Ma

a) (i) Show that $f(x) = \dfrac{4x}{x^2 + 1}$ is an odd function.

(ii) Find the co-ordinates of any stationary points of $f(x) = \dfrac{4x}{x^2 + 1}$

(iii) Determine the nature of the stationary points.

(iv) Describe the behaviour of $f(x)$ as x approaches positive or negative infinity.

(v) Sketch the graph of $y = f(x)$

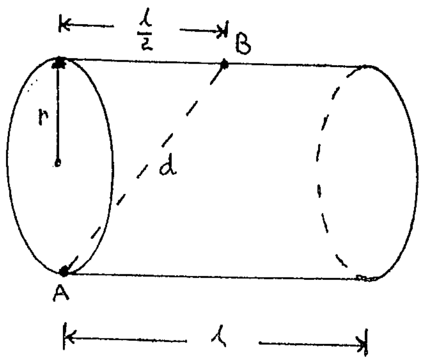
b) Show that $\dfrac{1}{x-1} - \dfrac{1}{x-2} = \dfrac{-1}{(x-1)(x-2)}$

and hence show that $\displaystyle\int \frac{dx}{x^2 - 3x + 2} = \log_e \frac{(x-2)}{(x-1)} + C$

Question 4 (8 marks) **Marks**

a) Find the second derivative of x^2e^{-x} 3

b)



The diagram shows a cylindrical barrel of length l and radius r. The point A is at one end of the barrel, at the very bottom of the rim. The point B is at the very top of the barrel, half-way along its length. The length of AB is d.

(i) Show that the volume of the barrel is:

$$V=\frac{\pi l}{4}\left(d^2-\frac{l^2}{4}\right)$$

(ii) Find l in terms of d if the barrel has maximum volume for the given d. [d being constant] 3

Question 5 (10 marks) **Marks**

a) A spherical balloon is expanding so that its volume $V\text{mm}^3$ increases at a constant rate of 72mm^3 per second.

(i) Show that $\dfrac{dr}{dt}=\dfrac{18}{\pi r^2}$ 2

(ii) What is the rate of increase of its surface area $A\text{mm}^2$, when the radius is 12mm? 3

[Given: $V=\frac{4}{3}\pi r^3$ and $A=4\pi r^2$ for a sphere]

b) (i) Show that the equation of the chord joining $P(2p,p^2)$ and $Q(2q,q^2)$ on the parabola $x^2=4y$ is $y=\dfrac{p+q}{2}x-pq$ 2

(ii) Find the co-ordinates of the midpoint $M(x, y)$ of the interval PQ. 1

(iii) If PQ passes through the external point $A(-2,-1)$, show that $(p+q)+pq-1=0$.

Hence, show that the equation of the locus of M is $(x+1)^2=2(y+\frac{3}{2})$ 2

Assessment Task 1

Extension 1

Question 1

$\frac{dy}{dx} = x^2 - 2x + 1 \qquad (1, 5)$

$y = \frac{x^3}{3} - x^2 + x + C$

b. $5 = \frac{1}{3} - 1 + 1 + C$

$C = 4\frac{2}{3}$

[illegible]

equation of the curve

$\int [illegible]$

$\int [illegible] = \frac{-1}{12}\int_0^1 -12x^2(2-x^3)^3\,dx$

$= -\frac{1}{12}\left[(2-x^3)^4\right]_0^1$

$= -\frac{1}{12}(1 - 16)$

$= \frac{15}{12}$

$= \frac{5}{4}$

(i) $\frac{d}{dx}\left(\frac{1}{\sqrt{1+x^2}}\right) = \frac{d}{dx}\left((1+x^2)^{-\frac{1}{2}}\right)$

$= -\frac{1}{2}(1+x^2)^{-3/2}\cdot 2x$

$= \frac{-x}{\sqrt{(1+x^2)^3}}$

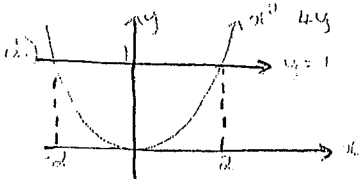
$= \frac{-x}{(1+x^2)\sqrt{1+x^2}}$

(ii) $-\int_0^{\sqrt{3}} \frac{-x\,dx}{(1+x^2)\sqrt{1+x^2}} = -\left[\frac{1}{\sqrt{1+x^2}}\right]_0^{\sqrt{3}}$

