St George Girls' High School

Trial Higher School Certificate Examination

2001

Mathematics Extension 1

Time Allowed: Two hours
(Plus 5 minutes reading time)

Directions to Candidates

- Attempt ALL questions.
- All questions are of equal value.
- All necessary working should be shown in every question.
- Marks may be deducted for careless or badly arranged work.
- A table of standard integrals is attached.
- Board approved calculators may be used.
- Start each question on a new page.
- Return your answers in one bundle with a cover sheet

This is a Trial paper only and does not necessarily reflect either the content or format of the final Higher School Certificate Examination in this subject.

Question 1 – (12 Marks) **Marks**

a) (i) On the same set of axes draw the graphs of **3**

$y=|x+2|$ and $y=2-x$.

(ii) Hence, or otherwise, solve $|x+2|<2-x$

b) Solve: $x\geq\frac{4}{x}$ **3**

c) Write down the general solution of the equation

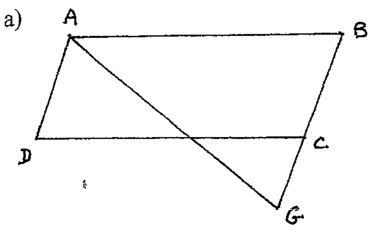
$2\sin\theta=\sqrt{3}$ **2**

d) Evaluate $\int_0^{\sqrt{2}}\frac{1}{\sqrt{4-x^2}}\,dx$ **2**

e) An approximate solution to the equation $\cos x=x$ is 0.7. **2**
Using one application of Newton's method find a better approximation (correct to 2 decimal places).

Marks

Question 2 – (12 Marks)

a) 

Given that $AB = AG$ and $ABCD$ is a parallelogram prove: **3**

(i) $ACGD$ is cyclic

(ii) and hence that $A\hat{C}D = A\hat{G}D$

b) Evaluate: $\int_0^3 \sqrt{9-x^2}\,dx$ using the substitution $x = 3\sin\theta$ for $0 \le \theta \le \frac{\pi}{2}$. **4**

c) A particle moves in a straight line so that its acceleration as a function of displacement is given by: **5**

$$\frac{d^2x}{dt^2} = 1 - 4x$$

Initially: $x = 1.25$ cm, $v = 0$.

(i) Show that the motion is simple harmonic.

(ii) Find the amplitude. Write down the period and position of the centre.

(iii) Find the velocity when $t = 0.2$ seconds.

Marks

Question 3 – (12 Marks)

a) (i) The letters of the word PERSEVERE are arranged in a row. How many different arrangements are possible? **3**

(ii) Out of all the different arrangements in part (i) one is chosen at random. Find the probability that this particular arrangement will have all the E's together and all the R's together in another group.

b) (i) Prove that $\frac{d}{dx}(x\log x) = \log x + 1$ **5**

(ii) A particle moves according to $\ddot{x} = 1 + \log x$. Initially it is stationary at $x=1$. Find v^2 as a function of x.

(iii) Explain why v is always positive for $t > 0$ and find v when $x = e^2$.

c) Prove, using Mathematical Induction, that $\cos(x + n\pi) = (-1)^n \cos x$ for all positive integers $n \ge 1$. **4**

Question 4 – (12 Marks) **Marks**

a) Differentiate **2**

(i) $y = \cos^{-1} 2x$

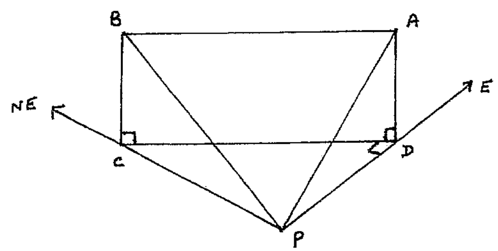
(ii) $y = \log_e \sqrt{2x+1}$

b) (i) Show that: $\sin x - \cos x = \sqrt{2}\sin\left(x - \frac{\pi}{4}\right)$ **6**

(ii) Hence sketch the graph of $y = \sin x - \cos x$ for $0 \le x \le 2\pi$.

