ST. GEORGE GIRLS HIGH SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

MATHEMATICS 1996

3 UNIT (ADDITIONAL)

AND

3/4 UNIT (COMMON)

Time Allowed - Two Hours
(Plus 5 minutes reading time.)

Directions to Candidates

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
- A table of standard integrals is attached.
- Board-approved calculators may be used.
- Start each question on a new page.
- Return your answers in one bundle with a cover sheet. On your cover sheet write your name, mathematics class and teacher's name as well as

.... Q1
.... Q2
.... Q3
.... Q4
.... Q5
.... Q6
.... Q7

This is a trial paper only and does not necessarily reflect either the content or format of the final Higher School Certificate examination in this subject.

QUESTION 1

Marks

(a) Find $\lim_{x \to 0} \dfrac{\tan 3x}{4x}$ **1**

(b) On a number plane indicate the region specified by **4**

$y \le \sqrt{4-x}^{\,2}$ and $y \ge x - 4$

Your diagram should clearly show any points of intersection of the boundaries with the co-ordinate axes and with each other.

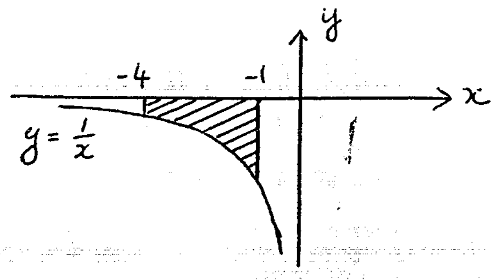
(c) Find the value(s) of k for which $y = e^{kx}$ is a solution of the equation **3**

$$\frac{d^2y}{dx^2} + \frac{3dy}{dx} - 4y = 0$$

(d) For what values of x does the geometric series **2**

$x + \dfrac{x}{x+2} + \dfrac{x}{(x+2)^2} + \dfrac{x}{(x+2)^3} +$ have a limiting sum?

(e)



Find the shaded area **2**

QUESTION 2

MARKS

(a) Find the largest possible (natural) domain of the following functions **3**

(i) $y = \sqrt{5 - \sqrt{20 - x}}$

(ii) $y = \sin^{-1}(\log_e x)$

(b) Use the substitution $u = 3x - 1$ to find **4**

$$\int_0^1 \frac{1}{\sqrt{4 - (3x-1)^2}}\,dx$$ (Give your answer in terms of π)

(c) $f(x) = x^3 + 3x^2 - 9x + 3$ **5**

(i) Show that $f(x) = 0$ has a root between $x = 1$ and $x = 2$

(ii) Taking $x = 2$ as a first approximation, use Newton's method to find a second approximation to this root.

(iii) By means of a diagram, or otherwise, give a geometrical interpretation of the process used in (ii) and hence explain why $x = 1$ is not suitable as a first approximation to the root.

QUESTION 3 **MARKS**

(a) For the curve $y = f(x)$ it is given that $f'(x) = \cos^2 x$ and that it passes through the point $(\frac{\pi}{2}, \frac{\pi}{2})$. Find the equation of the curve. **3**

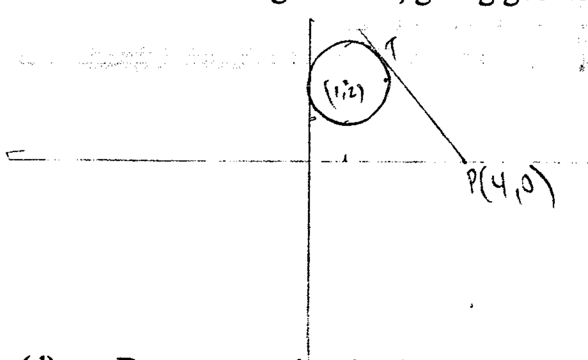
(b) (i) How many 2- digit even numbers are there with different digits? **3**

(ii) Hence, or otherwise, find the number of 3- digit even numbers which have no repeated digits.

(c) From the point $P(4,0)$ a tangent is drawn to touch the circle **4**

$(x - 1)^2 + (y - 2)^2 = 1$ at the point T.

Find the length of PT, giving geometric reason(s) for your calculations.



