Name:________________________

Class:________________________

SYDNEY TECHNICAL HIGH SCHOOL

MATHEMATICS

HSC ASSESSMENT TASK 3

JUNE 2006

Time Allowed: 70 minutes

Instructions

- Write your name and class at the top of this page, and at the top of each answer sheet.
- At the end of the examination, this examination paper must be attached to the front of your answers.
- Attempt all questions.
- All necessary working must be shown. Marks will be deducted for careless or badly arranged work.
- Marks indicated are a guide only and may be varied if necessary.
- A table of standard integrals is supplied.

Q1	Q2	Q3	Q4	Q5	Total
11	12	12	12	10	57

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1,\ x \neq 0,\ \text{if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1 (11 Marks)

a) Find the exact value of $\sin\frac{2\pi}{3}$ 1

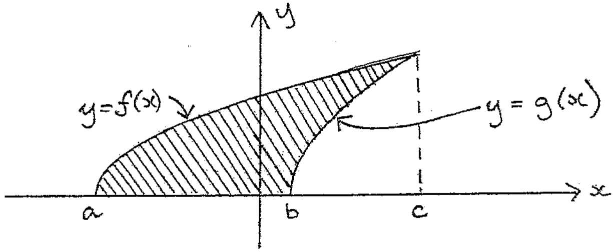
b) Find $\cos 1.5^c$ correct to 3 decimal places. 1

c) Express 2.25π radians in degrees. 1

d) Find $\lim\limits_{x \to 0} \frac{\sin x}{2x}$ 1

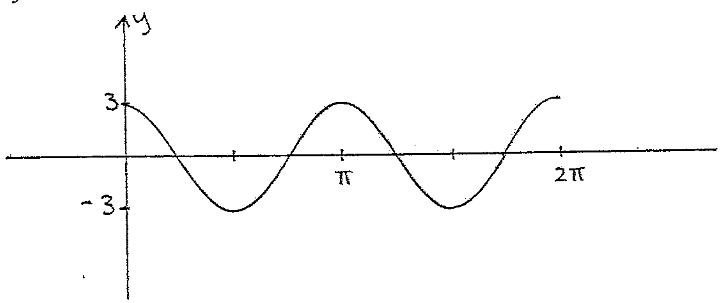
e) Express the shaded area below as either the <u>sum</u> or <u>difference</u> of two integrals

(correct notation must be used) 2



f) The curve below has been drawn from $x = 0$ to $x = 2\pi$. The curve has equation in the form

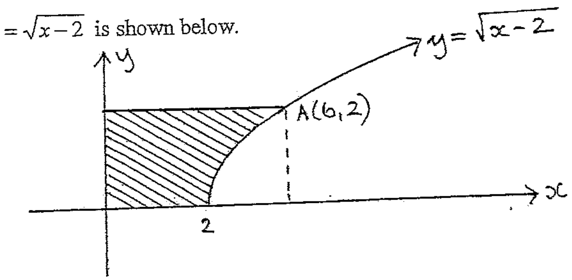
$y = a\cos bx$. Find a and b. 2



g) Draw a neat sketch of $y = f(x)$ in the domain $a \le x \le b$ given that $f'(x) > 0$ and $f''(x) > 0$ in the domain and $f(a) = 0$ 3

Question 2 (start a new page) (12 marks)

a) The curve $y = \sqrt{x-2}$ is shown below. 3

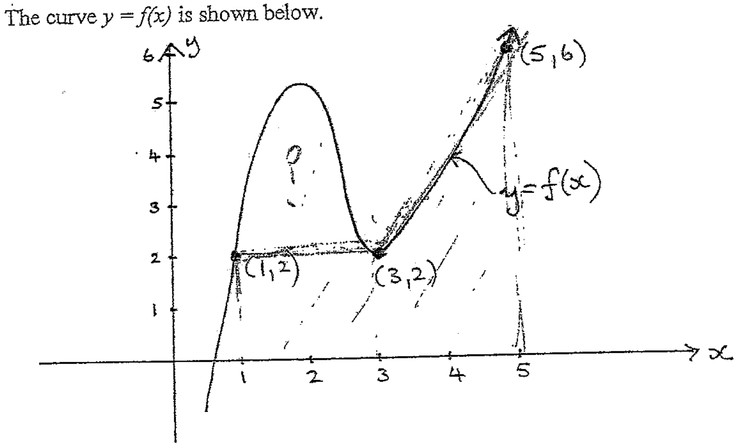


A(6,2) lies on the curve.

