St George Girls High School

Year 12

Mid-HSC Course Examination

2007

Mathematics

General Instructions

- Working time: 1½ hours
- Reading time: 5 minutes
- Write using blue or black pen.
- Board-approved calculators may be used.
- A table of standard integrals is provided.

Total marks – 90

- Attempt Questions 1 – 6
- All questions are of equal value
- Begin each question in a new booklet
- All necessary working should be shown in every question.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Marks

Question 1 – (15 marks) – (Start a new page)

a) Write down the equation of the parabola with

(i) vertex at the origin and directrix $y=3$ 2

(ii) vertex $(-1, 2)$ and focus $(1, 2)$ 2

b) A point $P(x, y)$ moves so that it is equidistant from the line $y=-2$ and the fixed pint $S(3, 4)$. 4

Find the equation of the locus of P.

c) (i) Express the equation of the parabola $2y=x^2-4x+10$ in the form $(x-h)^2=4a(y-k)$ 2

(ii) Give the co-ordinates of the focus and vertex for the parabola in (i). 2

(iii) Sketch the parabola in (i), clearly indicating its focus, vertex, directrix and axis of symmetry. 3

Marks

Question 2 – (15 marks) – (Start a new page)

a) Sketch the graph of the function $g(x)=|x-2|$ 3

State the value of x where the function is not differentiable.

b) A function is given by the rule $f(x)\begin{cases} =\dfrac{x^2-9}{x-3}, & x<3 \\ =2x, & x\geq 3 \end{cases}$

(i) Find the $\lim\limits_{x\to 3} f(x)$ by considering the two different parts of the function. 2

(ii) Is the function continuous at $x=3$?

Give mathematical reasons for your answer. 2

c) Differentiate with respect to x

(i) $y=3x^2(x-1)^2$ 2

(ii) $y=\dfrac{(x+1)^3}{x}$ 3

d) If $f(x)=x^3-3x^2-24x$, find the values of x for which the function is:

(i) increasing 2

(ii) decreasing 1

Question 3 – (15 marks) – (Start a new page) Marks

a)

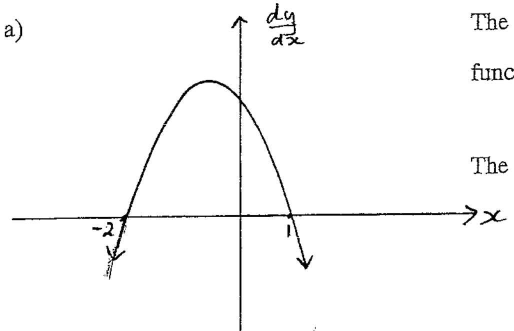

fig I.

The curve $y = f(x)$ has a gradient function $\frac{dy}{dx}$

The graph of $\frac{dy}{dx}$ is given in fig I.

(i) Find where $f(x)$ is a decreasing function. 1

(ii) State the nature of the point P with x co-ordinate -2 on the curve $y = f(x)$. Give mathematical reasons. 2

b) A curve has equation $y = x^3 - 6x^2 + 9x$

(i) Find the coordinates and the nature of any turning points. 4

(ii) Find the coordinates of the point of inflexion on the curve. 2

(iii) Sketch the curve in the domain $0 \le x \le 4$ 3

c) Find the area of the region bounded by the curve $y = 9 - x^2$ and the x-axis. 3

Question 4 – (15 marks) – (Start a new page) Marks

a) (i) Show that the function $f(x) = x^3 - 4x$ is an ODD function 1

(ii) Without integrating, evaluate the definite integral $\int_{-2}^{2} \left(x^3 - 4x\right) dx$ 2

b) Given $\int_{-1}^{2} \left(x^2 + Kx - 1\right) dx = \frac{9}{2}$

Find the value of K.

c) The region bounded by the curve $x = y^2$ and the y-axis between $y = 1$ and $y = 2$ is rotated through 360° about the y-axis. Find the volume of the solid of revolution generated. 3

d)

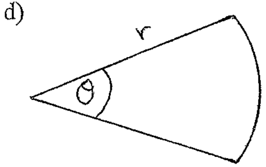

fig II

A wire framework enclosing an area of 144cm^2 is to be made in the shape of a sector of a circle.

