Angelina 1.

St George Girls High School

Year 12

Assessment Task 3

2005

Mathematics Extension 1

General Instructions

- Time allowed – 65 minutes
- Write using blue or black pen
- Board-approved calculators may be used.
- A table of standard integrals is provided.
- All necessary working should be shown in every question.
- Write on one side of the page only.
- Start each question on a new page.

Total marks – 72

- Attempt Questions 1 – 6
- All questions are of equal value

Question 1 – (12 marks) – Start a new page **Marks**

a) Show using addition formulae that $\sin 75 = \cos 15$ 2

b) If A and B are acute angles and $\sin A = \frac{4}{5}$ and $\cos B = \frac{12}{13}$ evaluate $\tan(A+B)$. 3

c) (i) Solve the equation $\sin 2x = 2\sin^2 x$ for $0 < x < \pi$ 2

(ii) Show that if $0 < x < \frac{\pi}{4}$ then $\sin 2x > 2\sin^2 x$ 2

$2\sin x \cos x > 2\sin^2 x$

$2\cos x > 2\sin x$

$1 > \tan x$

d) Prove that $\frac{\cos 3\theta}{\sin\theta} + \frac{\sin 3\theta}{\cos\theta} = 2\cot 2\theta$ 3

Question 2 – (12 marks) – Start a new page **Marks**

a) Use the t formulae to prove:

$$\frac{1-\cos\theta}{\sin\theta}=\frac{\sin\theta}{1+\cos\theta}$$ 3

b) Evaluate $\int_0^{\frac{\pi}{4}} \cos^2 x\,dx$ 3

c) Solve $\cos\left(2x-\frac{\pi}{6}\right)=\frac{\sqrt{3}}{2}$ for $0\le x\le 2\pi$ 3

d) For what value of k is $\int_0^k \sin^2 x\,dx=\frac{3\pi}{2}$? 3

$\frac{3\pi}{4}$ =

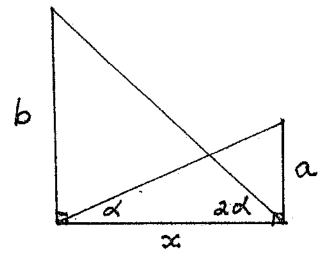
$3\pi - k = \frac{1}{2}\sin 2k.$

$6\pi - 2k = \sin 2k.$

$2\sin k \cos k.$

Question 3 – (12 marks) – Start a new page **Marks**

a)



(i) Use the diagram above to write expressions for $\tan\alpha$ and $\tan 2\alpha$ 1

(ii) Use these to show that $x=a\sqrt{\frac{b}{b-2a}}$ 2

b) Solve for $0\le x\le 2\pi$

(i) $8\cos x-\sin x=4$ 3

(ii) $\sqrt{3}\sin x-\cos x=1$ 3

c) From a point P due south of a vertical tower, the angle of elevation of the top of the tower is 20° and from a point Q due east of the tower it is 35°. If the distance from P to Q is 40 metres, find the height of the tower. 3

Question 4 – (12 marks) – Start a new page **Marks**

a) If $P(x) = 3x^3 - 7x^2 + 5x - 1$ and $Q(x) = 5x^3 + 8x$

Find:

(i) $Q(x) - P(x)$ 1

(ii) $P(x) \cdot Q(x)$ 1

b) Without division find the remainder when $P(x) = x^3 + 2x^2 - 4x + 5$ is divided by $3x - 2$ 1

c) A polynomial is given by $Q(x) = x^3 + ax^2 + bx - 18$

Find the values of a and b if $(x+2)$ is a factor of $Q(x)$ and if -24 is the remainder when $Q(x)$ is divided by $(x-1)$ 3

d) Write $6x^3 - 23x^2 - 6x + 8$ as a product of its linear factors. 4

e) Draw a sketch graph that shows the general form of the polynomial function 2

$y = x(x-2)^3(x+1)^4$

Question 5 – (12 marks) – Start a new page **Marks**

a) When the polynomial $P(x)$ is divided by $x^2 - 1$ the remainder is $3x - 1$.