(iii) Show that $x = \frac{\pi}{2}$ is a solution to $\sin x - \cos x = 1$ and hence, with the aid of your sketch, solve $\sin x - \cos x > 1$ for $0 \le x \le 2\pi$.

c) A plane is flying at a constant height h, and with constant speed. An observer at P sighted the plane due east at an angle of elevation of 45°. Soon after it was sighted again in a north-easterly direction at an angle of elevation of 60°. **4**



(i) Write down expressions for PC and PD in terms of h.

(ii) Show that $CD^2 = \frac{1}{3}h^2(4-\sqrt{6})$

Question 5 – (12 Marks) **Marks**

a) Prove that $\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{10}} = \frac{\pi}{4}$. **3**

b) A committee of 3 men and 4 women must be chosen from a group of 5 men and 7 women. How many ways can this be done? **2**

c) For the graph of $y = f(x)$ where $f(x) = \frac{6}{3+x^2}$ **7**

(i) Show that $f(x)$ is an even function.

(ii) Find the coordinates of the stationary point and determine its nature.

(iii) Sketch the graph showing all essential features.

(iv) Find the exact area between the graph, the x axis and the ordinates $x = -1$ and $x = 1$.

Question 6 – (12 Marks) **Marks**

a) Sand is poured into a conical heap at a constant rate of $0.6\text{m}^3/\text{sec}$ so that the height of the heap is always equal to twice the radius of the base. When the heap is 5m high, how fast is the height increasing? **3**

b) The points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. **5**

(i) If PQ passes through $(2a, 0)$ show that $pq = p + q$.

(ii) hence, find the locus of M, the midpoint of PQ.

c) When a polynomial $P(x)$ is divided by $(x-4)(x-3)$ explain why the remainder is of the form $ax + b$. **4**

If the polynomial $P(x)$ is divided by $x-3$ and $x-4$ the remainders are 3 and 4 respectively. Find the remainder when $P(x)$ is divided by $(x-4)(x-3)$.

Question 7 – (12 Marks) **Mar.**

a) A projectile is fired with initial velocity Vm/s at an angle of projection θ from a point O on horizontal ground. After 2 seconds it just passes over a 10 metre high wall that is 12m from the point of projection. **6**

Assume acceleration due to gravity is $10m/\sec^2$. Assume the equations of displacement are:

$x = Vt\cos\theta$ and $y = -5t^2 + Vt\sin\theta$.

(i) Find V and θ, to the nearest degree.

(ii) Find the maximum height reached by the projectile.

(iii) Find the range in the horizontal plane through the point of projection.

b) Assume that at time t years the population $P(t)$ of a town is given by $P(t) = 50000 + Ae^{kt}$ where A and k are constants. **6**

(i) Show that $P(t)$ satisfies the equation $P'(t) = k[P(t) - 50000]$

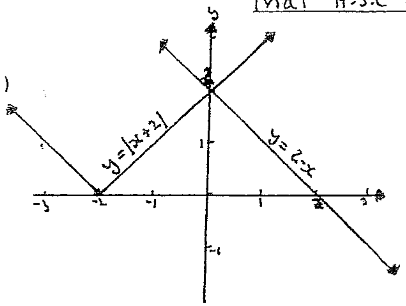
(ii) Given that $P(0) = 70000$, evaluate A.

(iii) If $P(8) = 150000$, find k correct to 4 decimal places.

(iv) Find, to the nearest year, the value of t for which $P(t+1) = 1.2 \times P(t)$.