The length of wire required is L metres. If the sector angle is θ degrees and the radius r centimetres.

(i) Show that $L = 2r + \frac{288}{r}$ 2

[Hint: use arc length l of a circle is $l = \frac{\theta}{360}(2\pi r)$]

(ii) Find the minimum length of wire required to make the framework. 4

Question 5 – (15 marks) – (Start a new page) Marks

a) The graph of $g(x)=\sqrt{4-x^2}$ is given in fig III.

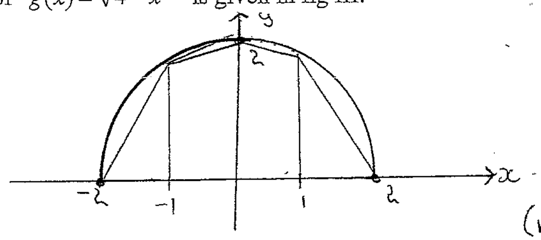

fig III (not to scale)

(i) Use the Trapezoidal rule with 5 function values to approximate the area between the curve and the x-axis, giving your answer correct to four decimal places. 3

(ii) Explain whether the approximation in (i) is an underestimate or an overestimate by reference to fig III. 2

b) (i) Find the indefinite integral $\int (x^2-1)^2\, dx$ 2

c) Calculate the area bounded by the line $y=6-2x$ and the curve $y=x^2-3x$. 4

Answer correct to 2 decimal places.

d) The area bounded by the curves $y=2x^2$, $x+y=3$ and the x-axis is rotated about the x-axis [fig IV]. 4

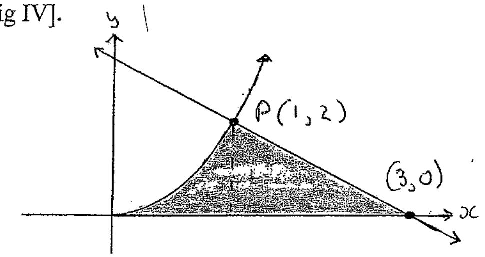

fig IV

Calculate the volume of the solid generated (give answer in terms of π).

$P(1, 2)$ is point of intersection in the first quadrant.

Question 6 – (15 marks) – (Start a new page) Marks

a) (i) Evaluate, to four significant figures e^{-3} 1

(ii) Simplify $\dfrac{2^{2x+1}}{2^{2x+2}}$ 1

b) (i) Sketch the graph of $y=e^x-1$ 2

(ii) State the range for $y=e^x-1$ 1

c) Find the derivative of $f(x)=xe^{2x}$ 2

d) (i) Give the first and second derivatives of $y=x-e^x$ 2

(ii) Use part (i) to find any stationary points for $y=x-e^x$ and determine their nature. 3

e) Show that the equation of the normal to the curve $y=e^{2x+1}$ at the point where $x=1$ is

$$x+2e^3y-2e^6-1=0$$ 3

End of Paper

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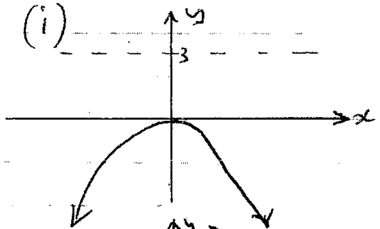
Solutions.

