Angelina

St George Girls High School

Year 12

Common Test 3

2005

Mathematics

General Instructions

1. Working time – 75 minutes.
2. Use only blue or black pen.
3. Board approval calculators may be used.
4. A table of standard integrals is provided.
5. All necessary working should be shown in every question.
6. Start each question on a new page.

Total Marks

1. Attempt Question 1 – 5.
2. All questions are of equal value.

Question 1 – (13 marks) – Start a New Page

a) Evaluate the definite integrals 9

(i) $\int_{-1}^{1} 3x^2 - 4x + 1 \, dx$

(ii) $\int_{1}^{4} \frac{2}{x^2} \, dx$

(iii) $\int_{0}^{2} (2x-1)^3 \, dx$

(iv) $\int_{1}^{e} \frac{2x+1}{x} \, dx$

b) Find each of the following indefinite integrals. 4

(i) $\int x^2(x-5) \, dx$

(ii) $\int \frac{6}{2-3x} \, dx$

Question 2 – (13 marks) – Start a New Page

a) Evaluate $\log_5 50 - \log_5 2$ 2

b) (i) Show that the curve $y = x^2$ intersects the line $y = 8 - 2x$ at the point $(2, 4)$. 5

(ii) Find the area bounded by $y = x^2$, $y = 8 - 2x$ and the x-axis.

c) (i) Show that $\dfrac{d}{dx} x\sqrt{x-3} = \dfrac{3x-6}{2\sqrt{x-3}}$

(ii) Hence find $\displaystyle\int \frac{x-2}{\sqrt{x-3}}\,dx$

d) If the function $y = f(x)$ has a stationary point at $(-1, 2)$ and $\dfrac{d^2y}{dx^2} = 6x - 4$

find the equation of the curve. 3

Question 3 – (13 marks) – Start a New Page

a) Consider the curve $y = e^{-2x}$. 5

(i) Find the equation of the tangent to the curve at the point where $x = 0$.

(ii) Find y'' and hence determine the concavity of $y = e^{-2x}$ at $x = 0$.

b) Differentiate $\sin^2 5x$ with respect to x 2

c) For the function $y = \log_e(x+1)$ 6

(i) Give the domain of the function.

(ii) Sketch the graph of $y = \log_e(x+1)$

(iii) Using Simpson's Rule with 3 function values find an approximation for $\int_0^2 \log_e(x+1)\,dx$

Question 4 – (13 marks) – Start a New Page

a) Solve for x: 4

(i) $3^x = 51$

(ii) $\cos 2x = \frac{1}{2}, \quad 0 \le x \le \pi$

b) For the function $y = e^{x^2}$ 3

(i) find $\frac{dy}{dx}$

(ii) find $\frac{d^2y}{dx^2}$

c) The length of an arc in a circle of radius 12cm is 10π cm. 2

(i) Calculate the size of the angle (in radians) which is subtended at the centre by this arc.

(ii) Find the area of the sector cut off by this arc.

d) Find the coordinates of the stationary point on $y = \frac{e^x}{x^2}$ and determine its nature. 4

Question 5 – (13 marks) – Start a New Page

a) For the curve $y = 3\cos 2x$ for $0 \le x \le \pi$ 8

(i) State the period and amplitude.

(ii) Sketch the curve showing clearly where it crosses the x-axis.

(iii) Evaluate $\int_0^{\frac{\pi}{4}} 3\cos 2x \, dx$

(iv) By adding a line to your sketch in part (ii) determine how many solutions there are for $\cos 2x = \frac{x}{3}, \quad 0 \le x \le 2\pi$

b) Find the volume of the solid formed when the curve $y = e^{2x}$ is rotated about the x-axis from $x = 0$ to $x = 2$. 3

c) Find $\int \tan^2 x \, dx$ 2

-1- AT

Question 1

(a) (i) $\int_{-1}^{1} 3x^2 - 4x + 1\,dx = \left[x^3 - 2x^2 + x\right]_{-1}^{1}$

$= (1-2+1) - (-1-2-1)$

$= 0 - (-4) = \underline{4}$

2

(ii) $\int_1^4 2x^{-2}\,dx = \left[-2x^{-1}\right]_1^4 = \left(-\frac{2}{4}\right) - \left(-\frac{2}{1}\right)$

$= \underline{\frac{3}{2}}$

2

(iii) $\int_0^2 (2x-1)^3\,dx = \left[\frac{1}{8}(2x-1)^4\right]_0^2 = \left(\frac{1}{8}(3)^4 - \frac{1}{8}(-1)^4\right)$

$= \underline{10}$

2

(iv) $\int_1^e 2 + \frac{1}{x}\,dx = \left[2x + \log_e x\right]_1^e = 2e + 1 - (2 + 0)$