What is the remainder when $P(x)$ is divided by $x - 1$? 2

b) (i) The polynomial equation $P(x) = 0$ has a double root at $x = a$.
By writing $P(x) = (x-a)^2 \, . \, Q(x)$ where $Q(x)$ is a polynomial, show that $P'(a) = 0$. 2

(ii) Hence or otherwise find the values of a and b if $x = 1$ is a double root of $x^4 + ax^3 + bx^2 - 5x + 1 = 0$ 4

c) If α, β and γ are the roots of the equation $x^3 + 2x^2 - 11x - 12 = 0$

Find:

(i) $(\alpha+1)(\beta+1)(\gamma+1)$ 2

$\alpha\beta + \alpha + \beta + 1$

$\alpha\beta\gamma + \alpha\gamma + \beta\gamma + h$

(ii) $(\alpha\beta)^2\gamma + (\alpha\gamma)^2\beta + (\beta\gamma)^2\alpha$ 2

Question 6 – (12 marks) – Start a new page **Marks**

a) A particle starts at the origin and moves in a straight line with velocity $V m/s$, such that $V = 3 - 2\sin 2t$, where t is the time elapsed in seconds.

(i) Find the initial velocity. 1

(ii) Sketch a graph of V as a function of t for $0 \le t \le 2\pi$ 2

(iii) Using your graph or otherwise find:

1. the time when the acceleration is first zero. 1

2. the velocity at this time. 1

(iv) Find the distance travelled by the particle in the first $\frac{\pi}{2}$ seconds. 2

b) Find the change in displacement during the 2nd second of motion of a particle whose velocity is $V = \dfrac{4}{t+1}$ 2

c) The acceleration of a particle is given by $a = 6t - 12 (ms^{-2})$. If the particle is initially at rest $2m$ to the left of the origin. Find:

(i) Its velocity as a function of t. 1

(ii) Its displacement after 5 seconds. 2

End of Paper

Extension 1 - Task 2

Question 1

a.) $\sin 75° = \sin(45° + 30°)$

$= \sin 45° \cdot \cos 30° + \cos 45° \cdot \sin 30°$

$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$

$\cos 15° = \cos(45° - 30°)$

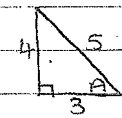
$= \cos 45° \cos 30° + \sin 45 \cdot \sin 30$

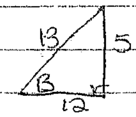
$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2}$

$= \frac{\sqrt{3}+1}{2\sqrt{2}}$

$\therefore$ LHS = RHS

b.) $\tan(A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$





$\tan A = \frac{4}{3}$ $\tan B = \frac{5}{12}$

$= \dfrac{\frac{4}{3} + \frac{5}{12}}{1 - \frac{4}{3} \cdot \frac{5}{12}}$

$= \dfrac{\frac{21}{12}}{16/36}$

$= \frac{63}{16}$

c.)(i) $\sin 2x = 2\sin^2 x$ $0 < x < \pi$

$2\sin^2 x - 2\sin x \cos x = 0$

$2\sin x(\sin x - \cos x) = 0$

$2\sin x = 0$ $\sin x = \cos x$

$\sin x = 0$ $\tan x = 1$

$x = 0$ but $0 < x < \pi$ $x = \frac{\pi}{4}$

(ii) If $0 < x < \frac{\pi}{4}$ then $\tan x < 1$

$\tan x < 1$ $\sin x < \cos x$

$\frac{\sin x}{\cos x} < 1$ $\sin^2 x < \sin x \cos x$

$2\sin^2 x < 2\sin x \cos x$

d.) $\frac{\cos 3\theta}{\sin\theta} + \frac{\sin 3\theta}{\cos\theta} = 2\cot 2\theta$

LHS $= \frac{\cos 3\theta.\cos\theta + \sin 3\theta.\sin\theta}{\sin\theta \times \cos\theta}$

$= \frac{\cos(3\theta - \theta)}{\frac{1}{2}\sin 2\theta}$

$= \frac{2\cos 2\theta}{\sin 2\theta}$

$= 2\cot 2\theta$

= RHS

Question 2

a.) $\frac{1-\cos\theta}{\sin\theta} = \frac{\sin\theta}{1+\cos\theta}$

LHS $= \frac{1-\left(\frac{1-t^2}{1+t^2}\right)}{\frac{2t}{1+t^2}} \times \frac{1+t^2}{1+t^2}$

$= \frac{1+t^2-1+t^2}{2t}$

$= \frac{2t^2}{2t}$

$= t$

RHS $= \frac{\frac{2t}{1+t^2}}{1+\frac{1-t^2}{1+t^2}} \times \frac{1+t^2}{1+t^2}$

$= \frac{2t}{1+t^2+1-t^2}$

$= \frac{2t}{2}$

$= t$

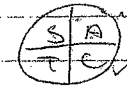
$\therefore$ LHS = RHS

b.) $\int_0^{\pi/4} \cos^2 x \, dx = \frac{1}{2}\int_0^{\pi/4} \cos 2x + 1 \, dx$