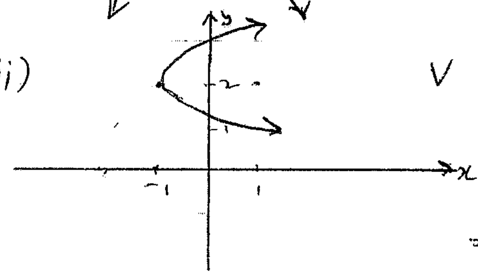
Q1 a) (i)



$x^2 = -4ay$
$\therefore x^2 = -12y$

(2)

(ii)



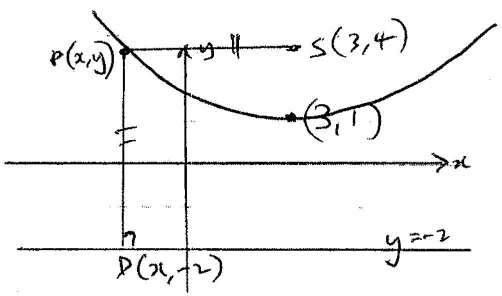
V (-1, 2) F (1, 2) a = 2

$(y-k)^2 = 4a(x-h)$

$\Rightarrow (y-2)^2 = 8(x+1)$

(2)

b)



$PS = PD$

$\therefore \sqrt{(x-3)^2+(y-4)^2} = \sqrt{(y+2)^2}$

Squaring gives

$x^2-6x+9+y^2-8y+16 = y^2+4y+4$

i.e. $x^2-6x+9 = 12y-12$

$(x-3)^2 = 12(y-1)$

(4)

OR quote $(x-h)^2 = 4a(y-k)$ with

$(h,k) \equiv (3,1)$ and $a = 3$.

$\Rightarrow (x-3)^2 = 12(y-1)$

c)(i) $2y = x^2-4x+10$
$= x^2-4x+4+6$

$\therefore (x-2)^2 = 2y-6$
$= 4 \times \frac{1}{2}(y-3)$

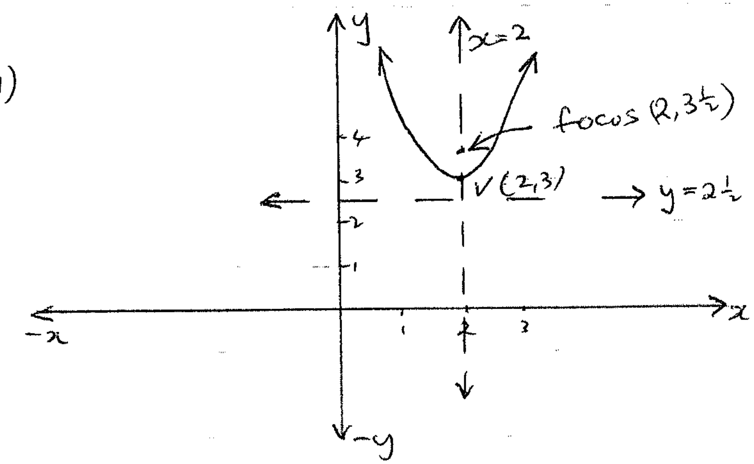
(2)

i.e. $(h,k) = (2,3)$
$a = \frac{1}{2}$.

(ii) Vertex is $(2,3)$ 1
Focus is $(2,3\frac{1}{2})$ 1

(2)

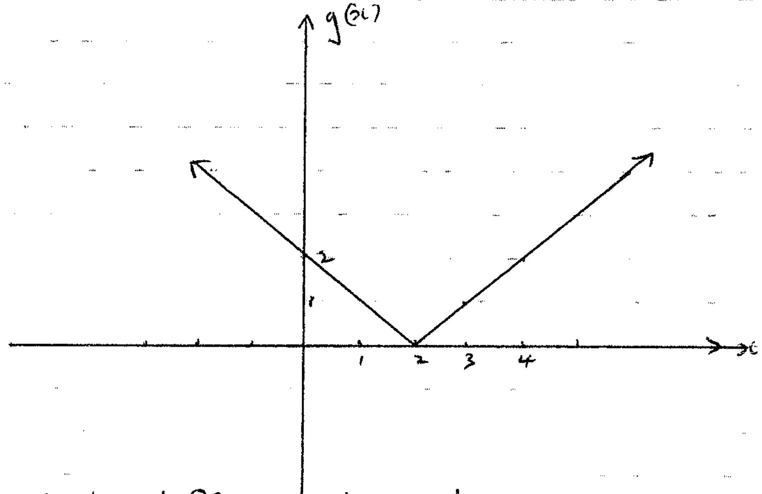
(iii)



