Name: .... Maths Class: ....

SYDNEY TECHNICAL HIGH SCHOOL

Year 11

Mathematics Extension 1

Preliminary HSC Course

Assessment 2

July, 2015

Time allowed: 90 minutes

General Instructions:

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A set of Standard Integrals is provided at the rear of this Question Booklet, and may be removed at any time.

Section 1 Multiple Choice
Questions 1-5
5 Marks

Section II Questions 6-11
60 Marks

SECTION I

Choose the most appropriate answer from the choices, and fill in the circle on the multiple-choice answer sheet provided in your answer booklet

1 The gradient of the tangent to the curve $y = 5x - x^3 - 2$ at the point (2, 0) is

A. 6 B. 2 C. -2 D. -7

2 $\lim\limits_{x \to 2} \frac{x^2-6x+8}{x-2} =$

A. 0 B. -2 C. 6 D. ∞

3. The acute angle between the lines $x = 3$ and $3x - 2y - 5 = 0$, to the nearest degree, is:

A. 56^o B. 124^o C. 34^o D. 144^o

4 If $y = 5t$ and $x = t^2$ then $\frac{dy}{dx} =$

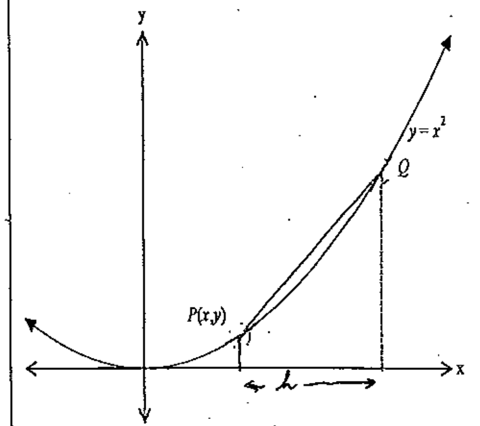
A. 5 B. 2t C. $\frac{5}{2t}$ D. $\frac{1}{2t}$

5 In the diagram at right, $f(x) = x^2$

P is the point (x, y) on the curve

Q is another point on the curve which has an x value of $x + h$

The slope of the secant PQ is given by



A. $\frac{(x+h)^2 - x^2}{x+h}$

B. $\frac{(x+h)^2}{h}$

C. $\frac{h^2 - x^2}{h}$

D. $2x + h$

SECTION II

Start each new question on a new page

QUESTION 6: (10 Marks)

Marks

(a) If $0^o \le \theta \le 90^o$ and $cos\theta = x$ find $sin2\theta$ in terms of x **2**

(b) Find the distance between the lines $2x + 3y = 6$ and $2x + 3y + 4 = 0$ as a simplified surd. **3**

(c) The interval joining the points A (-1, 5) to B(2, -1) is divided by the point M <u>externally</u> in the ratio 3:2. **2**

Find the co-ordinates of M.

(d) Show that $\tan x = \frac{1-cos2x}{\sin 2x}$ **3**

QUESTION 7: (10 Marks) *(Start a New Page)*

Marks

(a) Find derivatives of

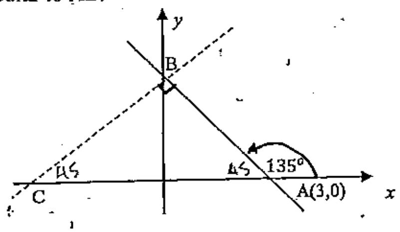
(i) $y = \frac{3}{x}$ **1**

(ii) $y = 5\sqrt{x}$ **1**

(iii) $y = (2x^3 - 1)(x^2 + 1)^3$ (give the answer in fully factored form) **2**

(b) A is the point (3, 0) and B is on the y axis.
AB makes an angle of 135^o with the positive x-axis.

BC is drawn perpendicular to AB.



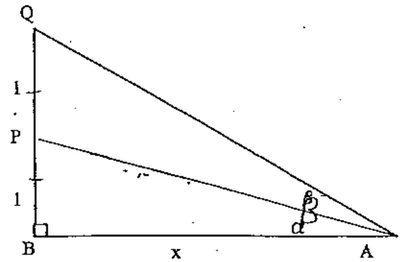
(i) Find the equation of the line BC **2**

(ii) You are further given that C lies on the (negative) x-axis. Find the area of ΔABC **1**

(c) In the diagram at right, ΔABQ is right-angled at B. **3**

$\angle QAP = \beta$ and $\angle PAB = \alpha$
PQ = PB = 1 unit
AB = x units

Prove that $\tan \beta = \frac{x}{x^2+2}$



QUESTION 8: (10 Marks) *(Start a New Page)*

Marks

(a) Show that the gradient of the tangent to the curve $y = \frac{x^3}{1+x^2}$ is always positive except at the origin. **2**