(d) Draw a neat sketch of the curve **2**

$y = 2\cos^{-1}(2x)$

QUESTION 4 **MARKS**

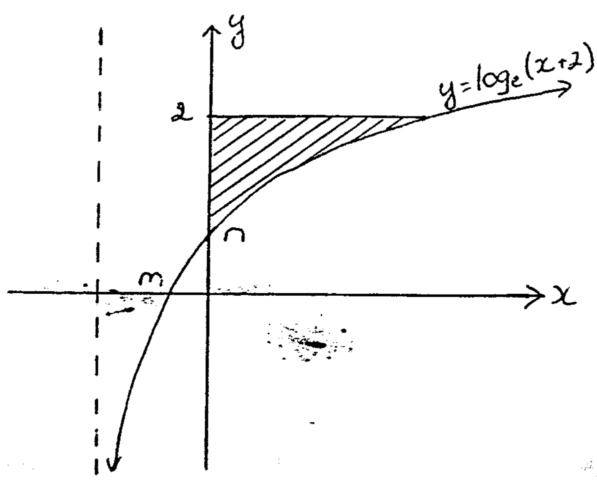
(a) A particle moves along a straight line so that it has displacement x, velocity v and acceleration a at time t. **(6)**

If $v = 2(1+3t)^{-\frac{1}{3}}$ *and* $x = 3$ *when* $t = 0$

(i) find an expression for x in terms of t and hence find x when $t = 21$

(ii) find an expression for a in terms of t and hence show that it can be expressed as $a = kv^n$ by finding the constants k and n.

(b)



A sketch (not necessarily to scale) of $y = \log_e(x+2)$ is shown. **(4)**

(i) Write down the values of m and n (the x- and y- intercepts of $y = \log_e(x+2)$)

(ii) Find the exact value of the shaded area.

(c) By means of a diagram and written explanation, show that the equation $\tan x = x + 1$ has an infinite number of solutions. **(2)**

QUESTION 5 **MARKS**

(a) According to the Theory of Relativity, the mass, M *kg*, of an object travelling at a speed of v *km/s* is given by

$$M = m\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

where m *kg* is the mass of the object when at rest and c *km/s* is the speed of light (a constant).

When at rest, the captain of a spaceship has a mass of 80 kg. If the speed of the spaceship is increasing at a rate of 0.025c km/s/s, at what rate is the captain's mass increasing when the space ship has a speed of 0.6c km/s (ie 0.6 times the speed of light)?

(b) Let $M = \int_1^3 \left(\frac{2}{x} - x\right)dx$ **(4)**

(i) Find the exact value of M

(ii) Use one application of Simpson's rule to find an approximate value of M.

(c) (i) Show that for a particle moving with acceleration a, velocity v and displacement x, acceleration can be expressed as **(4)**

$$a = \frac{d\left(\frac{1}{2}v^2\right)}{dx}$$

(ii) A particle moves in a straight line so that its acceleration is given by

$$a = 2x - 2x^3$$

If $v = \sqrt{3}$ when $x = 0$ find an expression for v in terms of x.

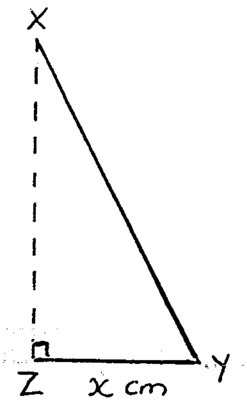
QUESTION 6 **MARKS**

(a) Solve $2\sin\theta - 3\sin 2\theta = 0 \text{ for } 0 \le \theta \le \pi$ **(3)**

(b) When a polynomial $P(x)$ is divided by $x^2 - 1$ the remainder is $R(x)$. **(3)**

(i) Explain why $R(x)$ can be expressed in the form
$R(x) = ax + b$ (a and b constant)

(ii) It is known that 6 is the remainder when $P(x)$ is divided by $x - 1$ and 4 is the remainder when $P(x)$ is divided by $x + 1$.
Find $R(x)$.

(c) 

A piece of wire 9cm long is bent to form the hypotenuse and one side of right angled triangle XYZ (as shown in the diagram). **(6)**

(i) If $ZY = x\text{cm}$ show that the area of ΔXYZ is given by

$$A = \frac{3x\sqrt{9-2x}}{2}$$

(ii) Hence, or otherwise show that the maximum area of the triangle occurs when the wire is bent at an angle of $60°$.