(3)

Q.2

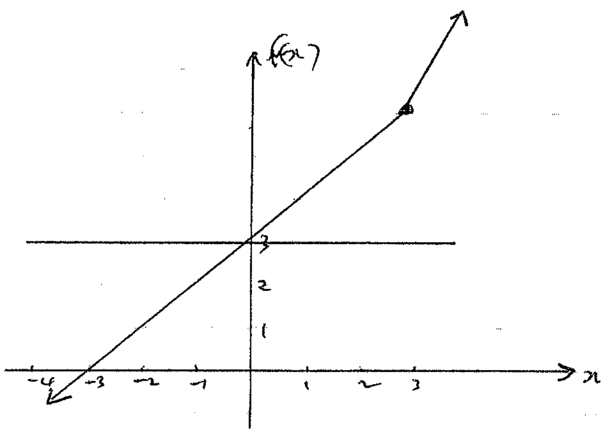
a)



(3)

Not differentiable at $x=2$.

b)



(graph not necessary)

$f(x) = \frac{x^2-9}{x-3}, \quad x<3$ i.e. $y = x+3,\ x \neq 3$

$= 2x, \quad x \geq 3$ i.e. $f(3) = 6$.

(i) $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} x+3 = 6$

$\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 2x = 6$. (2)

(ii) Yes – $f(3)$ exists, $\lim_{x\to 3^+} f(x)$ and $\lim_{x\to 3^-} f(x)$ both exist and all are equal. (2)

c) (i) $\frac{d}{dx}\, 3x^2(x-1)^2 = \frac{d}{dx}\, 3x^2(x^2-2x+1)$

$= \frac{d}{dx}\, 3x^4 - 6x^3 + 3x^2$

$= 12x^3 - 18x^2 + 6x$ (2)

OR use product rule & f'n of f'n rule

(ii) $y = \frac{(x+1)^3}{x}$

$\therefore y' = \frac{x \cdot 3(x+1)^2 - (x+1)^3 \cdot 1}{x^2}$

$= \frac{(x+1)^2(3x-x-1)}{x^2}$

$= \frac{(x+1)^2(2x-1)}{x^2}$ (3)

d) $f(x) = x^3 - 3x^2 - 24x$

$\therefore f'(x) = 3x^2 - 6x - 24 = 3(x^2-2x-8)$

(i) increasing where $f'(x) > 0$,

i.e. $x^2-2x-8 > 0$

$(x-4)(x+2) > 0$

i.e. $x>4,\ x<-2$. (2)

(ii) decreasing where $f'(x) < 0$

i.e. $-2 < x < 4$. (1)

Q3

(1) a) (i) $f(x)$ decreasing for $x < -2$, $x > 1$.

(ii) At P, $\frac{dy}{dx}$ changes from negative to zero to positive as we increase in x values
i.e. there is a minimum turning
(2) point at P.

b) (i) $y = x^3 - 6x^2 + 9x. = x(x^2 - 6x + 9) = x(x-3)^2$

$y' = 3x^2 - 12x + 9$
$= 3(x^2 - 4x + 3)$
$= 3(x-3)(x-1)$

St. pts where $y' = 0$ i.e. $x = 1$, $x = 3$.

(4) $y'' = 6x - 12$
when $x = 1$, $y'' < 0$ $\therefore$ Max. t. pt at $(1, 4)$
when $x = 3$, $y'' > 0$ $\therefore$ Min. t. pt at $(3, 0)$

(ii) For point of inflection, $y'' = 0$ & y'' changes sign.
$y'' = 6x - 12$
$y'' = 0 \Rightarrow x = 2$.