Find the shaded area.

b) i) Find the approximate area enclosed by the curve $y = f(x)$, the x axis and the lines $x = 1$ and $x = 5$, by using 3 function values and the Trapezoidal Rule. 2

The curve $y = f(x)$ is shown below.



ii) Is your answer in part i) an under or over estimate of the exact area. 1

Explain your answer.

c) The curve $y=\sqrt{\cos \pi x}$ from $x=0$ to $x=\frac{1}{2}$ is rotated around the x axis. What is the volume of the solid of revolution generated? 3

d) Evaluate $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sec^2 x \, dx$ 3

Question 3 (start a new page) (12 marks)

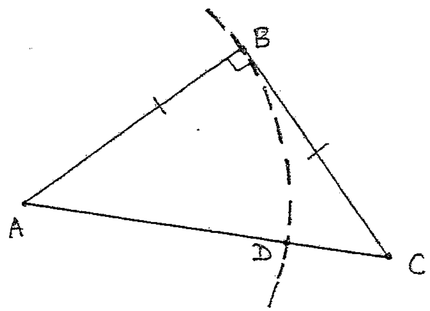
a) Solve $2\cos^2 x + 3\cos x - 2 = 0$ for x, if $0 \le x \le 2\pi$ 3

b) i) Find $\frac{d}{dx}(\sin^2 x)$ 2

ii) Find $\frac{d}{dx}(\sin x.\cos 2x)$ 2

iii) Find $\int \sin(2x+1)dx$ 2

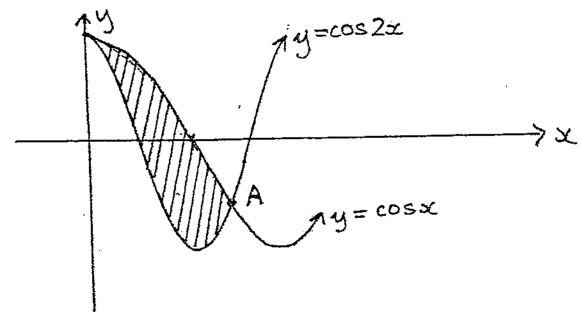
c) ABC is an isosceles right angled triangle. AB=BC=4cm. An arc, centre A and radius 4cm is drawn to cut the side AC at D.



Show the area of the portion BDC is $2(4-\pi)cm^2$ 3

Question 4 (start a new page) (12 marks)

a) The diagram shows parts of the curves $y=\cos x$ and $y=\cos 2x$ 4



The coordinates of A are $(\frac{2\pi}{3}, -\frac{1}{2})$

Show that the shaded area is $\frac{3\sqrt{3}}{4} unit^2$.

b) Prove that the curve $y = x + 2\cos x$ has a maximum turning point at $x=\frac{\pi}{6}$ in the domain $0 \le x \le \frac{\pi}{2}$ (do not sketch the curve). 4

c) i) Sketch the parabola $y = x^2 - x$ indicating where it cuts the x axis. 1

ii) The area enclosed by the parabola $y = x^2 - x$ and the x axis is rotated around the x axis. Find the volume of the solid generated. 3

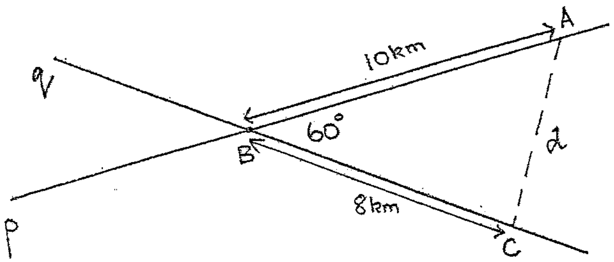
Question 5 (start a new page) **10 marks**

a) A couple borrow \$320,000 at 6% p.a. The interest on the loan is compounded monthly on the balance owing. The loan is to be repaid in equal monthly instalments over 30 years. Let the monthly instalment be \$M and the amount owing after n months be $\$A_n$.

i) Find an expression for $\$A_1$, the amount owing after one month. 1

ii) Find the monthly instalment if the loan is to be fully repaid in 30 years. 4

b) Two streets p and q intersect at B at an angle pf $60°$. Andrew is at A, 10 km from B and walks towards B at 5km/h. Con is at C, 8km from B and walks towards B at 6km/h.