QUESTION 7

MARKS

(a) $y = \tan^{-1}(\tan x)$ 5

(i) Write down the domain and range of this function.

(ii) Show that $\frac{dy}{dx} = 1$ for all values of x in the domain.

(iii) Sketch the graph of $y = \tan^{-1}(\tan x)$.

(b) Let T be the temperature of an object at time t and let D be the constant temperature of the surrounding medium. Newton's law of cooling states that the rate of change of T is proportional to $T - D$ 7

i.e. $$\frac{dT}{dt} = k(T - D)$$

(i) Show that $T = D + Ce^{kt}$ (where C and k are constants) satisfies Newton's law of cooling.

(ii) A packet of meat with an initial temperature of 20°C is placed in a freezer whose temperature is kept constant at -15°C. It takes 12 minutes for the temperature of the meat to drop to 10°C.

How much additional time does it take for the temperature of the meat to fall a further 10°C (i.e. to reach freezing point, 0°C)? Give your answer in minutes, correct to 1 decimal place.

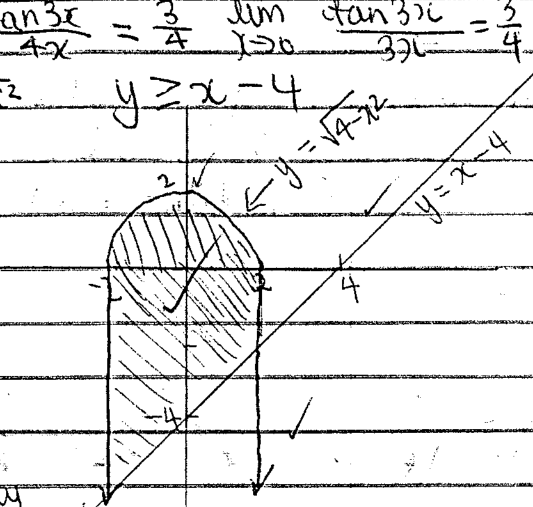
(iii) Draw a neat sketch of the graph of T (temperature of the meat) against time (t).

St. George Girls High School Trial HSC 3 Unit Maths 1996

Question 1

(a) $\lim_{x\to 0} \frac{\tan 3x}{4x} = \frac{3}{4} \lim_{x\to 0} \frac{\tan 3x}{3x} = \frac{3}{4}$ ✓

(b) $y \le \sqrt{4-x^2}$ $\quad y \ge x-4$



✓

(c) $\frac{d^2y}{dx^2} + \frac{3dy}{dx} - 4y = 0$

$y = e^{kx}$ $\quad \frac{dy}{dx} = ke^{kx}$ $\quad \frac{d^2y}{dx^2} = k^2e^{kx}$ ✓

$\therefore k^2e^{kx} + 3ke^{kx} - 4e^{kx} = 0$

$e^{kx}(k^2+3k-4) = 0$

$e^{kx}(k+4)(k-1) = 0$ ✓

$\therefore k = -4$ or $k = 1$ ✓

(d) $x + \frac{x}{x+2} + \frac{x}{(x+2)^2} + \frac{x}{(x+2)^3} \ldots$

$S = \frac{a}{1-r}$ $\quad a = x$ $\quad r = \frac{1}{(x+2)}$

$S = \frac{x}{1-\frac{1}{x+2}} = \frac{x(x+2)}{x+2-1} = \frac{x(x+2)}{x+1}$

$x \ne -1$ For limiting sum $-1 < \frac{1}{x+2} < 1 \Rightarrow \boxed{x > -1}$ $\boxed{x < -3}$

(e) $y = \frac{1}{x}$ $\quad A = \int_{4}^{1} \frac{1}{x}\,dx = \left[\ln|x|\right]_{4}^{1}$ ✓

$= 0 - \ln 4$

$\therefore$ Since area is positive, area is $[\ln 4]$ units² ✓

Question 2.