(2)

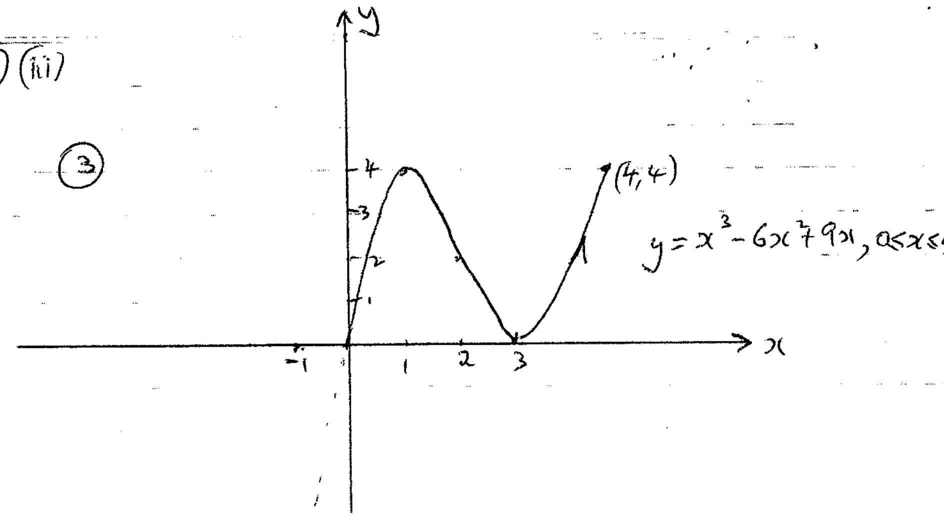
x	2^-	2	2^+
y''	$-$	0	$+$

$\therefore$ pt of inflection at $(2, 2)$

(iii) P.T.O.

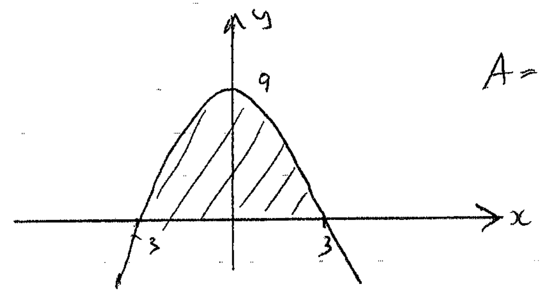
b) (iii)

(3)



c)

(3)



$A = \int_{-3}^{3} 9 - x^2 \, dx$
$= 2\int_0^3 9 - x^2 \, dx$
$= 2\left[9x - \frac{x^3}{3}\right]_0^3$
$= 2(27 - 9 - 0)$
$= 36 \, u^2$.

Q4

a)(i) $f(x) = x^3 - 4x$

∴ $f(-x) = (-x)^3 - 4.(-x)$

$= -x^3 + 4x$ (1)

$= -(x^3 - 4x)$

$= -f(x)$ ∴ ODD fn.

(ii) $\int_{-2}^{2} x^3 - 4x \, dx = 0$ since odd function with symmetrical limits. (2)

b) $\int_{-1}^{2} x^2 + Kx - 1 \, dx = \frac{9}{2}$

∴ $\left[\frac{x^3}{3} + \frac{Kx^2}{2} - x \right]_{-1}^{2} = 9/2$

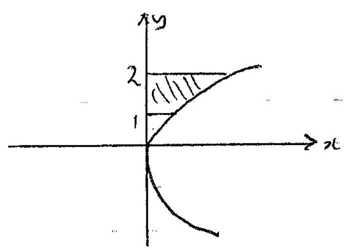
ie $\frac{8}{3} + 2K - 2 - \left(-\frac{1}{3} + \frac{K}{2} + 1\right) = 9/2$

ie $\frac{2}{3} + 2K - 2/3 - \frac{K}{2} = 9/2$

∴ $\frac{3K}{2} = \frac{9}{2}$ (3)