$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right]_0^{\pi/4}$

$= \frac{1}{2}\left(\frac{\pi}{4} + \frac{1}{2} - 0\right)$

c) $\cos\left(2x - \frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}$ $\quad 0 \le x \le 2\pi$



$2x - \frac{\pi}{6} = -\frac{\pi}{6}, \frac{\pi}{6}, \frac{11\pi}{6}, \frac{13\pi}{6}, \frac{23\pi}{6}$ $\quad -\frac{\pi}{6} \le 2x - \frac{\pi}{6} \le \frac{23\pi}{6}$

$2x = 0, \frac{\pi}{3}, 2\pi, \frac{7\pi}{3}, 4\pi$

$x = 0, \frac{\pi}{6}, \pi, \frac{7\pi}{6}, 2\pi$

d.) $\int_0^k \sin^2 x \, dx = \frac{3\pi}{2}$

$\int_0^k \sin^2 x \, dx = \frac{1}{2}\int_0^k 1 - \cos 2x \, dx$

$= \frac{1}{2}\left[x - \frac{\sin 2x}{2}\right]_0^k$

$= \frac{1}{2}\left(k - \frac{\sin 2k}{2} - 0\right)$

$\therefore \frac{k}{2} - \frac{\sin 2k}{4} = \frac{3\pi}{2}$

$2k - \sin 2k = 6\pi$

$\sin 2k = 2k - 6\pi$

$2\sin k.\cos k = 2(k - 3\pi)$

$\sin k.\cos k = k - 3\pi$

if LHS = 0 then RHS = 0

$\therefore k = 3\pi$

Question 3

a.) (i) $\tan\alpha = \frac{a}{x}$ $\tan 2\alpha = \frac{b}{x}$

(ii) $\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}$

$$\frac{b}{x} = \frac{\frac{2a}{x}}{1-\frac{a^2}{x^2}}$$

$$b - \frac{a^2 b}{x^2} = 2a$$

$$x^2 b - a^2 b = 2ax^2$$

$$x^2(b-2a) = a^2 b$$

$$x^2 = \frac{a^2 b}{b-2a}$$

$$x = a\sqrt{\frac{b}{b-2a}}$$

b.) (i) $8\cos x - \sin x = 4$

$$8\left(\frac{1-t^2}{1+t^2}\right) - \frac{2t}{1+t^2} = 4$$

$$8 - 8t^2 - 2t = 4 + 4t^2$$

$$12t^2 + 2t - 4 = 0$$

$$6t^2 + t - 2 = 0$$

$$6t^2 - 3t + 4t - 2 = 0$$

$$3t(2t-1) + 2(2t-1) = 0$$

$$(2t-1)(3t+2) = 0$$

$$t = \frac{1}{2}, -\frac{2}{3}$$

$\tan\frac{x}{2} = \frac{1}{2}$ or $\tan\frac{x}{2} = -\frac{2}{3}$

$x/2 = 0.4636\ldots,$ $x/2 = 2.55\ldots,$

$3.6052\ldots$ $5.695\ldots$

$x = 0.927, 7.21$ $x = 5.11, 11.39$

test $x = \pi$

$8\cos\pi - \sin\pi = -8 - 0$

$\neq 4$

$\therefore \pi$ not a solution

$\therefore x = 0.927, 5.11.$

(ii) $\sqrt{3}\sin x - \cos x = R\sin(x-\alpha)$

$= R(\sin x\cos\alpha - \cos x\sin\alpha)$

$\therefore R\cos\alpha = \sqrt{3}$ ① $R\sin\alpha = 1$

①^2 + ②^2

$$R^2(\cos^2\alpha + \sin^2\alpha) = 3 + 1$$

$$R^2 = 4$$

$$R = 2, \quad R > 0$$

② ÷ ① $\frac{R\sin\alpha}{R\cos\alpha} = \frac{1}{\sqrt{3}}$

$$\tan\alpha = \frac{1}{\sqrt{3}}$$

$$\alpha = \pi/6$$

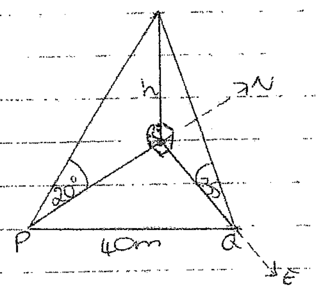
$\therefore 2\sin(x - \pi/6) = 1$

$$\sin(x - \pi/6) = \frac{1}{2}$$

$$x - \frac{\pi}{6} = \frac{\pi}{6}, \frac{5\pi}{6}$$

$$x = \frac{\pi}{3}, \pi$$

c.)