$= \underline{2e - 1}$

3

(b) (i) $\int x^3 - 5x^2\,dx = \underline{\frac{1}{4}x^4 - \frac{5}{3}x^3 + c}$

2

(ii) $\int \frac{6}{2-3x}\,dx = \frac{6}{-3}\int \frac{-3}{2-3x}\,dx$

$= -2\log_e(2-3x) + C$

2

13

-2- AT

Question 2

(a) $\log_5 50 - \log_5 2 = \log_5 25 = 2\log_5 5$

$= \underline{2}$

2

NB, $\log_5 50 - \log_5 2 = \log_5\left(\frac{50}{2}\right)$ and $\log_5 25 = \log_5$

(b) (i) $y = x^2$ $\quad y = 8 - 2x$

$\therefore x^2 = 8 - 2x$

$x^2 + 2x - 8 = 0$

$(x+4)(x-2) = 0$

$\therefore x = 2, -4$

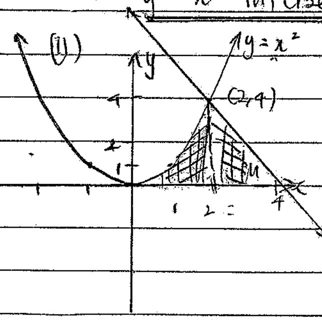
when $x = 2$, $y = 4$

$\therefore$ $(2, 4)$ is a point of intersection of curve and line

$\therefore$ $\underline{y = x^2 \text{ intersects line at } (2, 4)}$

1

(ii)



$A = \int_0^2 x^2\,dx + \int_2^4 8 - 2x\,dx$

$= \left[\frac{1}{3}x^3\right]_0^2 + \left[8x - x^2\right]_2^4$

$= \left(\frac{1}{3}\cdot 8 - 0\right) + \left(32 - 16 - (16 - 4)\right)$

$= \frac{8}{3} + 4$

$= \underline{6\frac{2}{3} \text{ units}^2}$

4

(c) (i) $\frac{d}{dx} x\sqrt{x-3} = \frac{d}{dx} \overset{u}{x} \overset{v}{(x-3)^{\frac{1}{2}}}$

$= (x-3)^{\frac{1}{2}} + x \cdot \frac{1}{2}(x-3)^{-\frac{1}{2}}$

$= \sqrt{x-3} + \frac{x}{2\sqrt{x-3}}$

$= \frac{2(x-3)+x}{2\sqrt{x-3}} = \frac{2x+x-6}{2\sqrt{x-3}}$

$= \frac{3x-6}{2\sqrt{x-3}}$

2

$\therefore \frac{d}{dx} x\sqrt{x-3} = \frac{3x-6}{2\sqrt{x-3}}$

(ii) $\int \frac{3x-6}{2\sqrt{x-3}}\,dx = x\sqrt{x-3} + C$

$\int \frac{3(x-2)}{2(\sqrt{x-3})}\,dx = x\sqrt{x-3} + C$

$\therefore \frac{2}{3} \times \int \frac{3(x-2)}{2(\sqrt{x-3})}\,dx = \frac{2}{3} x\sqrt{x-3} + C$

1

$\therefore \int \frac{x-2}{\sqrt{x-3}}\,dx = \frac{2}{3}x\sqrt{x-3} + C.$

(d) $\frac{d^2y}{dx^2} = 6x - 4 \quad \therefore \frac{dy}{dx} = 3x^2 - 4x + C$

when $\frac{dy}{dx} = 0,\ x = -1 \quad \therefore \quad 0 = 3(-1)^2 + 4 + C.$

$C = -7$

$\therefore \frac{dy}{dx} = 3x^2 - 4x - 7$

3

$y = x^3 - 2x^2 - 7x + k \quad$ sub $(-1, 2)$

$2 = -1 - 2 + 7 + k$

$k = -2$

$\therefore$ equ of curve: $y = x^3 - 2x^2 - 7x - 2$

Question 3

(a) (i) $y = e^{-2x}$

$y' = -2e^{-2x}.$

When $x = 0$, $y = 1$, $m = -2e^0 = -2$

$\therefore$ equ of tangent: $y - 1 = -2(x - 0)$

$y = -2x + 1$ ✓ 3

or in general form: $2x + y - 1 = 0$

(ii) $y'' = 4e^{-2x}$

when $x = 0$, $y'' = 4e^0 = 4$

$\therefore$ since $y'' > 0$, the concavity of $y = e^{-2x}$ at $x = 0$ is concave up ✓ 2

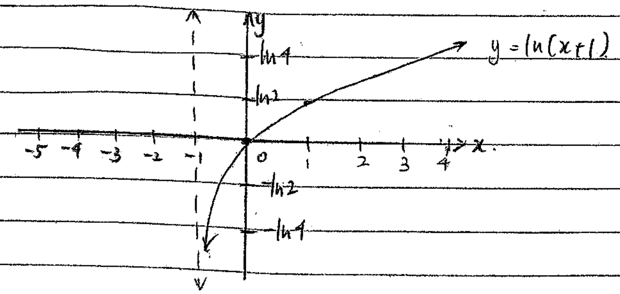
(b) $\frac{d}{dx}(\sin 5x)^2 = 2\sin 5x \times 5\cos 5x$ ✓ 2

$= 10\sin 5x \cos 5x$

(c) $y = \log_e(x+1)$

(i) $D: \{x > -1\}$ ✓ 1

(ii)