(a) i) $y = \sqrt{5 - \sqrt{20-x}}$

$\sqrt{20-x} \le 5$ $\quad \cap \quad 20 - x \ge 0$

$20 - x \le 25$ $\quad x \le 20$

$-x \le 5$ ✓

$-x \ge -5$

$\therefore D_x : \{x \ge -5\}$ $\boxed{-5 \le x \le 20}$

(ii) $y = \sin^{-1}(\log_e x)$

$-1 \le \log_e x \le 1$

$e^{-1} \le x \le e$ ✓

(b) $\int_0^1 \frac{dx}{\sqrt{4-(3x-1)^2}}$ $\quad$ let $u = 3x-1$

$du = 3dx$

$\frac{1}{3}\int_{-1}^{2} \frac{du}{\sqrt{4-u^2}}$ ✓

$= \frac{1}{3}\left[\sin^{-1}\frac{u}{2}\right]_{-1}^{2}$ ✓

$= \frac{1}{3}\left[\frac{\pi}{2} - -\frac{\pi}{6}\right] = \frac{1}{3}\left[\frac{2\pi}{3}\right] = \frac{2\pi}{9}$ ✓

(c) i) $f(x) = x^3 + 3x^2 - 9x + 3$

$f(1) = -2 < 0$

$f(2) = 5 > 0$ ✓

$\therefore$ There is a root between $x = 1$ & $x = 2$

(ii) $x = 2$

$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$

Now $f(x_1) = 5$

$f'(x) = 3x^2 + 6x - 9$

$\therefore f'(2) = 15$

$\therefore x_2 = 2 - \frac{5}{15}$ ✓

$\therefore x_2 = 1\frac{2}{3}$ ✓

(iii) $x_2 = 1 - \frac{-2}{0}$ $\quad$ i.e. the line is horizontal hence will not intersect the x-axis.

Now denominator cannot equal 0

$\therefore x = 1$ is not a suitable first approximation.

Question 3.

(a) $y = f(x)$ $\quad f'(x) = \cos^2 x$

passes $\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\int \cos^2 x\,dx = \frac{1}{2}\int (1+\cos 2x)\,dx$ ✓

$= \frac{1}{2}\left[x + \frac{\sin 2x}{2}\right] + c$

$\therefore y = \frac{x}{2} + \frac{\sin 2x}{4} + C$

Sub $\left[\frac{\pi}{2}, \frac{\pi}{2}\right]$ into eq^n

$\frac{\pi}{2} = \frac{\pi}{4} + 0 + C$ ✓

$\therefore C = \frac{\pi}{4}$

$\therefore y = \frac{1}{2}\left[x + \frac{\sin 2x}{2} + \frac{\pi}{2}\right]$ ✓

(b) i) — $\boxed{9\ \boxed{8}\,1}$ — 0 $= 9$ $\quad \boxed{1}$ ✓

$\boxed{8\,1}$ — 2 $= 8 \times 4$ $= \boxed{41}$

* (iii) see page 4

(c) $(x-1)^2 + (y-2)^2 = 1$

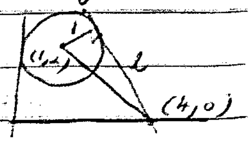
$y = \sqrt{1-(x-1)^2} + 2$

$\frac{dy}{dx} = \frac{1}{2}(1-(x-1)^2)^{-1/2} \times 2(x-1)$

$= \frac{x-1}{\sqrt{1-(x-1)^2}}$

try again

use Pyth. Thm



sub $(4,0)$ $0 = \frac{3}{\sqrt{1-9}}$

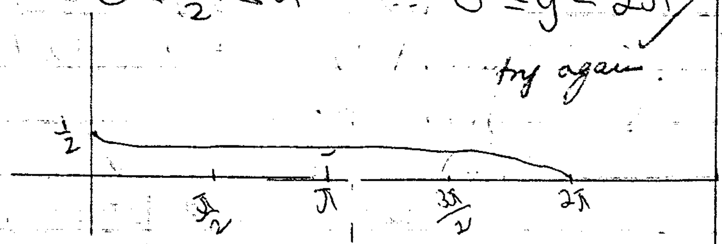
(d) $y = 2\cos^{-1}(2x)$

Domain
$-1 \le 2x \le 1$
$-\frac{1}{2} \le x \le \frac{1}{2}$

$0 \le 2x \le 1$ ~~$\therefore 0 \le x \le \frac{1}{2}$~~

$0 \le \frac{y}{2} \le \pi$ $\quad \therefore 0 \le y \le 2\pi$ ✓

try again



Question 4.