St. George Girls
Trial H.S.C 2001 - Mathematics Ext1 Solutions

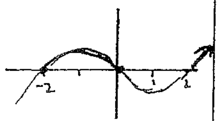
(a)(i)



(ii) From graph, $|x+2| < 2-x$ for $x<0$

(b) $x^3 \geq 4x$

$x(x-2)(x+2) \geq 0$



critical values $0, 2, -2$

Soln $-2 \leq x < 0$ and $x \geq 2$

(c) $\sin\theta = \frac{\sqrt3}{2}$

General Soln $\theta = n\pi + (-1)^n \sin^{-1}\left(\frac{\sqrt3}{2}\right)$

$\theta = n\pi + (-1)^n . \frac{\pi}{3}$

(d) $\int_0^{\sqrt2} \frac{1}{\sqrt{4-x^2}}\,dx = \left[\sin^{-1}\left(\frac{x}{2}\right)\right]_0^{\sqrt2}$

$= \sin^{-1}\left(\frac{1}{\sqrt2}\right) - \sin^{-1}(0)$

$= \frac{\pi}{4}$

(e) Let $f(x) = \cos x - x$; $f(0.7) = 0.648\ldots$

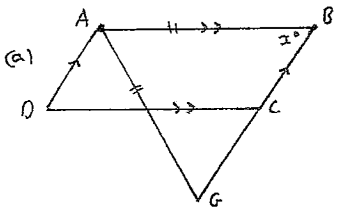
then $f'(x) = -\sin x - 1$; $f'(0.7) = -1.644\ldots$

Next approx.

$x_1 = 0.7 - \frac{f(0.7)}{f'(0.7)}$

$x_1 = 0.74$ [2 d.pl.]

Q2. (a)



(i) let $\angle ABC = x°$

then $\angle ADC = x°$ (opp ∠'s of //gram)

and $\angle AGB = x°$ (equal ∠'s opp equal sides)

∴ AC subtends equal angles at D and G.

Thus, ACGD is cyclic

(ii) Given ACGD is cyclic; AD subtends equal angles

∴ $\angle ACD = \angle AGD$.

(b) $\int_0^3 \sqrt{9-x^2}\,dx$

let $x = 3\sin\theta$

then $dx = 3\cos\theta . d\theta$

and $x=0, \theta=0$

$x=3, \theta=\frac{\pi}{2}$

Substitute

$= \int_0^{\frac{\pi}{2}} \sqrt{9-9\sin^2\theta} . 3.\cos\theta\, d\theta$

$= 9\int_0^{\frac{\pi}{2}} \cos^2\theta . d\theta$ (use $\cos^2\theta = \frac12(\cos2\theta+1)$)

$= \frac92 \int_0^{\frac{\pi}{2}} (\cos2\theta + 1)\, d\theta$

$= \frac92 \left[\frac{\sin2\theta}{2} + \theta\right]_0^{\frac{\pi}{2}}$

$= \frac92 \left[\left(0+\frac{\pi}{2}\right) - 0\right]$

$= \frac{9\pi}{4}$

(c)(i) given $\frac{d^2x}{dt^2} = 1-4x$

then $a = -4\left(x - \frac14\right)$

ie acceleration as a function of displacement is form $a = -n^2(x-b)$ gives motion as S.H.M.

P.T.O.

2(c) cont.

(ii) Since starts from rest then

$x = b + a\cos nt$

ie $1.25 = .25 + a\cos 2\times0$

$a = 1$

Amplitude is 1 and Centre $b = \frac14$

Period: $T = \frac{2\pi}{2}$

$= \pi$

(iii) $\frac{dx}{dt} = \dot{x} = -2\sin 2t$

at $t=0.2$, $\dot{x} = -2\sin 0.4$

$= 0.8$ cm/sec (1 d.p.)

Q3 (a)(i) No. of arrangements

$\frac{9!}{4!\times2!} = 7560$

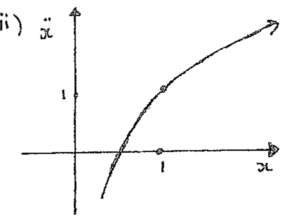
(ii) Arrangements with E's and R's together is $5!$

∴ Prob. $\frac{5!}{7560} = \frac{1}{63}$

(b) (i) $\frac{d}{dx}(x.\log x) = 1.\log x + x.\frac1x$

$= \log x + 1$

(ii)



Acceleration Diagram.