$\tan 20° = \frac{h}{PR}$ $\tan 35° = \frac{h}{RQ}$

$PR = \frac{h}{\tan 20°}$ $RQ = \frac{h}{\tan 35°}$

$$PQ^2 = PR^2 + RQ^2$$

$$40^2 = \frac{h^2}{\tan^2 20°} + \frac{h^2}{\tan^2 35°}$$

$$= h^2\frac{(\tan^2 35° + \tan^2 20°)}{\tan^2 20° \cdot \tan^2 35°}$$

$$h^2 = \frac{40^2 \cdot \tan^2 20° \cdot \tan^2 35°}{\tan^2 35° + \tan^2 20°}$$

$$h = \frac{40 \cdot \tan 20° \tan 35°}{\sqrt{\tan^2 35° + \tan^2 20°}}$$

Question 4.

a) (i) $Q(x) - P(x) = 5x^3 + 8x - (3x^3 - 7x^2 + 5x - 1)$

$= 2x^3 + 7x^2 + 3x + 1$

(ii) $P(x).Q(x) = (3x^3 - 7x^2 + 5x - 1)(5x^3 + 8x)$

$= 15x^6 + 24x^4 - 35x^5 - 56x^3 + 25x^4 + 40x^2 - 5x^3 - 8x$

$= 15x^6 - 35x^5 + 49x^4 - 61x^3 + 40x^2 - 8x$

b) $P\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^3 + 2\left(\frac{2}{3}\right)^2 - 4\left(\frac{2}{3}\right) + 5$

$= \frac{8}{27} + \frac{8}{9} - \frac{8}{3} + 5$

$= 3\frac{14}{27}$

c) $Q(x) = x^3 + ax^2 + bx - 18$

$Q(-2) = 0 \quad Q(1) = -24$

$0 = (-2)^3 + a(-2)^2 + b(-2) - 18$ $\qquad$ $-24 = (1)^3 + a(1)^2 + b(1) - 18$

$= -8 + 4a - 2b - 18$ $\qquad$ $= 1 + a + b - 18$

$4a - 2b = 26$ ① $\qquad$ $a + b = -7$ ②

$4a - 2b = 26$ ①

$2\times$② $\quad 2a + 2b = -14$ ③

①+③ $\quad 6a = 12$

$\therefore a = 2$

sub. in ②

$2 + b = -7$

$\therefore b = -9$

d) $P(x) = 6x^3 - 23x^2 - 6x + 8$

$P(4) = 0 \quad \therefore (x-4)$ is a factor

$P(x) = (x-4)(6x^2 + x - 2)$

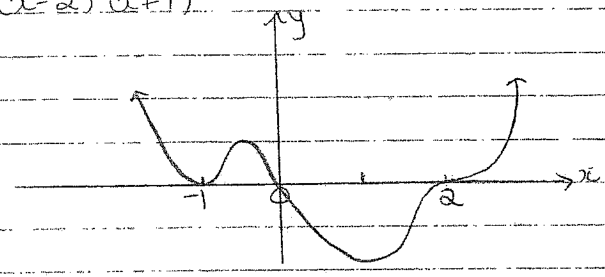
$= (x-4)(6x^2 - 3x + 4x - 2)$

$= (x-4)(3x(2x-1) + 2(2x-1))$

$= (x-4)(2x-1)(3x+2)$

$$\begin{array}{r} 6x^2 + x - 2 \\ x-4 \overline{)\, 6x^3 - 23x^2 - 6x + 8} \\ \underline{6x^3 - 24x^2} \\ x^2 - 6x \\ \underline{x^2 - 4x} \\ -2x + 8 \\ \underline{-2x + 8} \\ 0 \end{array}$$

e) $y = x(x-2)^3(x+1)^4$



Question 5.

a) $P(x) = (x^2 - 1).Q(x) + (3x - 1)$

$\frac{P(x)}{(x-1)(x+1)} = Q(x) + \frac{(3x-1)}{(x-1)(x+1)}$

$\frac{P(x)}{(x-1)} = Q(x).(x+1) + \frac{3x-1}{x-1}$

$\therefore$ remainder $= \frac{3x-1}{x-1}$ but the remainder must have degree zero.

$= \frac{3(x-1)}{x-1} + \frac{2}{x-1}$

$= 3 + \frac{2}{x-1}$

$\therefore \frac{P(x)}{(x-1)} = Q(x+1).(x+1) + 3 + \frac{2}{x-1}$

$\therefore$ remainder is 2

b) (i) $P(x) = (x-a)^2.Q(x)$

$P'(x) = Q(x).2(x-a) + (x-a)^2.Q'(x)$

$P'(a) = Q(a).2(a-a) + (a-a)^2.Q'(a)$

$= 0.$

(ii) $x^4+ax^3+bx^2-5x+1=0$. $P'(1)=0$, $P(1)=0$.

