Su Yee.

St George Girls' High School

Year 12

Assessment Task 1

November 2000

Extension Mathematics

Time Allowed: 65 minutes

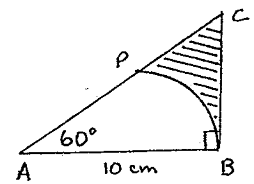
Instructions:

1. Attempt ALL questions.
2. Each question is of equal value.
3. Start each question on a new page.
4. Show all working.
5. Complete the cover sheet on the front of your solutions clearly showing:

   - your name
   - your mathematics class and teacher.

Question 2 – (12 Marks)

a) In the diagram ABC is a triangle that is right angled at B, $AB = 10$cm and angle A is 60°. The circular arc BP has centre A and radius AB. 5



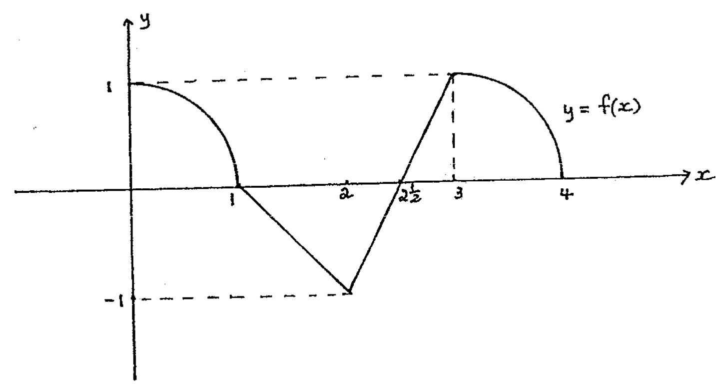
(i) Find the exact area of sector APB.

(ii) hence find the exact area of the shaded portion BCP.

b) A point $P(x, y)$ moves so that it is equidistant from $(-1, 2)$ and the line $x = 3$. Derive the equation of the locus. 3

c) (i) Use the graph to evaluate $\int_0^4 f(x)dx$ 4

(ii) Calculate the area between the graph and the coordinate axes.



Question 1 – (12 Marks) **Marks**

a) Differentiate 3

(i) $y = x\cos x$

(ii) $y = \sin^3 2x$

b) For what values of k has the equation $x^2 - 2kx + 8k - 15 = 0$ two different roots. 3

c) Find the limit: $\lim\limits_{x \to 0} \dfrac{\sin 2x + \tan 3x}{x}$ 2

d) Solve: $(x + \frac{1}{x})^2 - 5(x + \frac{1}{x}) + 6 = 0$ 3

e) Give exact value of: $\sin\frac{3\pi}{4} - \cos\frac{7\pi}{6}$ 1

Question 3 – (12 Marks)

a) (i) State the period and amplitude of $y = 3\sin\frac{x}{2}$ 5

(ii) On the same set of axes sketch the graphs $y = 3\sin\frac{x}{2}$ and $y = x$ for $-2\pi \le x \le 2\pi$.

(iii) How many solutions does the equation $3\sin\frac{x}{2} = x$ have in the interval $-2\pi \le x \le 2\pi$?

b) Show that the expression $(a^2 + b^2)x^2 - 2(a+b)x + 2$ is positive definite if a and b are unequal. 3

c) Evaluate $\int_0^8 f(x)dx$ using 2 applications of Simpson's rule and the following table of values. 2

x	0	2	4	6	8
$f(x)$	1.2	3.7	9.4	7.2	6.3

d) Differentiate $\cos^4 x$ and hence integrate $\sin x \cos^3 x$ 2

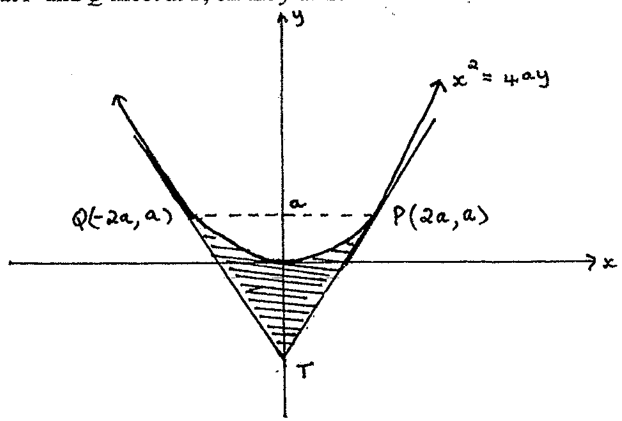
Question 4 (12 Marks)