Now, $\frac{d}{dx}\left(\frac12 v^2\right) = 1 + \log x$

$\frac12 v^2 = x.\log x + C$ [from (i)]

at $x=1$, $v=0$ so $C=0$

∴ $v^2 = 2x.\log x$

(iii) Initially at rest at $x=1$. Acceleration always positive, from therefore velocity will always move to the right

$\left[v^2=0,\ 2x=0 \text{ or } \log x = 0;\ x=0,\ x=1\right]$

So $v>0$ for $t>0$ since $x \geq 1$.

When $x=e^2$, $v^2 = 2e^2.\log e^2$

$v^2 = 4e^2$

$v = 2e$ ($v>0$

(c) • let $n=1$

then $\cos(x+\pi) = \cos x\cos\pi - \sin x \sin\ldots$

$= -\cos x$

$= (-1)^1\cos x$

True for $n=1$

• Assume result true for $n=k$ (k positive integer)

ie $\cos(x+k\pi) = (-1)^k\cos x$

• Test result for next n, ie $n=k+1$

$\cos(x+(k+1)\pi) = \cos[(x+k\pi)+\pi]$

$= \cos(x+k\pi)\cos\pi - \ldots$

$= \cos(x+k\pi)\times-1$

From above $= (-1)^k\cos x\times-1$

$= (-1)^{k+1}\cos x$ (as req

• From above, if result is true for $n=k$ then it is true for next n, n= ... Since true for $n=1$, it is true for $n=2$. By induction true for n= and all n

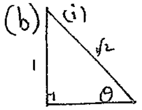
Q4 (a) (i) $y = \cos^{-1} 2x$

$y' = \dfrac{-2}{\sqrt{1-4x^2}}$

(ii) $y = \frac{1}{2} \ln(2x+1)$

$y' = \dfrac{1}{2x+1}$

(b) (i)



$\sin\theta = \frac{1}{\sqrt{2}}$

$\cos\theta = \frac{1}{\sqrt{2}}$

$\tan\theta = 1$

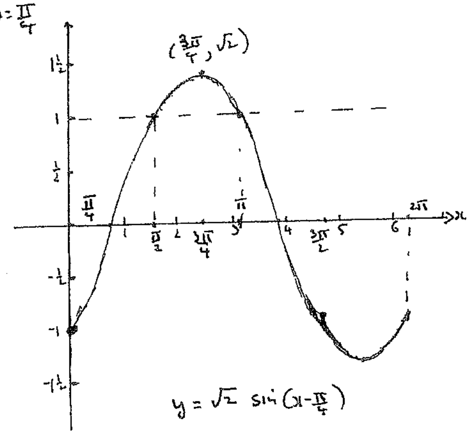
$\therefore \theta = \frac{\pi}{4}$

$\sin x - \cos x = \sqrt{2}(\sin x \cos\theta - \cos x \sin\theta)$

$= \sqrt{2} \sin(x - \theta)$

$= \sqrt{2} \sin(x - \frac{\pi}{4})$

(ii)



$x = 0, y = -1$; $x = \frac{\pi}{4}, y = 0$; $x = \frac{\pi}{2}, y = 1$

Amplitude $\sqrt{2}$

Period 2π

let $\sqrt{2} = \sqrt{2}\sin(x - \frac{\pi}{4})$

$1 = \sin(x - \frac{\pi}{4})$

$\therefore x - \frac{\pi}{4} = \frac{\pi}{2}$

$x = \frac{3\pi}{4}$

(iii)