$P(1)=(1)^4+a(1)^3+b(1)^2-5(1)+1$

$\therefore a+b-3=0$ (1)

$P'(x)=4x^3+3ax^2+2bx-5$

$P'(1)=4(1)^3+3a(1)^2+2b(1)-5$

$\therefore 3a+2b-1=0$ (2)

(1)×2 $2a+2b-6=0$ (3)

$3a+2b-1=0$ (2)

(2)−(3) $a+5=0$

$\therefore a=-5$

sub. in (1) $-5+b-3=0$

$\therefore b=8$.

c.) $x^3+2x^2-11x-12=0$.

$\alpha+\beta+\gamma=\frac{-b}{a}$ $\quad \alpha\beta+\alpha\gamma+\beta\gamma=\frac{c}{a}$ $\quad \alpha\beta\gamma=\frac{-d}{a}$

$=-2$ $\quad =-11$ $\quad =12$.

(i) $(\alpha+1)(\beta+1)(\gamma+1)=(\alpha\beta+\alpha+\beta+1)(\gamma+1)$

$=\alpha\beta\gamma+\alpha\beta+\alpha\gamma+\alpha+\beta\gamma+\beta+\gamma+1$

$=\alpha\beta\gamma+(\alpha\beta+\alpha\gamma+\beta\gamma)+(\alpha+\beta+\gamma)+1$

$=12+(-11)+(-2)+1$

$=0$

(ii) $(\alpha\beta)^2\gamma+(\alpha\gamma)^2\beta+(\beta\gamma)^2\alpha=(\alpha\beta\gamma).\alpha\beta+(\alpha\beta\gamma).\alpha\gamma+(\alpha\beta\gamma).\beta\gamma$

$=\alpha\beta\gamma(\alpha\beta+\alpha\gamma+\beta\gamma)$

$=12\times-11$

$=-132$

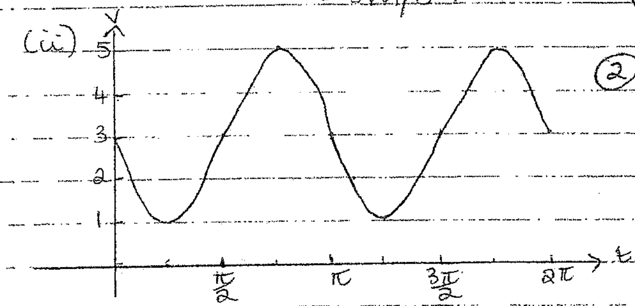
Question 6

a.) (i) $V=3-2\sin 2t$

when $t=0$, $V=3-2\sin(0)$

$=3$ m/s (1)

(ii) (2)



½ - period

½ - amplitude

½ - negative

½ - up 3

(iii) 1. $a=0$ when v is max or min.

$\therefore t=\frac{\pi}{4}$ secs (1)

2. when $t=\frac{\pi}{4}$, $V=1$ m/s. (1)

(iv) $x=\int_0^{\pi/2} 3-2\sin 2t \, dt$ (2)

$=\left[3t+\frac{2\cos 2t}{2}\right]_0^{\pi/2}$ 1.

$=\frac{3\pi}{2}+\frac{2\cos\pi}{2}-\left(0+\frac{2\cos 0}{2}\right)$

$=\left(\frac{3\pi}{2}-2\right)$ m 1.

b.) $V=\frac{4}{t+1}$ (2)

$x=\int_1^2 \frac{4}{t+1}\,dt$

$=\left[4\log(t+1)\right]_1^2$ 1

$=4\log 3-4\log 2$

$=4\log\left(\frac{3}{2}\right)$ 1

c) $a = 6t - 12$ when $t=0, v=0, x=-2$

(i) $v = \frac{6t^2}{2} - 12t + c_1$ $\frac{1}{2}$ 1

when $t=0, v=0 \therefore c_1 = 0$

$v = 3t^2 - 12t$ $\frac{1}{2}$

(ii) $x = \frac{3t^3}{3} - \frac{12t^2}{2} + c_2$ $\frac{1}{2}$ 2

when $t=0, x=-2 \therefore c_2 = -2$ $\frac{1}{2}$

$x = t^3 - 6t^2 - 2$ $1/2$

when $t=5$

$x = 5^3 - 6(5)^2 - 2$

$= -27$ $\frac{1}{2}$

$\therefore$ the displacement is 27m to the left of the origin after 5 seconds.