a) Prove by Mathematical Induction that 4

$$1+4+4^2+\dots\dots+4^n=\frac{1}{3}(4^{n+1}-1)\text{ for } n\geq 0$$

b) Find $\int_0^{\frac{\pi}{4}}\tan^2\theta\,d\theta$ 2

c) In the diagram P, Q are the points $(2a, a)$ and $(-2a, a)$ on the parabola $x^2=4ay$. The tangents at P and Q meet at T, on the y axis. 6



(i) Show that the equation of the tangent PT is $y=x-a$.

(ii) Find the coordinates of T.

(iii) hence, find the exact area enclosed between $x^2=4ay$ and the tangents at P and Q.

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Q. 2

Question 1

a.) i.) $y=x\cos x$

$\frac{dy}{dx}=x(-\sin x)+1(\cos x)$

$=\cos x-x\sin x$

ii.) $y=\sin^3 2x$

$=(\sin 2x)^3$

$\frac{dy}{dx}=3(\sin 2x)^2(2\cos 2x)$

$=6\cos 2x\sin^2 2x$.

b.) $x^2-2kx+8k-15=0$

$\Delta>0$.

$\Delta=b^2-4ac$

$=(-2k)^2-4\times1\times(8k-15)$

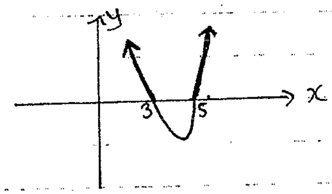
$=4k^2-32k+60$.

$4k^2-32k+60>0$.

$k^2-8k+15>0$.

$(k-5)(k-3)>0$.

$\therefore k<3$ or $k>5$



c.) $\lim_{x\to0}\frac{\sin 2x+\tan 3x}{x}=\lim_{x\to0}\frac{\sin 2x}{x}+\lim_{x\to0}\frac{\tan 3x}{x}$

$=\lim_{x\to0}\frac{2\sin 2x}{2x}+\lim_{x\to0}\frac{3\tan 3x}{3x}$

$=2\lim_{x\to0}\frac{\sin 2x}{2x}+3\lim_{x\to0}\frac{\tan 3x}{3x}$

$\left[\text{Since }\lim_{x\to0}\frac{\sin x}{x}=1\text{ and }\lim_{x\to0}\frac{\tan x}{x}=1\right]$ $=2(1)+3(1)$

$=5$.

Question 5 (12 Marks)

a) Express $2x^2+3x+1$ in the form $A(x-1)^2+B(x-1)+C$ **3**

b) Find the value of k if one root is two more than the other in the equation $x^2-(k+2)x+4k=0$ **4**

c) The area between the two curves $x+y=6$ and $xy=5$ is rotated about the x axis, creating a solid of revolution. Find the volume of this solid of revolution. **5**

d.) $\left(x+\frac{1}{x}\right)^2-5\left(x+\frac{1}{x}\right)+6=0.$

Let $m=x+\frac{1}{x}$

$m^2-5m+6=0$

$(m-2)(m-3)=0$

$m=2,3$

$\therefore\ x+\frac{1}{x}=2$ or $x+\frac{1}{x}=3.$

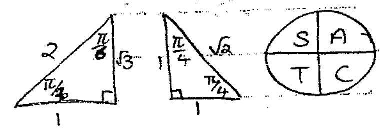
$x^2+1=2x$ $\qquad$ $x^2+1=3x$

$x^2-2x+1=0$ $\qquad$ $x^2-3x+1=0.$

$(x-1)(x-1)=0.$ $\qquad$ $x=\frac{3\pm\sqrt{9-4\times1\times1}}{2\times1}$

$x=1$ $\qquad$ $=\frac{3\pm\sqrt{5}}{2}$

e.) $\sin\frac{3\pi}{4}-\cos\frac{7\pi}{6}$



$\sin\frac{\pi}{4}=\frac{1}{\sqrt2}$ $\qquad$ $\cos\frac{\pi}{6}=\frac{\sqrt3}{2}$