(c) (i) $\frac{h}{PC} = \tan 60°$ $\frac{h}{PD} = \tan 45°$

$\therefore PC = \frac{h}{\sqrt{3}}$ --(A) $PD = h$ --(B)

(ii) Cosine rule, noting $\angle = 45°$

$CD^2 = h^2 + \left(\frac{h}{\sqrt{3}}\right)^2 - 2 \cdot h \times \frac{h}{\sqrt{3}} \cos 45°$

$= h^2 \left[1 + \frac{1}{3} - \frac{2}{\sqrt{3}} \times \frac{1}{\sqrt{2}}\right]$

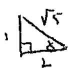
$= h^2 \left[\frac{4}{3} - \frac{2}{\sqrt{6}}\right]$

$= 2h^2 \left[\frac{4}{6} - \frac{\sqrt{6}}{6}\right]$

$CD^2 = \frac{1}{3}h^2(4 - \sqrt{6})$ as required

Q5 (a) let $X = \sin^{-1}\left(\frac{1}{\sqrt{5}}\right)$ and $Y = \sin^{-1}\left(\frac{1}{\sqrt{10}}\right)$

then $\sin X = \frac{1}{\sqrt{5}}$ and $\sin Y = \frac{1}{\sqrt{10}}$



$\cos X = \frac{2}{\sqrt{5}}$ $\cos Y = \frac{3}{\sqrt{10}}$

Now, $\sin(X + Y)$

$= \sin X \cos Y + \cos X \sin Y$

$= \frac{1}{\sqrt{5}} \times \frac{3}{\sqrt{10}} + \frac{2}{\sqrt{5}} \times \frac{1}{\sqrt{10}}$

$= \frac{5}{5\sqrt{2}}$

$= \frac{1}{\sqrt{2}}$

ie $\sin(X + Y) = \frac{1}{\sqrt{2}}$

$X + Y = \sin^{-1}\left(\frac{1}{\sqrt{2}}\right)$

$= \frac{\pi}{4}$

$\therefore \sin^{-1}X + \sin^{-1}Y = \frac{\pi}{4}$

(b) ${}^5C_3 \times {}^7C_4 = 350$ ways.

(c) (i) $f(-x) = \dfrac{6}{3 + (-x)^2}$

$= f(x)$

Even function since $f(-x) = f(x)$

P.T.O

(c) Cont.

(ii) $f'(x) = \dfrac{-12x}{(3+x^2)^2}$

St. pts when $f'(x) = 0$ at $x = 0$

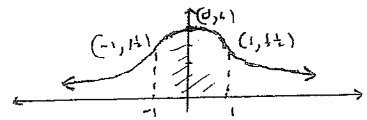
Stat. at (0, 2)

x	$-\frac{1}{2}$	0	$\frac{1}{2}$
$f'(x)$	+ve	–	–ve
	6	0	-6

Local max.

Local max. at (0, 2)

(iii) As $x \to \pm\infty$, $f(x) \to 0$



(iv) Area $= 2\displaystyle\int_0^1 \frac{6}{3+x^2}\,dx$

$= \left[\frac{12}{\sqrt{3}} \tan^{-1}\left(\frac{x}{\sqrt{3}}\right)\right]_0^1$

$= \frac{12}{\sqrt{3}}\left(\frac{\pi}{6} - 0\right)$

$= \frac{2\pi}{\sqrt{3}}$ sq units

Q6. (a)

$h = 2r$

$\therefore \frac{h}{2} = r$

Given $\frac{dV}{dt} = 0.6$ m³/sec.

Now $V = \frac{1}{3}\pi r^2 h$

$= \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h$

$= \frac{1}{12}\pi h^3$ --- (A)

So $\frac{dV}{dh} = \frac{1}{4}\pi h^2$, Require $\frac{dh}{dt}$

and $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$

$0.6 = \frac{\pi h^2}{4} \cdot \frac{dh}{dt}$

$\therefore \frac{dh}{dt} = 0.031$ m/sec.