After t hours Andrew has walked 5t km towards B and Con has walked 6t km towards B.

i) Use the cosine rule to show the distance d between Andrew and Con can be given by $d^2 = 31t^2 - 96t + 84$ 2

ii) Hence find how many hours (correct to 2 decimal places) until Andrew and Con are the least distance apart. 3

TEACHERS COPY - showing suggested marks

STHS 2UNIT HSC ASSESSMENT TASK 3 JUNE 2006

Question 1

a) $\sin\frac{2\pi}{3} = \sin(\pi - \frac{\pi}{3})$

$= \sin \pi/3$

$= \sqrt{3}/2$ 1

b) $\cos 1{\cdot}5 = 0{\cdot}071$ 1

c) $2{\cdot}25\pi = 405°$ 1 d) $1/2$ 1

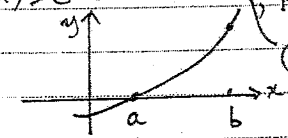
e) $A = \int_a^c f(x)\,dx - \int_b^c g(x)\,dx$ 1 1

f) $a=3$ $\quad \frac{2\pi}{b} = \pi$ $\quad \therefore b=2$ 2

$\therefore y = 3\cos 2x$ 1

g) $f'(x) > 0$ ∴ +ve gradient 1

$f''(x) > 0$ ∴ conc. ↑, pt$(a,0)$ 1 1



Question 2

a) $A_y = \int_0^2 (y^2+2)\,dy$ 1

$y = \sqrt{x-2}$

$\therefore y^2 = x-2$

$y^2 + 2 = x$ 1

$\therefore A_y = \left[\frac{y^3}{3} + 2y\right]_0^2$

$= \frac{8}{3} + 4$ 1

$= 6\frac{2}{3}$ unit^2

bi) $A = \frac{2}{2}\left[2 + 6 + 2(2)\right]$ 1

$= 12$ unit^2 1

ii) under estimate as area from $x=1$ to $x=3$ becomes approximated to 4 and area above line $y=2$ is not accounted for. 1

c) $V = \pi \int_0^{1/2} \left(\sqrt{\cos \pi x}\right)^2 dx$ 1

$= \pi \int_0^{1/2} \cos \pi x \, dx$

$= \pi \left[\frac{1}{\pi} \sin \pi x\right]_0^{1/2}$

$= \sin\frac{\pi}{2} - \sin 0$ 1

$= 1$ unit^3 1

d) $\int_{\pi/6}^{\pi/3} \sec^2 x \, dx$ 1

$= \left[\tan x\right]_{\pi/6}^{\pi/3}$

$= \tan\frac{\pi}{3} - \tan\frac{\pi}{6}$ 1

$= \sqrt{3} - \frac{1}{\sqrt{3}}$ 1

Question 3

a) $2\cos^2 x + 3\cos x - 2 = 0$

1 $(2\cos x - 1)(\cos x + 2) = 0$

$\cos x = \frac{1}{2}$ $\qquad \cos x = -2$ no solutio

1 $x = \frac{\pi}{3}, \frac{5\pi}{3}$ 1

b) i) $\frac{d}{dx}(\sin x)^2 = 2\cos x . \sin x$ 2

ii) $u = \sin x$ $\quad v = \cos 2x$

$u' = \cos x$ $\quad v' = -2\sin 2x$

2 $\frac{dy}{dx} = \cos x . \cos 2x - 2\sin x . \sin$

iii) $\int \sin(2x+1)\,dx$ 2

$= \frac{1}{2}\cos(2x+1) + c$

do not take off mk if no "C"

c) $A\hat{B}C = \frac{\pi}{4}$

Area BDC = Area ΔABC − sector ABD 1

$= \frac{1}{2}\times 4 \times 4 - \frac{1}{2}\times 4^2 \times \frac{\pi}{4}$

$= 8 - 2\pi$

Question 4

a) $A = \int_0^{2\pi/3} (\cos x - \cos 2x)\,dx$ (1)

$= \left[\sin x - \frac{1}{2}\sin 2x\right]_0^{2\pi/3}$ (1)

$= \sin\frac{2\pi}{3} - \frac{1}{2}\sin\frac{\pi}{3}$

$= \frac{\sqrt{3}}{2} + \frac{1}{2}\cdot\frac{\sqrt{3}}{2}$

(1) $= \frac{2\sqrt{3}+\sqrt{3}}{4}$

$= \frac{3\sqrt{3}}{4}$ units^2 (1)

b) $y = x + 2\cos x$

$\frac{dy}{dx} = 1 - 2\sin x$ — (1)

$\frac{d^2y}{dx^2} = -2\cos x$

st pt $\frac{dy}{dx} = 0$ $\quad 1 - 2\sin x = 0$ (1)