$\therefore \sin\frac{3\pi}{4}-\cos\frac{7\pi}{6}=\frac{1}{\sqrt2}+\frac{\sqrt3}{2}$

$=\frac{\sqrt2}{2}+\frac{\sqrt3}{2}$

$=\frac{\sqrt2+\sqrt3}{2}$

Question 2.

a.) i.) Area of APB $= \frac{\pi}{3} \times 10^2 \times \frac{1}{2}$

$$= \frac{50\pi}{3} \text{ cm}^2$$

ii.) Area of BCP = ΔABC − sector APB

$\tan \frac{\pi}{3} = \frac{BC}{10}$

$BC = 10 \tan \pi/3$

$= 10 \times \sqrt{3}$

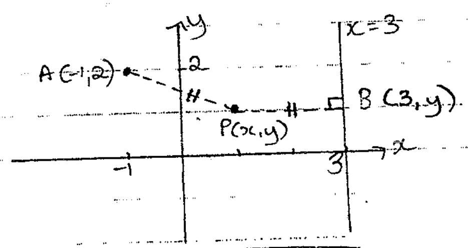
$= 10\sqrt{3}$

$\Delta ABC = \frac{1}{2} \times 10 \times 10\sqrt{3}$

$= 50\sqrt{3}$

$\therefore$ Area BCP $= (50\sqrt{3} - \frac{50\pi}{3}) \text{ cm}^2$

b.)



PA = PB

$PA = \sqrt{(x+1)^2 + (y-2)^2}$

$PB = \sqrt{(x-3)^2 + (y-y)^2}$

$$\sqrt{(x+1)^2 + (y-2)^2} = \sqrt{(x-3)^2}$$

$$x^2 + 2x + 1 + y^2 - 4y + 4 = x^2 - 6x + 9$$

$$8x - 4 + y^2 - 4y = 0.$$

$$y^2 - 4y + 4 = -8x + 4 + 4$$

$$(y-2)^2 = -8(x-1).$$

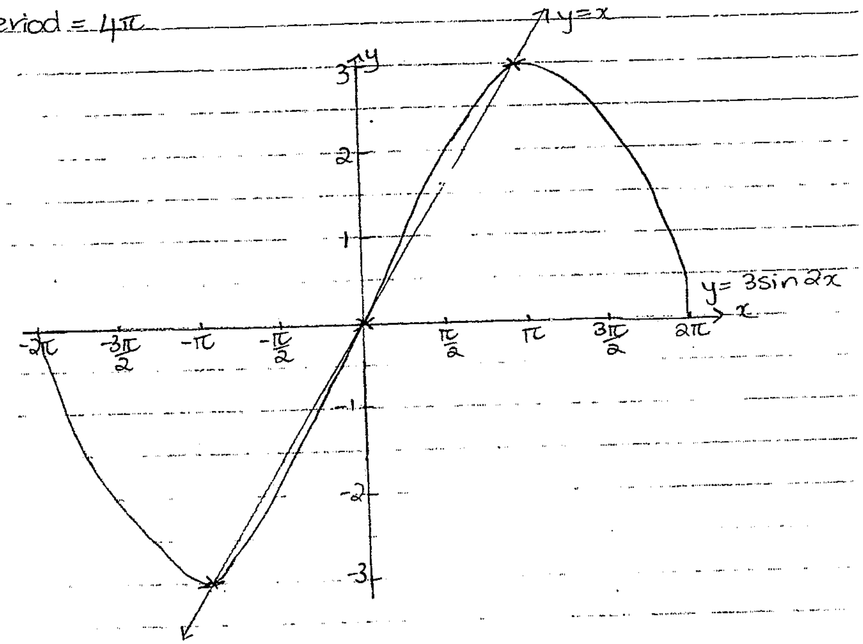
Question 3.