(a)(i) $v = 2(1+3t)^{-1/3}$ $\quad x = 3$ when $t = 0$

$v = \frac{dx}{dt}$ $\quad \therefore x = 2\int (1+3t)^{-1/3}\, dt$

let $u = 1+3t$ $\quad du = 3dt$

$x = \frac{2}{3}\int u^{-1/3}\, du$ ✓

$= \frac{2}{3} \times \frac{3}{2} u^{2/3} = [1+3t]^{2/3} + C$

when $t = 0$ $\quad x = 3$

$3 = 1 + C$ $\quad \therefore C = 2$ ✓

$\therefore x = [1+3t]^{2/3} + 2$ ✓

when $t = 21$ $\quad x = 18$. ✓

(ii) $a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$

$\therefore v = 2(1+3t)^{-1/3}$

$\frac{dv}{dt} = -\frac{2}{3}(1+3t)^{-4/3} \times 3$

$\therefore a = -2(1+3t)^{-4/3}$ ✓

arrange in form $a = kv^n$

$a = -\frac{1}{8}(2(1+3t)^{-1/3})^4$

$\therefore k = -\frac{1}{8}$ $\quad n = 4$

$a = -\frac{1}{8}v^4$ ✓

(b)(i) $y = \log_e(x+2)$ at m, $y = 0$

$\therefore x+2 = 1$ $\quad \therefore x = -1$ ✓

at n, $x = 0$ $\quad \therefore y = \ln 2$

$\therefore m(-1, 0)$ $\quad n(0, \ln 2)$ ✓

(ii) Area $= \int y\, dy$

$y = \log_e(x+2)$ $\quad x = \log_e(y+2)$

$e^x = y+2$ $\quad \therefore y = e^x - 2$

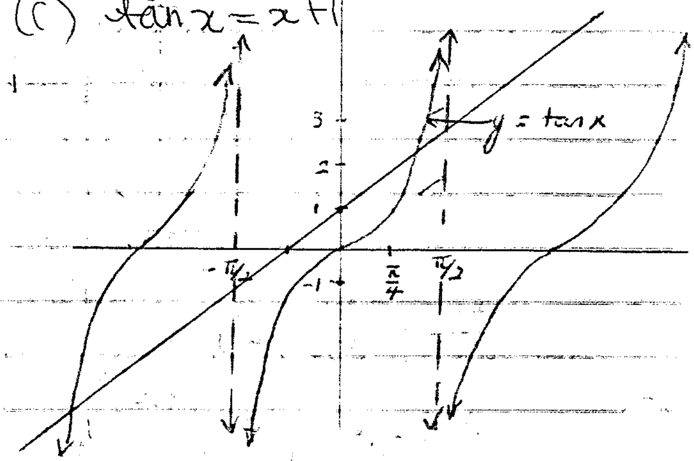
Area $= \int_2^{\ln 2} e^x - 2\, dx$

Area $= \left[e^x - 2x\right]_2^{\ln 2}$

$= [2 - 2\ln 2] - [e^2 - 4]$

$= [6 - 2\ln 2 - e^2]$ units^2

(c) $\tan x = x + 1$



The line $y = x+1$ intersects $y = \tan x$ in infinite points.

Question 5.

(a) $M = m\left(1 - \frac{v^2}{c^2}\right)^{-1/2}$

$V = 0$ $\quad m = 80$kg $\quad \frac{dv}{dt} = 0.025c$

Find $\frac{dM}{dt}$ when $v = 0.6c$ km/s.

$\frac{dM}{dt} = \frac{dM}{dv} \times \frac{dv}{dt}$

$= 80 \cdot -\frac{1}{2}\left(1 - \frac{v^2}{c^2}\right)^{-3/2}\left(\frac{-2v}{c^2}\right) \times 0.025c$

$= 80\left(1 - \frac{v^2}{c^2}\right)^{-3/2}\left(\frac{v}{c^2}\right)(0.025c)$

$= 80\left(1 - \frac{(0.6c)^2}{c^2}\right)^{-3/2}\left(\frac{0.6c}{c}\right)(0.025)$

$= 2.34$ (to 2 dp)