$\therefore \sin x = 1/2$

(1) — $x = \pi/6$ if $0 \le x \le \pi/2$

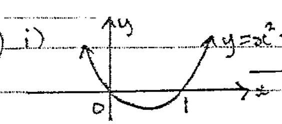
test max/min for $x = \pi/6$

$\frac{d^2y}{dx^2} = -2\cos\frac{\pi}{6} < 0$

$\therefore$ max — (1)

$\therefore$ max turning pt at $x = \frac{\pi}{6}$

c) i)



— (1)

ii) $V = \pi \int_0^1 (x^2 - x)^2\,dx$ (1)

$= \pi \int_0^1 x^4 - 2x^3 + x^2\,dx$

$= \pi\left[\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3}\right]_0^1$ — (1)

$= \frac{\pi}{...}$ unit^2

Question 5

a) \$320,000

6% pa $\Rightarrow$ ·5% p.m

30 yrs $\Rightarrow$ 360 months

i) $\$A_1 = 320{,}000\left(1 + \frac{\cdot 5}{100}\right)^1 - M$

$= 320{,}000\,(1.005)^1 - M$ (1)

ii) $\$A_2 = \left(320{,}000\,(1.005) - M\right)1.005 - M$

(1) $= 320{,}000\,(1.005)^2 - 1.005M - 1$

$\therefore$

$\$A_{360} = 320{,}000\,(1.005)^{360} - M(1 + 1.005 + \ldots + 1.005^{359})$

(1)

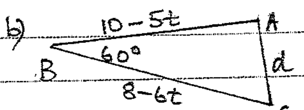
G.P. $a = 1$ $\quad r = 1.005$ $\quad n = 360$

$\$A_{360} = 0$ as loan repaid

$M\left[\frac{1(1.005^{360} - 1)}{1.005 - 1}\right] = 320{,}000\,(1.005)^{360}$ — (1)

$M = \$1918.56$ — (1)

b)



(1)

i) $d^2 = (10-5t)^2 + (8-6t)^2 - 2(10-5t)(8-6t)\cos 60$

$= 100 - 100t + 25t^2 + 64 - 96t + 36t^2$

$- (80 - 100t + 30t^2)$

$\therefore d^2 = 84 - 96t + 31t^2$ (1)

ii) $\frac{d(d^2)}{dt} = -96 + 62t$ — (1)

$\frac{d^2(d^2)}{dt^2} = 62 > 0$ $\quad \therefore$ minimum

st pt $-96 + 62t = 0$ $\quad \therefore t = 1.548$ hrs

(1) $\quad \therefore t = 1.55$ hr (2 dec pl)

(1)

Question 1

a) $\sin\frac{2\pi}{3} = \sin\left(\pi - \frac{\pi}{3}\right)$

$= \sin\pi/3$

$= \sqrt{3}/2$

b) $\cos 1.5 = 0.071$

c) $2.25\pi = 405°$ $\qquad$ d) $1/2$

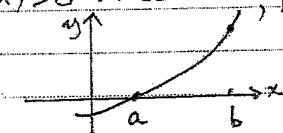
e) $A = \int_a^c f(x)\,dx - \int_b^c g(x)\,dx$

f) $a = 3$ $\quad \frac{2\pi}{b} = \pi$ $\quad \therefore b = 2$

$\therefore y = 3\cos 2x$

g) $f'(x) > 0$ $\therefore$ +ve gradient

$f''(x) > 0$ $\therefore$ conc. ↑, pt$(a, 0)$



c) $V = \pi \int_0^{1/2} \left(\sqrt{\cos\pi x}\right)^2 dx$

$= \pi \int_0^{1/2} \cos\pi x\,dx$

$= \pi\left[\frac{1}{\pi}\sin\pi x\right]_0^{1/2}$

$= \sin\frac{\pi}{2} - \sin 0$

$= 1$ units^3

d) $\int_{\pi/6}^{\pi/3} \sec^2 x\,dx$

$= \left[\tan x\right]_{\pi/6}^{\pi/3}$

$= \tan\frac{\pi}{3} - \tan\frac{\pi}{6}$

$= \sqrt{3} - \frac{1}{\sqrt{3}}$

Question 2

a) $A_y = \int_0^2 (y^2 + 2)\,dy$

$y = \sqrt{x-2}$

$\therefore y^2 = x - 2$

$y^2 + 2 = x$

$\therefore A_y = \left[\frac{y^3}{3} + 2y\right]_0^2$

$= \frac{8}{3} + 4$

$= 6\frac{2}{3}$ unit^2

b i) $A = \frac{2}{2}\left[2 + 6 + 2(2)\right]$

$= 12$ unit^2

ii) under estimate as area from $x = 1$ to $x = 3$ becomes approximated to 4 and area above line $y = 2$ is not accounted for.

