Su Yee

St George Girls' High School

Higher School Certificate Course

Assessment Task 1

December 2002

Mathematics

General Instructions

- Working time – 70 minutes
- Write using black or blue pen
- Attempt all questions.
- Start each question on a new page.
- Show ALL working.
- Marks for each question are shown in right column
- Complete cover sheet clearly showing
  - your name
  - mathematics class and teacher

MARKS

Question 1 (12 marks)

(a) Find the first and second derivative of:

(i) $f(x) = \dfrac{1}{1-x}$ **2**

(ii) $y = (3x-1)^4$ **2**

(b) The curve $y = x^3 + ax^2 + bx + c$ has a relative minimum at $x = 5$ and a point of inflexion at the point (2, 3). Find the values of a, b and c. **5**

(c) For what values of x is the curve $y = x^4 - 4x^3$ concave down. **2**

(d) What is meant by a 'normal' to a curve? **1**

Question 2 (12 marks) START A NEW PAGE

For $y = x^3 + 2x^2 - 4x - 6$,

(a) find:

(i) the coordinates of any stationary points and determine their nature. **4**

(ii) any points of inflexion. **2**

(b) Sketch the graph of the above function over the domain $-3 \le x \le 2$. Show any turning points, points of inflexion and the end points. **3**

(c) Find the centre and radius of the circle **3**

$$x^2 + y^2 + 6x - 4y - 23 = 0$$

Question 3 (13 marks) START A NEW PAGE

(a) One of the roots of $2kx^2+11x-10k$ is $x=2$. Find:

(i) the value of k. **3**

(ii) the other root.

(b) If α and β are the roots of $2x^2-5x-6=0$, evaluate:

(i) $\alpha\beta$ **1**

(ii) $\alpha+\beta$ **1**

(iii) $\alpha^2+\beta^2$ **2**

(iv) $(3-\alpha)(3-\beta)$ **2**

(v) $\alpha^2\beta+\alpha\beta^2$ **2**

(c) Determine whether the expression $2x^2+3x+5$ is positive definite, negative definite or indefinite. Justify your answer. **2**

Question 4 (12 marks) START A NEW PAGE

(a) Express x^2+3x+4 in the form $A(x-1)^2+B(x-1)+C$ **3**

(b) Find the value of k so that the equation $3x^2+4x-2k=0$ has:

(i) equal roots. **3**

(ii) one root zero.

(iii) roots that are the reciprocals of each other.

(c) Find the primitive of:

(i) $3x-2$ **1**

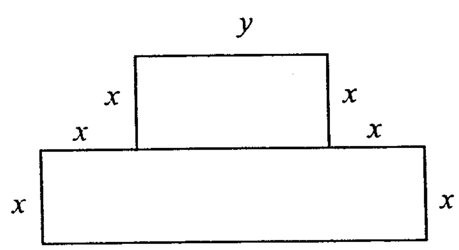
(ii) $4x^7$ **1**

(iii) $\frac{6}{x^2}$ **1**

(d) Find the equation of the locus of a point P moving so that its distance from A(-1, 3) is equal to the distance from the line $y = -3$. **3**

Question 5 (12 marks) START A NEW PAGE

(a) The fields below are to be enclosed using 120 metres of fencing (all lines are fences)



(i) Show that $8x + 3y = 120$ **1**

(ii) If the total area enclosed is A m^2 , show that A = $2x^2 + 2xy$ and express this in terms of x alone. **2**

(iii) Hence find the values of x and y that gives the greatest enclosed area and calculate this maximum area. **4**

(b) Given $\frac{dy}{dx} = 6x - 2$ and $y = 4$ when $x = 1$, find y when $x = -2$. **3**

(c) Find the equation of the parabola with vertex (4, -1) and focus (2, -1). **2**