$= -\left(\frac{1}{2} - 1\right)$

$= \frac{1}{2}$

(d)



$V = 2\pi\int_0^2 \left(1 - \frac{x^4}{16}\right)dx \qquad A = 2\int_0^2 \left(1 - \frac{x^2}{4}\right)dx$

$= 2\pi\left[x - \frac{x^5}{80}\right]_0^2$

$2\pi\left(2 - \frac{2}{5} - 0\right)$

$\frac{16}{5}\pi$ units^3

Question 2

a) (i) $y = e^x \quad (1)$

$y = x^2 e^x \quad (2)$

$(2) \div (1) \quad x^2 = 1$

$\therefore x = \pm 1$

when $x = 1$, $y = e$

when $x = -1$, $y = e^{-1}$

$\therefore (1, e)$ and $(-1, e^{-1})$ lie on both curves

(ii) $A = \int_{-1}^{1} e^x\,dx$

$= \left[e^x\right]_{-1}^{1}$

$= e - e^{-1}$

$= \left(e - \frac{1}{e}\right)$ units^2

(iii)

x	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1
y	$\frac{1}{e}$	$\frac{1}{4\sqrt{e}}$	0	$\frac{\sqrt{e}}{4}$	e

$A \approx \frac{1}{6}\left[\frac{1}{e} + 4\left(\frac{1}{4\sqrt{e}}\right) + 0\right] + \frac{1}{6}\left[0 + 4\left(\frac{\sqrt{e}}{4}\right) + e\right]$

$= \frac{1}{6}\left(\frac{1}{e} + \frac{1}{\sqrt{e}} + \sqrt{e} + e\right)$

$= 0.89$ (2 d.p.)

(iv) Shaded Area $\approx \left(e - \frac{1}{e}\right) - 0.89$

$= 1.46$ units^2 (2 d.p.)

(i) $f(x) = \dfrac{4x}{x^2+1}$.

$f(-x) = \dfrac{-4x}{x^2+1}$

$-f(x) = -\left(\dfrac{4x}{x^2+1}\right)$

$= \dfrac{-4x}{x^2+1}$.

since $f(-x) = -f(x)$ then $f(x)$ is odd.

$f'(x) = \dfrac{(x^2+1)(4)-(4x)(2x)}{(x^2+1)^2}$

$= \dfrac{-4x^2+4}{(x^2+1)^2}$

stat. pts occur when $f'(x)=0$

$\dfrac{4-4x^2}{(x^2+1)^2} = 0$

$4 = 4x^2$

$x^2 = 1$

$x = \pm 1$

when $x = 1$, $y = 2$

when $x = -1$, $y = -2$

∴ stat. pts occur at $(1,2)$ and $(-1,-2)$

)

x	0	1	2
$f'(x)$	4	0	$\frac{-12}{25}$

∴ maximum at $(1,2)$

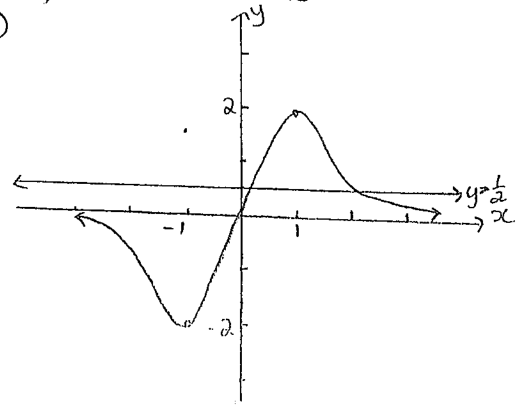
x	-2	-1	0
$f'(x)$	$\frac{-12}{25}$	0	4

∴ minimum at $(-1,-2)$

(iv.) $f(x) \to 0$ as $x \to \infty$

$f(x) \to 0$ as $x \to -\infty$

(v.)



b.) $\dfrac{1}{x-1} - \dfrac{1}{x-2} = \dfrac{(x-2)-(x-1)}{(x-1)(x-2)}$

$= \dfrac{-1}{(x-1)(x-2)}$

$\displaystyle\int \frac{dx}{x^2-3x+2} = \int \frac{dx}{x-1} - \int \frac{dx}{x-2}$