∴ $K = 3$

c)



$V = \pi \int_{1}^{2} x^2 \, dy$

$= \pi \int_{1}^{2} y^4 \, dy$

$= \pi \left[\frac{y^5}{5} \right]_{1}^{2}$

$= \pi \left(\frac{32}{5} - \frac{1}{5} \right)$ (3)

$= \frac{31\pi}{5} \; u^3$

d)(i) $A = \frac{\theta}{360} \times \pi r^2 = 144$

∴ $\frac{\theta}{360} = \frac{144}{\pi r^2}$

Arc length of framework $= \frac{\theta}{360} \times 2\pi r$

$= \frac{144}{\pi r^2} \times 2\pi r$

$= \frac{288}{r}$.

∴ total perimeter $= L = 2r + \frac{288}{r}$. (2)

(ii) We wish to minimise L ∴ find $\frac{dL}{dr}$ first.

$\frac{dL}{dr} = 2 - 288 r^{-2}$

$\frac{dL}{dr} = 0 \Rightarrow 2 = \frac{288}{r^2}$

∴ $2r^2 = 288$

$r = 12 \quad (r > 0)$

$\frac{d^2L}{dR^2} = 576 r^{-3}$ (4)

when $r = 12$, $\frac{d^2L}{dR^2} > 0$

∴ Min. value of L at $r = 12$

ie perimeter minimised when radius is 12 cm.

Q5

a) (i) use trapezoidal rule

x	-2	-1	0	1	2
$g(x)$	0	$\sqrt{3}$	2	$\sqrt{3}$	0

$A \doteqdot \frac{1}{6}(2)[0+2+4\sqrt{3}] + \frac{1}{6}(2)[2+0+4\sqrt{3}]$

$= \frac{2}{3}(2+4\sqrt{3})$

(3) $\doteqdot 5.9521$ (to 4 dec. pl.)

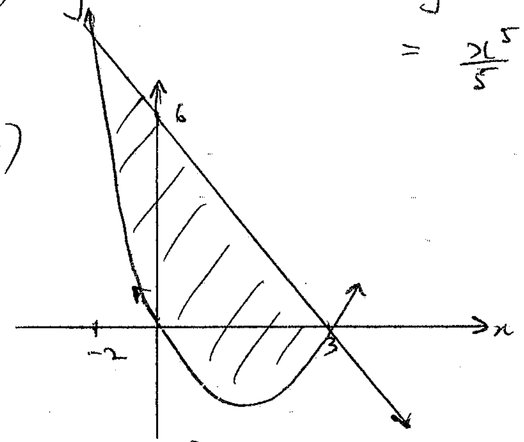
(ii) This would be an underestimate as the trapezia would lie below the semi-circle.

(2)

b) $\int (x^2-1)^2\,dx = \int x^4 - 2x^2 + 1\,dx$

$= \frac{x^5}{5} - \frac{2}{3}x^3 + x + C$

c)



$6 - 2x = x^2 - 3x$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = -2, 3$

$A = \int_{-2}^{3} 6 - 2x - (x^2 - 3x)\,dx$

$= \int_{-2}^{3} -x^2 + x + 6\,dx$

$= \left[-\frac{x^3}{3} + \frac{x^2}{2} + 6x \right]_{-2}^{3}$

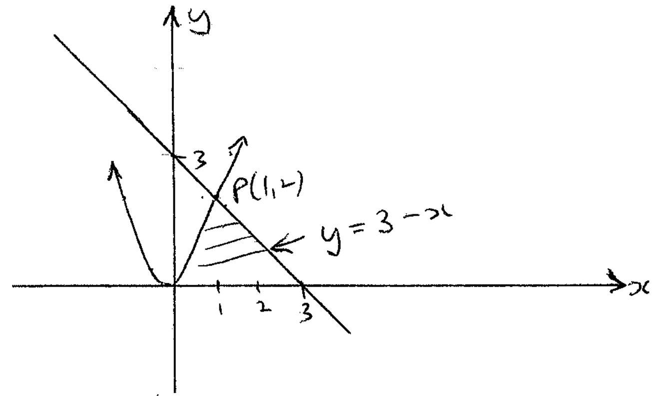
$= -9 + \frac{9}{2} + 18 - \left(+\frac{8}{3} + 2 - 12\right)$