Question 3

a) $2\cos^2 x + 3\cos x - 2 = 0$

$(2\cos x - 1)(\cos x + 2) = 0$

$\cos x = \frac{1}{2}$ $\qquad$ $\cos x = -2$ no solutio

$x = \frac{\pi}{3}, \frac{5\pi}{3}$

b) i) $\frac{d}{dx}(\sin x)^2 = 2\cos x.\sin x$

ii) $u = \sin x$ $\qquad$ $v = \cos 2x$

$u' = \cos x$ $\qquad$ $v' = -2\sin 2x$

$\frac{dy}{dx} = \cos x.\cos 2x - 2\sin x\sin...$

iii) $\int \sin(2x+1)\,dx$

$= -\frac{1}{2}\cos(2x+1) + C$

c) $\hat{ABC} = \frac{\pi}{4}$

Area BDC = Area $\triangle$ABC − sector ABD

$= \frac{1}{2}\times 4\times 4 - \frac{1}{2}\times 4^2\times\frac{\pi}{4}$

$= 8 - 2\pi$

Question 4

a) $A = \int_0^{2\pi/3} (\cos x - \cos 2x)\, dx$

$= \left[\sin x - \frac{1}{2} \sin 2x \right]_0^{2\pi/3}$

$= \sin \frac{2\pi}{3} - \frac{1}{2} \sin \frac{\pi}{3}$

$= \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{\sqrt{3}}{2}$

$= \frac{2\sqrt{3} + \sqrt{3}}{4}$

$= \frac{3\sqrt{3}}{4}$ units^2

b) $y = x + 2\cos x$

$\frac{dy}{dx} = 1 - 2\sin x$

$\frac{d^2y}{dx^2} = -2\cos x$

st pt $\frac{dy}{dx} = 0$ $\quad 1 - 2\sin x = 0$

$\therefore \sin x = 1/2$

$x = \pi/6$ if $0 \le x \le \pi/2$

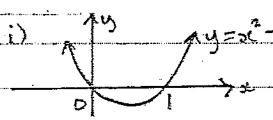
test max/min for $x = \pi/6$

$\frac{d^2y}{dx^2} = -2\cos\frac{\pi}{6} < 0$

$\therefore$ max

$\therefore$ max turning pt at $x = \frac{\pi}{6}$

c) i)



ii) $V = \pi \int_0^1 (x^2 - x)^2\, dx$

$= \pi \int_0^1 x^4 - 2x^3 + x^2\, dx$

$= \pi \left[\frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3} \right]_0^1$

$= \frac{\pi}{...}$ unit^2

Question 5

a) \$320,000

6% pa $\Rightarrow$ .5% p.m

30 yrs $\Rightarrow$ 360 months

i) $\$A_1 = 320{,}000\left(1 + \frac{.5}{100}\right)^1 - M$

$= 320{,}000(1.005)^1 - M$

ii) $\$A_2 = \left(320{,}000(1.005) - M\right)1.005 - M$

$= 320{,}000(1.005)^2 - 1.005M - 1$

$\therefore$

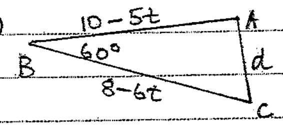
$\$A_{360} = 320{,}000(1.005)^{360} - M(1 + 1.00 + \dots + 1.005^{359})$

G.P. $a = 1$ $\quad r = 1.005$ $\quad n = 360$

$\$A_{360} = 0$ as loan repaid

$M\left[\frac{1(1.005^{360} - 1)}{1.005 - 1}\right] = 320{,}000(1.005)^{360}$ $\quad$ $M = \$1918.56$

b)



i) $d^2 = (10-5t)^2 + (8-6t)^2 - 2(10-5t)(8-6t)\cos 60°$

$= 100 - 100t + 25t^2 + 64 - 96t + 36t^2$

$- (80 - 100t + 30t^2)$

$\therefore d^2 = 84 - 96t + 31t^2$

ii) $\frac{d(d^2)}{dt} = -96 + 62t$

$\frac{d^2(d^2)}{dt^2} = 62 > 0$ $\therefore$ minimum

st pt $-96 + 62t = 0$ $\quad \therefore t = 1.548$ hrs

$\therefore t = 1.55$ hr (2 dec pl)