✓ 2

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(iii)

x	0	1	2.
$f(x)$	$\log_e 1$	$\log_e 2$	$\log_e 3$

3 ✓

$\therefore \int_0^2 \log_e(x+1)\,dx \doteqdot \frac{2-0}{6}(\log_e 1 + \log_e 3 + 4\log_e 2)$

$\doteqdot 1.2904\ldots \doteqdot 1.29$ (to 2 dp)

Question 4

(a) (i) $3^x = 51$

$x\log_e 3 = \log_e 51$

$x = \dfrac{\log_e 51}{\log_e 3}$

$= 3.5789\ldots$

$= 3.58$ (to 2dp) ✓ 2

(ii) $\cos 2x = \frac{1}{2}$ $\quad 0 \le 2x \le 2\pi$

$2x = \frac{\pi}{3}, \frac{5\pi}{3}$

$x = \frac{\pi}{6}, \frac{5\pi}{6}$ ✓ 2

(b) $y = e^{x^2}$

(i) $\dfrac{dy}{dx} = 2x e^{x^2}$ ✓ 1

(ii) $\dfrac{d^2y}{dx^2} = 2e^{x^2} + 2x \cdot 2x e^{x^2}$

$= 2e^{x^2}(1 + 2x^2)$ ✓ 2

(c) $r = 12 \quad l = 10\pi$

(i) $l = r\theta$

$\therefore \theta = \dfrac{l}{r} = \dfrac{10\pi}{12} = \dfrac{5\pi}{6}$ ✓ 1

$\therefore$ size of angle subtended at centre is $\frac{5\pi}{6}$ radians

(ii) $A = \frac{1}{2}r^2\theta = \frac{1}{2} \cdot 144 \cdot \frac{5\pi}{6}$ ✓ 1

$= 60\pi$ units^2

(d) $y = \dfrac{e^x}{x^2}$ $\quad u = e^x \quad v = x^2$

$u' = e^x \quad v' = 2x$

$y' = \dfrac{e^x x^2 - e^x \cdot 2x}{x^4} = \dfrac{\cancel{x}e^x(x-2)}{x^{\cancel{4}}3}$

$= \dfrac{e^x(x-2)}{x^3}$ ✓

when $y' = 0$, $\dfrac{e^x(x-2)}{x^3} = 0$

$e^x(x-2) = 0$

$x = 2$ ✓ $\therefore$ coords of stationary point is $(2, \frac{1}{4}e^2)$ ✓

Test nature:

x	$1\frac{1}{2}$	2	$2\frac{1}{2}$
y'	-0.66	0	0.3898
	\	—	/

$\therefore (2, \frac{1}{4}e^2)$ is a minimum turning point ✓

4

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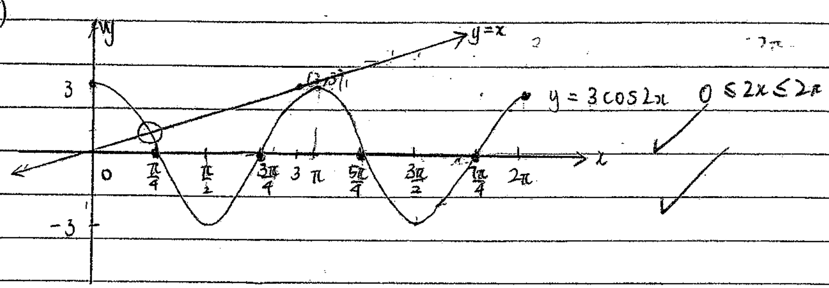
Question 5

(a) (i) $y = 3\cos 2x \quad 0 \leq x \leq \pi$.

$T = \frac{2\pi}{2} = \pi \quad \therefore$ period is π radians

Amplitude = 3 $\quad \therefore$ amplitude is 3 units

(ii)



(iii) $3\int_0^{\frac{\pi}{4}} \cos 2x \, dx = 3\left[\frac{1}{2}\sin 2x\right]_0^{\frac{\pi}{4}}$

$= 3\left(\frac{1}{2}\sin\frac{\pi}{2} - \frac{1}{2}\sin 0\right)$

$= \frac{3}{2}$

(iv) $3\cos 2x = x$

$\therefore$ sketch $y = x$.

$\therefore$ only one solution for $\cos 2x = \frac{x}{3} \quad 0 \leq x \leq 2\pi$

(b) $V = \pi\int y^2 \, dx = \pi\int_0^2 e^{4x} \, dx = \pi\left[\frac{1}{4}e^{4x}\right]_0^2$

$y^2 = e^{2x} \cdot e^{2x}$

$= e^{4x}$

$= \pi\left(\frac{1}{4}e^8 - \frac{1}{4}e^0\right)$

$= \pi\left(\frac{e^8}{4} - \frac{1}{4}\right)$ ~~units³~~

$= \frac{\pi}{4}(e^8 - 1)$ units³

(c) $\int \tan^2 x \, dx = \int \sec^2 x - 1 \, dx = \tan x - x + C$.

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