$= 13\frac{1}{2} - \left(-7\frac{1}{3}\right)$

(4) $= 20\frac{5}{6}\ u^2$

$= 20.83\ u^2$ (to 2 dec. pl.)

d)



$V = \pi \int_0^1 (2x^2)^2\,dx + \pi \int_1^3 (3-x)^2\,dx$

~~$= \pi [4x^4]_0^1 + \pi \left[-\frac{1}{3}(3-x)^3\right]_1^3$~~

$= \pi \int_0^1 4x^4\,dx + \pi \int_1^3 (3-x)^2\,dx$

$= \pi \left[\frac{4}{5}x^5\right]_0^1 + \pi \left[-\frac{1}{3}(3-x)^3\right]_1^3$

$= \frac{4}{5}\pi + \pi\left(0 + \frac{1}{3} \times 2^3\right)$

(4) $= \left(\frac{4}{5} + \frac{8}{3}\right)\pi\ u^3$

$= \frac{12+40}{15}\pi\ u^3$

$= 4\frac{7}{15}\pi\ u^3$.

Q6

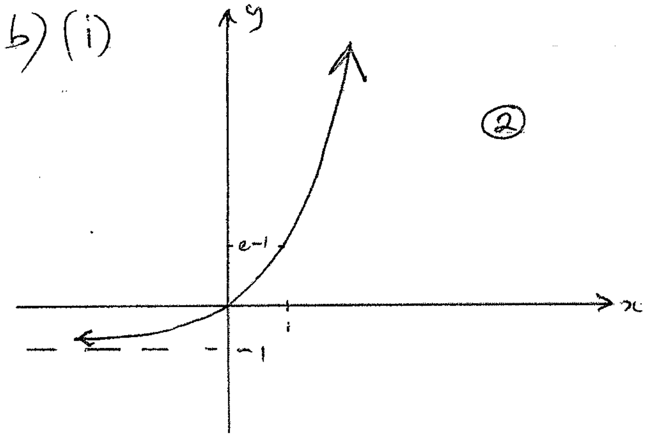
a) (i) $e^{-3} \doteqdot 0.04979$ (to 4 sig. figs) (1)

(ii) $\dfrac{2^{2x+1}}{2^{2x+2}} = 2^{2x+1-2x-2}$

$= 2^{-1}$

$= \frac{1}{2}$.

b) (i)



(2)

(ii) $y > -1$ (1)

c) $\dfrac{d}{dx}\, x e^{2x} = x.2e^{2x} + e^{2x}$

$= e^{2x}(2x+1)$ (2)

d) (i) $y = x - e^{x}$

$\therefore y' = 1 - e^{x}$ (2)

$y'' = -e^{x}$

(ii) For st. pts, $y' = 0$

ie $e^{x} = 1$

$x = 0$, $\therefore$ st. pt $(0,-1)$

(3) when $x = 0$, $y'' = -e^{0} = -1 < 0$

$\therefore$ Max t. pt at $(0,-1)$

e) $y = e^{2x+1}$

$\therefore y' = 2e^{2x+1}$

at $x = 1$, $y' = 2e^{3}$ & $y = e^{3}$

$\therefore$ slope of normal is $\dfrac{-1}{2e^{3}}$

$\therefore$ eqn of normal is

$y - e^{3} = \dfrac{-1}{2e^{3}}(x-1)$

ie $2e^{3}y - 2e^{6} = -x + 1$.

ie $x + 2e^{3}y - 2e^{6} - 1 = 0$.