(b)(i) $M = \int_1^3 \left(\frac{2}{x} - x\right) dx$

$= \left[2\ln x - \frac{x^2}{2}\right]_1^3$ ✓

$= \left(2\ln 3 - \frac{9}{2}\right) - \left[0 - \frac{1}{2}\right]$

$M = 2\ln 3 - 4$ ✓

(ii) $M = \frac{b-a}{6}\left\{f(a) + 4 \cdot f\left(\frac{a+b}{2}\right) + f(b)\right\}$

$= \frac{1}{3}\left\{1 + 4(-1) + -2\frac{1}{3}\right\}$ ✓

$= -1\frac{7}{9}$ ✓

(c)(i) $v=\frac{dx}{dt}$ $a=\frac{dv}{dt}$

$\frac{dv}{dt}=\frac{dx}{dt}\cdot\frac{dv}{dx}$

$\therefore a=v\frac{dv}{dx}$ ✓

$\frac{d}{dx}\left(\frac{1}{2}v^2\right)=\left(\frac{d}{dv}\frac{1}{2}v^2\right)\frac{dv}{dx}$

$=\frac{d}{dx}\frac{1}{2}v^2$

$=v\frac{dv}{dx}$ ✓

$\therefore a=\frac{d}{dx}\left(\frac{1}{2}v^2\right)$

(ii) $a=2x-2x^3$ $v=\sqrt{3}$ when $x=0$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right)=2x-2x^3$

integrate both sides

$\frac{1}{2}v^2=\int 2x-2x^3\,dx$

$\frac{1}{2}v^2=\left(x^2-\frac{x^4}{2}\right)$ ~~dx~~ C ✓

when $x=0$, $v=\sqrt{3}$

$\frac{3}{2}=C$

$\therefore \frac{1}{2}v^2=x^2-\frac{x^4}{2}+\frac{3}{2}$

$v^2=2\left[x^2-\frac{x^4}{2}+\frac{3}{2}\right]$

$v=\pm\sqrt{2\left(x^2-\frac{x^4}{2}+\frac{3}{2}\right)}$ ✓

Question 6.

(a) $2\sin\theta-3\sin 2\theta=0$ for $0\le\theta\le\pi$

$2\sin\theta-3(2\sin\theta\cos\theta)=0$

$2\sin\theta-6\sin\theta\cos\theta=0$ ✓

$2\sin\theta(1-3\cos\theta)=0$

$\sin\theta=0$ ✓ or $\cos\theta=\frac{1}{3}$ ✓

$\theta=0,\pi$ or $\theta=$ ~~70°32'~~ 0.6161

(b)(i) When polynomial is divided by (x^2-1) ie a quadratic, ∴ the remainder is one degree lower ie $R(x)=ax+b$ ✓

(ii) $P(1)=6$ ① ~~$R(x)=ax^2+bx+c$ since~~

$P(-1)=4$ ② ~~$P(x)$ divided by degree 1~~

∴ from ① $a+b+c=6$ ③

② $a-b+c=4$ ④

③−④ $2b=2$ $b=1$

③+④ $=2a+2c=10$

$a+c=5$ $c=5-a$

$R(x)=ax+b$

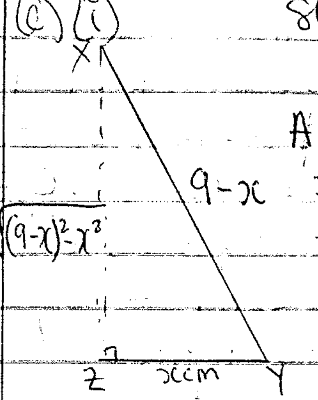
$R(1)=a+b=6$ . . . (i)

$R(-1)=-a+b=4$ . . . (ii)

$2b=10$

$b=5$, $a=1$

(c)(i) Show area $\triangle XYZ$ $A=\frac{3x\sqrt{9-2x}}{2}$



$A=\frac{1}{2}x\sqrt{(9-x)^2-x^2}$ ✓

$=\frac{x}{2}\sqrt{81-18x+x^2-x^2}$

$=\frac{3x}{2}\sqrt{9-2x}$ ✓

as req'd ✓

(ii) $\frac{dA}{dx}=\frac{3}{2}\sqrt{9-2x}+\frac{3x}{2}\cdot\frac{1}{2}(9-2x)^{-\frac{1}{2}}\times -2$