Solutions to CT 1 - Yr 12 2003

1. a) i) $f(x) = \frac{1}{1-x}$

$f'(x) = \frac{(1-x)\times 0 - 1\times -1}{(1-x)^2}$

$= \frac{1}{(1-x)^2}$

$f''(x) = \frac{(1-x)^2\times 0 - 1\times 2(1-x)^1\times -1}{(1-x)^4}$

$= \frac{2(1-x)}{(1-x)^4}$

$= \frac{2}{(1-x)^3}$

ii) $y = (3x-1)^4$

$y' = 4(3x-1)^3\times 3$

$= 12(3x-1)^3$

$y'' = 36(3x-1)^2\times 3$

$= 108(3x-1)^2$

(b) $y = x^3 + ax^2 + bx + c$

$y' = 3x^2 + 2ax + b$

$y'' = 6x + 2a$

At $x = 2$: $y'' = 0$

$\therefore 12 + 2a = 0$

$a = -6$

At $x = 5$: $y' = 0$

$75 + 2a\times 5 + b = 0$

$75 - 60 + b = 0$

$\therefore -15 = b$

(2,3) is on curve

$\therefore 3 = 8 + 4a + 2b + c$

$3 = 8 - 24 - 15 + c$

$\therefore c = 34$

(c) Concave down $y'' < 0$

$y = x^4 - 4x^3$

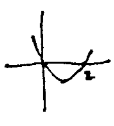
$y' = 4x^3 - 12x^2$

$y'' = 12x^2 - 24x$

$\therefore 12x^2 - 24x < 0$

$12x(x-2) < 0$

$0 < x < 2$



(d) normal is a line perpendicular to tangent.

2. a) $y = x^3 + 2x^2 - 4x - 6$

$y' = 3x^2 + 4x - 4$

$y'' = 6x + 4$

i) Stat pts when $y' = 0$

$3x^2 + 4x - 4 = 0$

$(3x-2)(x+2) = 0$

$x = \frac{2}{3}$ $\quad x = -2$

$y = -7\frac{13}{27}$ $\quad y = 2$

At $x = \frac{2}{3}$: $y'' = 8 > 0$

$\therefore$ Min t pt at $\left(\frac{2}{3}, -7\frac{13}{27}\right)$

At $x = -2$: $y'' = -8 < 0$

$\therefore$ Max t pt at $(-2, 2)$

(ii) Inflexions when $y'' = 0$ & sign change

$6x + 4 = 0$

$\therefore x = -\frac{2}{3}$ $\quad y = -2\frac{20}{27}$

x	-1	$-\frac{2}{3}$	0
y''	-2	0	4

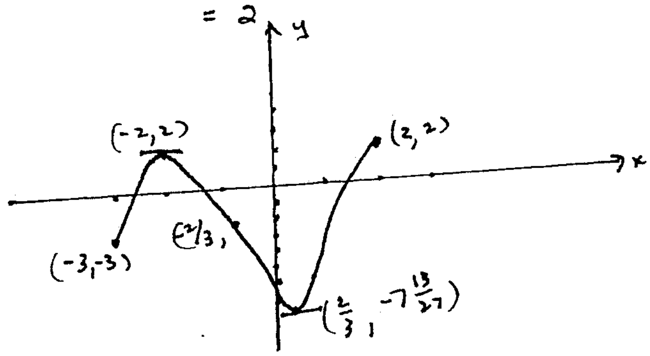
$\therefore$ Pt of inflexion at $x = -\frac{2}{3}$ since concavity changes.