$= \log_e(x-1) - \log_e(x-2) + C$

$= \log_e\left(\dfrac{x-1}{x-2}\right) + C$

Question 5.

a.) (i) $\dfrac{dV}{dt} = 72$ $\quad V = \frac{4}{3}\pi r^3$

$\dfrac{dV}{dr} = 4\pi r^2$

$\dfrac{dr}{dt} = \dfrac{dV}{dt} \cdot \dfrac{dr}{dV}$

$= 72 \cdot \dfrac{1}{4\pi r^2}$

$= \dfrac{18}{\pi r^2}$

(ii) $\dfrac{dA}{dt} = \dfrac{dA}{dr} \cdot \dfrac{dr}{dt}$ $\quad A = 4\pi r^2$

$\dfrac{dA}{dr} = 8\pi r$.

$= 8\pi r \cdot \dfrac{18}{\pi r^2}$

$= \dfrac{144}{r}$

when $r = 12$ $\quad \dfrac{dA}{dt} = \dfrac{144}{12}$

$= 12$ mm/sec.

∴ the rate of increase of its surface area is 12mm/sec.

b).(i) $P(2p, p^2)$ $\quad Q(2q, q^2)$

gradient of $PQ = \dfrac{q^2-p^2}{2q-2p}$

$= \dfrac{(q-p)(q+p)}{2(q-p)}$

$= \dfrac{p+q}{2}$

$y - p^2 = \dfrac{p+q}{2}(x-2p)$

$y = \dfrac{p+q}{2}x - p^2 - pq + p^2$

$y = \dfrac{p+q}{2}x - pq$ is equation

(ii) $x = \dfrac{2p+2q}{2}$, $y = \dfrac{p^2+q^2}{2}$

∴ mid-point of PQ is

$\left(p+q, \dfrac{p^2+q^2}{2}\right)$

(iii) $A(-2,-1)$.

$-1 = \left(\dfrac{p+q}{2}\right)(-2) - pq$.

∴ $(p+q) + pq - 1 = 0$.

$x = p+q$ $\quad y = \dfrac{p^2+q^2}{2}$

$pq = 1 - x$

$= \dfrac{(p+q)^2 - 2pq}{2}$

$2y = x^2 - 2pq$

$2y = x^2 - 2(1-x)$

$= x^2 - 2 + 2x$

$x^2 + 2x = 2y + 2$

$(x+1)^2 = 2y + 2 + 1$

$(x+1)^2 = 2y + 3$

∴ $(x+1)^2 = 2\left(y + \frac{3}{2}\right)$ is the equation of the locus.

Question 4

a.) $\frac{d}{dx}(x^2 e^{-x}) = (e^{-x})(2x) + (x^2)(-e^{-x})$.

$= 2xe^{-x} - x^2e^{-x}$

$\frac{d}{dx}(2xe^{-x} - x^2e^{-x}) = (e^{-x})(2) + (2x)(-e^{-x})$

$- (e^{-x}(2x) + (x^2)(-e^{-x}))$

$= 2e^{-x} - 2xe^{-x} - 2xe^{-x} + x^2e^{-x}$

$= e^{-x}(2 - 4x + x^2)$.

b.) (i) $V = \pi r^2 h$

$V = \pi l . \left(\frac{d^2 - \frac{l^2}{4}}{4}\right)$

$= \frac{\pi l}{4}\left(d^2 - \frac{l^2}{4}\right)$

$d^2 = \left(\frac{l}{2}\right)^2 + (2r)^2$

$= \frac{l^2}{4} + 4r^2$

$4r^2 = d^2 - \frac{l^2}{4}$

$r^2 = \frac{(d^2 - \frac{l^2}{4})}{4}$

(ii) $V = \frac{\pi l d^2}{4} - \frac{\pi l^3}{16}$

$\frac{dV}{dt} = \frac{\pi d^2}{4} - \frac{3\pi l^2}{16}$

when $\frac{dV}{dt} = 0$

$\frac{\pi}{4}\left(d^2 - \frac{3l^2}{4}\right) = 0$

$d^2 = \frac{3l^2}{4}$

$3l^2 = 4d^2$

$l = \sqrt{\frac{4d^2}{3}}$

$l = \frac{2d}{\sqrt{3}}$, since $l > 0$