a.) i.) $y = 3 \sin \frac{x}{2}$

amplitude = 3

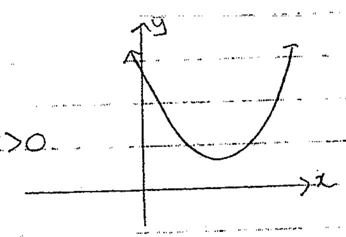
period = 4π

ii.)



iii.) from graph there are 3 solutions

b.) $(a^2+b^2)x^2 - 2(a+b)x + 2$



positive definite : $a \geq 0$, $\Delta < 0$

$a^2 + b^2 > 0$, since $a^2 > 0$ and $b^2 > 0$.

$\Delta = b^2 - 4ac$

$= (-2a-2b)^2 - 4(a^2+b^2)(2)$

$= 4a^2 - 8ab + 4b^2 - 8a^2 - 8b^2$

$= -4a^2 - 4b^2 - 8ab$

$= -4(a^2 + b^2 + 2ab).$

$\therefore \Delta < 0$ since it is multiplied by -4.

$\therefore$ positive definite if a and b are unequal.

c.) i.) $\int_0^4 f(x)\,dx = \frac{1}{2}\pi r^2 + (\frac{1}{2} \times \frac{1}{2} \times 1) - (\frac{1}{2} \times 1\frac{1}{2} \times 1)$

$= \frac{1}{2}\pi(1)^2 + \frac{1}{4} - \frac{3}{4}$

$= \left(\frac{\pi}{2} - \frac{1}{2}\right)$ units^2

ii.) Area $= \left(\frac{1}{4} \times \pi \times (1)^2\right) + \left(\frac{1}{2} \times 1\frac{1}{2} \times 1\right) + \left(\frac{1}{2} \times \frac{1}{2} \times 1\right) + \left(\frac{1}{4} \times \pi \times (1)^2\right)$

$= \frac{\pi}{4} + \frac{3}{4} + \frac{1}{4} + \frac{\pi}{4}$.

$= \left(1 + \frac{\pi}{2}\right)$ units^2.

c.) $A \doteqdot \frac{h}{6}\left\{f(1) + 4f(2) + f(3)\right\} + \frac{h}{6}\left\{f(3) + 4f(4) + f(5)\right\}$

$= \frac{4-0}{6}\left(1.2 + 4(3.7) + 9.4\right) + \frac{8-4}{6}\left(9.4 + 4(7.2) + 6.3\right)$

$= \frac{2}{3}\left(25.4 + 44.5\right)$

$= 46.6$ units^2

d.) $\frac{d}{dx}(\cos^4 x) = 4(\cos x)^3(-\sin x)$.

$= -4\sin x\cos^3 x$.

$\int \sin x\cos^3 x\,dx = -\frac{1}{4}\int -4\sin x\cos^3 x\,dx$.

$= -\frac{1}{4}\cos^4 x + c$.

uestion 4

i.) Prove $1+4+4^2+\dots+4^n = \frac{1}{3}(4^{n+1}-1) \quad n \geq 0.$

Test statement is true for $n=0$.

$LHS = 4^0 = 1$ $\qquad$ $RHS = \frac{1}{3}(4^{0+1}-1) = 1$

$LHS = RHS$

$\therefore$ true for $n=0$

Assume statement is true for $n=k$.

ie. $1+4+4^2+\dots+4^k = \frac{1}{3}(4^{k+1}-1)$.

Prove statement is true for $n=k+1$.

ie. $1+4+4^2+\dots+4^{k+1} = \frac{1}{3}(4^{k+2}-1)$.

$$S_{k+1} = S_k + T_{k+1}$$
$$= \frac{1}{3}(4^{k+1}-1)+4^{k+1}$$
$$= \frac{1}{3}\cdot 4^{k+1} + 4^{k+1} - \frac{1}{3}$$
$$= \frac{1}{3}\cdot 4^k \cdot 4^1 + 4^k\cdot 4^1 - \frac{1}{3}$$
$$= 4^k\left(\frac{4}{3}+4\right) - \frac{1}{3}$$
$$= \frac{16}{3}\cdot 4^k - \frac{1}{3}$$
$$= \frac{1}{3}(16\cdot 4^k - 1)$$
$$= \frac{1}{3}(4^2\cdot 4^k - 1)$$
$$= \frac{1}{3}(4^{k+2}-1)$$

$\therefore$ Statement is true for $n=k+1$ if it is true for $n=k$.