(b) $x = -3$: $y = -27 + 18 + 12 - 6$

$= -3$

$x = 2$: $y = 8 + 8 - 8 - 6$

$= 2$



(c) $x^2 + y^2 + 6x - 4y - 23 = 0$

$(x^2 + 6x + 9) + (y^2 - 4y + 4) = 23 + 9 + 4$

$= 36$

$(x+3)^2 + (y-2)^2$

Centre $(-3, 2)$

$r = 6$

. a) $2kx^2 + 11x - 10k = 0$

i) $\alpha + 2 = \frac{-11}{2k}$ $\qquad 2\alpha = \frac{-10k}{2k}$

$2\alpha = -5$

$\alpha = -\frac{5}{2}$

$\therefore -\frac{5}{2} + 2 = \frac{-11}{2k}$

$-\frac{1}{2} = \frac{-11}{2k}$

$2k = 22$

$k = 11$

ii) $\alpha = -2\frac{1}{2}$

) $2x^2 - 5x - 6 = 0$

i) $\alpha\beta = -3$

ii) $\alpha + \beta = \frac{5}{2}$

iii) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= \frac{25}{4} + 6$

$= 12\frac{1}{4}$

iv) $(3-\alpha)(3-\beta) = 9 - 3(\alpha+\beta) + \alpha\beta$

$= 9 - \frac{15}{2} + (-3)$

$= -1\frac{1}{2}$

v) $\alpha^2\beta + \alpha\beta^2 = \alpha\beta(\alpha+\beta)$

$= -3\left(\frac{5}{2}\right)$

$= -7\frac{1}{2}$

$2x^2 + 3x + 5$

$a = 2 > 0$

$\Delta = b^2 - 4ac$

$= 9 - 4 \times 2 \times 5$

$= -31$

$\therefore$ Since $a > 0$, $\Delta < 0$

$\therefore$ positive definite

4. a) $x^2 + 3x + 4 \equiv A(x-1)^2 + B(x-1) + C$

$\equiv A(x^2 - 2x + 1) + Bx - B + C$

$x^2 + 3x + 4 \equiv Ax^2 - 2Ax + A + Bx - B + C$

$\therefore A = 1$ $\qquad 3 = B - 2A$ $\qquad A - B + C = 4$

$3 = B - 2$ $\qquad 1 - 5 + C = 4$

$C = 8$

$\therefore B = 5$

b) $3x^2 + 4x - 2k = 0$

i) equal roots $\Delta = 0$

$16 - 4 \times 3 \times -2k = 0$

$16 + 24k = 0$

$k = \frac{-16}{24} = \frac{-2}{3}$

ii) one root zero $\therefore \frac{c}{a} = 0$

$\alpha\beta = 0$

$\frac{-2k}{3} = 0$

$\therefore k = 0$

iii) $\alpha\beta = 1$

$\frac{c}{a} = 1$

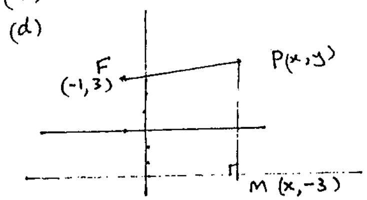
$\therefore \frac{-2k}{3} = 1$

$-2k = 3$

$k = \frac{3}{-2}$

(c) N.A.

(d)



Condition: $PF = PM$

$PF^2 = (y-3)^2 + (x+1)^2$

$PM^2 = (x-x)^2 + (y+3)^2$

$\therefore (y-3)^2 + (x+1)^2 = (y+3)^2$

$y^2 - 6y + 9 + (x+1)^2 = y^2 + 6y + 9.$

$(x+1)^2 = 12y$

5. Length of fences

i) $= 120 = x + x + y + 4x + 2x + y + y$

$\therefore 8x + 3y = 120$ ——— (1)

ii) area $= x(2x+y) + xy$

$= 2x^2 + 2xy$

using (1): $y = \frac{120 - 8x}{3}$

$\therefore$ area $= 2x^2 + 2x\left(\frac{120 - 8x}{3}\right)$

$= 2x^2 + \frac{2x}{3}(120 - 8x)$

$A = 2x^2 + 80x - \frac{16}{3}x^2$

$A' = 4x + 80 - \frac{32}{3}x$

$A'' = 4 - \frac{32}{3}$

$= -6\frac{2}{3}$

Max/min when $A' = 0$: $\qquad 4x - \frac{32}{3}x = -80$

$-\frac{20}{3}x = -80$

$x = \frac{80 \times 3}{20}$

$= 12$

$\underline{x = 12}$ $\qquad A''$ is always negative

$\therefore x = 12$ will give max. area.

$x = 12$, $y = \frac{120 - 96}{3}$

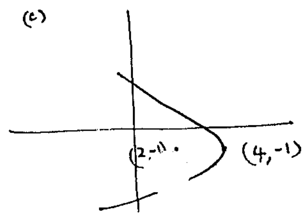
$= 8$

$\therefore$ Max area is $2 \times 12^2 + 2 \times 12 \times 8$

$= 480 \ m^2.$

(b) NA

(c)



of form: $-4a(x-h) = (y-k)^2$

$a = 2$

$(y+1)^2 = -8(x-4)$