(b) (i) Gradient 'PQ', $m = \dfrac{a(p-q)(p+q)}{2a(p-q)}$

$\therefore m = \frac{p+q}{2}$

Eqn. $y - ap^2 = \frac{p+q}{2}(x - 2ap)$

$\therefore y = \frac{p+q}{2}x - apq$

Substitute $(2a, 0)$

$0 = a(p+q) - apq$

$\therefore pq = p + q$ ---- (

(ii) Midpoint $M\left(a(p+q), \frac{a}{2}(p^2 + \ldots\right.$

then $X = a(p+q)$ --- $(p+q) = \frac{X}{a}$ ..

$Y = \frac{a}{2}(p^2 + q^2)$ --- $Y = \frac{a}{2}\left[(p+q)^2 - \ldots\right.$

Substitute I into II, use pq

$Y = \frac{a}{2}\left[\left(\frac{X}{a}\right)^2 - 2 \times \frac{X}{a}\right]$

$2Y = a\left[\frac{X^2}{a^2} - \frac{2X}{a}\right]$

$2Y = \frac{1}{a}\left[X^2 - 2aX\right]$

Locus is a parabola $\{ 2aY = ($

is $(X - a)^2 = 2a\left(Y + \frac{a}{2}\right)$

(c) $(x+4)(x-3) = x^2 - 7x + 12$.

Remainder must be of degree than 2. It must be linear

ie $(ax + b)$

• $P(x) = Q(x)(x-3)(x-4) + (ax \ldots$

$x = 3$, $P(3) = 3 = 3\_ + b$ ..

$x = 4$ $P(4) = 4 = 4 + b$ ..

$\therefore$ II - I, $a = 1$

so $b = 0$

Remainder is the linear polynomial $1x + 0 = x$

Q7

(a) (i) Given

$x = Vt\cos\theta$

$y = -5t^2 + Vt\sin\theta$

and at $t = 0$

$x = 12$

$y = 10$

then

$12 = 2V\cos\theta$ --- (i)

$10 = -20 + 2V\sin\theta$ --- (ii)

from (i) $V = \frac{6}{\cos\theta}$ sub $10 = -20 + 12\tan\theta$

$\tan\theta = \frac{5}{2}$

$\underline{\theta = 68^\circ}$ [nearest degree]

$\therefore$ $\underline{V = 16.2 \text{ m/s}}$ [1 dpl]

(ii) Max. height when $y' = 0$

ie $-10t + V\sin\theta = 0$

$t = \frac{V\sin\theta}{10}$

$= 1.5$ sec.

$\therefore$ $y = -5\times(1.5)^2 + 16.2\times1.5\times\sin 68$

$y = 11.3$ m (1 d.pl)

Max. height at $\underline{11.3\text{m}}$

(iii) Range, let $t = 2\times1.5$

(symmetry of projection)

$t = 3$

then $x = 16.2\times3\times\cos 68$

$x = 18.2$m

Range is $\underline{18.2}$ m

(b) (i) $P'(t) = k.Ae^{kt}$

since $A.e^{kt} = P(t) - 5000$

then $P'(t) = k[P(t) - 5000]$

(ii) $70000 = 50000 + Ae^0$

$\therefore$ $\underline{A = 20000}$

(iii) $150000 = 50000 + 20000e^{8k}$

$5 = e^{8k}$

$\therefore$ $k = \frac{1}{8}\ln 5$

$\doteqdot \underline{0.2012}$

(iv) let $P(t+1) = 1.2 \times P(t)$

then $50000 + 20000e^{k(t+1)} = 60000 + 24000e^{kt}$

$\therefore$ $e^{kt}[20000e^k - 24000] = 10000$

$t = \frac{1}{k}\cdot\ln\left[\frac{10000}{20000e^k - 24\ldots}\right]$

$= 15.33\ldots$

It would be after $\underline{15 \text{ years}}$.