Since the statement is true for $n=1$, it is true for $n=2$. If it is

Area between curve and line $= 2\left(a\times 2a - \int_0^{2a} \frac{x^2}{4a}\,dx\right)$
$$= 2\left(2a^2 - \left[\frac{x^3}{12a}\right]_0^{2a}\right)$$
$$= 2\left(2a^2 - \left(\frac{(2a)^3}{12a} - 0\right)\right)$$
$$= 2\left(2a^2 - \frac{8a^3}{12a}\right)$$
$$= 2\left(2a^2 - \frac{2a^2}{3}\right)$$
$$= \frac{8a^2}{3}$$

$\therefore$ shaded area $= 4a^2 - \frac{8a^2}{3}$
$$= \frac{4a^2}{3} \text{ units}^2$$

b.) $\int_0^{\pi/4} \tan^2\theta \, d\theta = \int_0^{\pi/4} (\sec^2\theta - 1) \, d\theta$ $\qquad 1+\tan^2\theta = \sec^2\theta$

$= \left[\tan\theta - \theta\right]_0^{\pi/4}$

$= \left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right) - (\tan 0 - 0)$

$= \left(1 - \frac{\pi}{4}\right) - 0$

$= 1 - \frac{\pi}{4}$

c.) i.) $x^2 = 4ay$.

$y = \frac{x^2}{4a}$

$\frac{dy}{dx} = \frac{2x}{4a}$

$= \frac{x}{2a}$

when $x = 2a$ $\frac{dy}{dx} = \frac{2a}{2a}$

$= 1$.

$(y - a) = 1(x - 2a)$.

$x - y - a = 0$.

$\therefore y = x - a$.

ii.) when $x = 0$ $y = 0 - a$.

$= -a$

$\therefore T$ is $(0, -a)$

iii.) $A =$ Area of ΔPQT − area between curve and $y = a$

Area of $\Delta PQT = \frac{1}{2} \times 4a \times 2a$.

$= 4a^2$.

Question 5

a.) $2x^2 + 3x + 1 \equiv A(x-1)^2 + B(x-1) + C$.

$A(x-1)^2 + B(x-1) + C = Ax^2 - 2Ax + A + Bx - B + C$

$= Ax^2 - (2A - B)x + (A - B + C)$

$\therefore A = 2$ ①

$-(2A - B) = 3$ ②

$A - B + C = 1$ ③

Sub. ① in ② $\quad 2A - B = -3$.

$2(2) - B = -3$

$-B = -7$.

$\therefore B = 7$.

Sub. $A + B$ in ③ $\quad 2 - 7 + C = 1$.

$-5 + C = 1$.

$\therefore C = 6$.

$\therefore 2x^2 + 3x + 1 = 2(x-1)^2 + 7(x-1) + 6$

b.) $x^2 - (k+2)x + 4k = 0$.

let roots be α and $\alpha + 2$

$\alpha + \beta = \frac{-b}{a}$ $\qquad \alpha\beta = \frac{c}{a}$

$\alpha + \alpha + 2 = \frac{k+2}{1}$ $\qquad \alpha(\alpha+2) = \frac{4k}{1}$

$2\alpha + 2 = k + 2$ $\qquad \alpha^2 + 2\alpha = 4k$.

$2\alpha = k$

$\alpha = \frac{k}{2}$ $\qquad$ Since $\alpha = \frac{k}{2}$, $\left(\frac{k}{2}\right)^2 + 2\left(\frac{k}{2}\right) = 4k$.

$\frac{k^2}{4} + k = 4k$.

$\frac{k^2}{4} - 3k = 0$.

$k^2 - 12k = 0$.

$k(k - 12) = 0$.