$=\frac{3}{2}\sqrt{9-2x}-\frac{3x}{2\sqrt{9-2x}}$

Now max Area when $\frac{dA}{dx}=0$

$0=\frac{3}{2}\left[\sqrt{9-2x}-\frac{x}{\sqrt{9-2x}}\right]$ ✓

$0=\sqrt{9-2x}-\frac{x}{\sqrt{9-2x}}$

$0=\frac{9-3x}{\sqrt{9-2x}}$ $x\ne\frac{9}{2}$

$0=9-3x$ $x=3$ ✓

$\therefore \frac{9-3}{\sin x}=\frac{\sqrt{(9-3)^2-3^2}}{\sin y}$

$6=\frac{\sqrt{27}}{\sin y}$ $\therefore y=60°$ ✓

Question 7.

(a)(i) $y=\tan^{-1}(\tan x)$

Dx: ~~$\{x: x\in R\}$~~ ~~$-\frac{\pi}{2}<x<\frac{\pi}{2}$~~

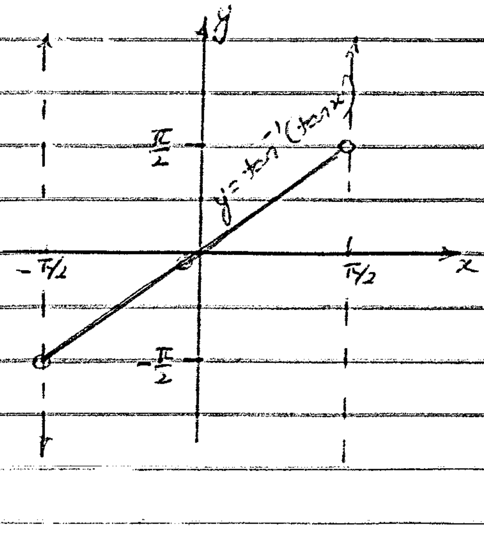
Ry: $\left\{-\frac{\pi}{2}<y<\frac{\pi}{2}\right\}$

(ii) Show $\frac{dy}{dx}=1$

$\frac{dy}{dx}=\frac{1}{(\tan x)^2+1}\times\sec^2x$

$=\frac{1}{\sec^2x}\times\sec^2x$

$=1$ as req'd.



(b)(i) $T - D = Ce^{kt}$ ✓

$\frac{dT}{dt} = kCe^{kt} = k(T-D)$ ✓

(ii) $t=0$, $T=20$, $D=-15$

$20 = -15 + C \quad \therefore C = 35$ ✓

Now $T = -15 + 35e^{kt}$

when $t=12$, $T=10$

$10 = -15 + 35e^{12k}$

$\ln\frac{5}{7} = 12k$ ✓

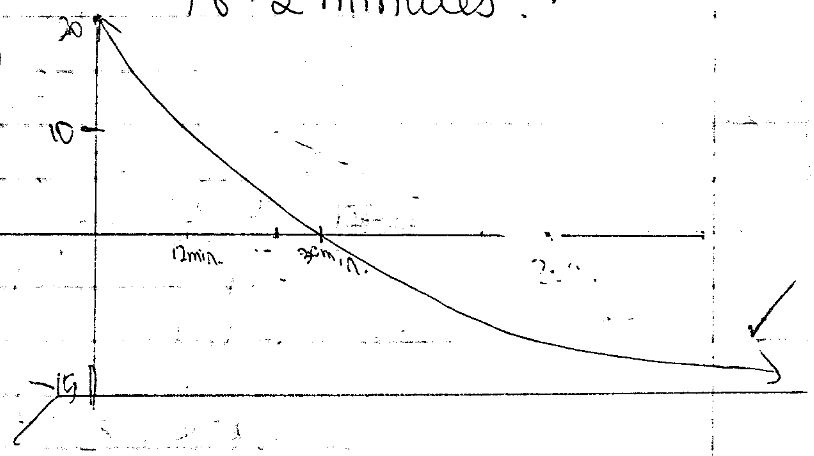
$\therefore k = -0.02803935$

$0 = -15 + 35e^{-0.028t}$

$t = 30.2181673 2$ minutes

$\therefore$ it takes a further 18.2 minutes. ✓

(iii)



(3)(b)(ii)

(iii)

9	8	0

= 72

+

8	8	2

= 64 × 4

} = 328

4

6

8