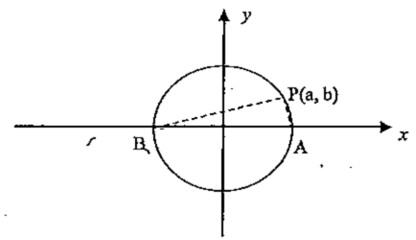
(b) (i) Find the exact value of $\tan 15^o$ in simplified form **2**

(ii) Hence find the value of $\cot 15^o + \tan 15^o$ **1**

(c) Solve the equation $2sin^2x + cosx - 2 = 0$, for $0^o \le x \le 360^o$ **3**

(d) For the circle $x^2 + y^2 = 16$, A and B are the points where the graph cuts the x-axis. **2**

P (a, b) is a point on the circle



(i) Find an expression for the gradient of PA

(ii) Hence prove that $\angle$ APB $= 90^o$

QUESTION 9: (10 Marks) *(Start a New Page)*

Marks

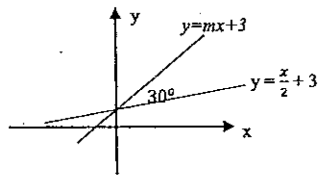
(a) (i) Find the value of $f'(8)$ if $f(x) = \frac{2}{\sqrt{x-4}}$ **2**

(ii) Hence, find the equation of the normal to the curve $y = f(x)$ at the point on it where $x = 8$ **2**

(b) Using the method of Differentiation from First Principles, find $\frac{dy}{dx}$ if **3**

$$y = x^2 + x$$

(c) The angle between the 2 lines shown below is 30^o. **3**



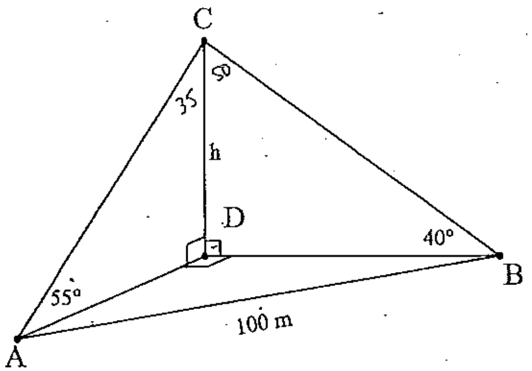
Show that m has the value $\frac{18+5\sqrt{3}}{11}$

QUESTION 10: (10 Marks) *(Start a New Page)*

			Marks
(a)	(i)	Express $\cos\theta - \sin\theta$ in the form $R\cos(\theta + \alpha)$, where $R > 0$ and $0^o \le \alpha \le 90^o$	2
	(ii)	Hence, or otherwise, solve $\cos\theta - \sin\theta = 1$ for $0^o \le \theta \le 360^o$	2
	(iii)	What is the maximum value that $\cos\theta - \sin\theta$ can take be? ***Explain your answer.***	1

(b) Two men, A and B, are standing on level ground at points 100 metres apart.

From A, who is due south of a perpendicular tower, the angle of elevation to the top of the tower is 55^o. B, who is due east of the tower, notes that the tower has an angle of elevation of 40^o



(i) If h is the height of the towers, prove that **3**

$$h^2 = \frac{10\,000\tan^2 40^o \tan^2 55^o)}{\tan^2 40^o + \tan^2 55^o}$$

(ii) Find the height of the tower, to the nearest metre. **2**

QUESTION 11: (10 Marks) *(Start a New Page)*

			Marks
(a)		Shade the area given by the relationship $\lvert x\rvert \le \lvert y\rvert$	3
(b)	(i)	Prove that $\cos 3A = 4\cos^3 A - 3\cos A$	2
	(ii)	Using the above, solve $4\cos^3 A - 3\cos A = 1$ for $0^o \le A \le 360^o$	2
(b)		AB is a diameter of the circle $(x-2)^2 + (y-2)^2 = 4$, where A is the closest point on the circle to the Origin (0, 0). Find an <u>unsimplified</u> expression for the x-co-ordinate of B.	3

END OF EXAMINATION PAPER

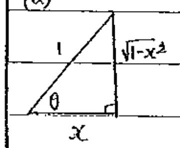
SECTION I

1	D
2	B
3	C
4	C
5	D

SECTION II

QUESTION 6

(a)



$\sin 2\theta = 2\sin\theta\cos\theta$

$= 2\sqrt{1-x^2}(x)$

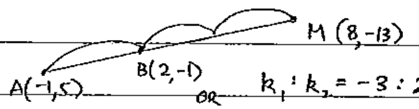
$= 2x\sqrt{1-x^2}$

(b) A point on $2x+3y=6$ is $(3,0)$

$P = \left|\dfrac{2\times 3 + 3\times 0 + 4}{\sqrt{2^2+3^2}}\right|$

$= \dfrac{10}{\sqrt{13}}$

(c)



OR $k_1 : k_2 = -3 : 2$

$M = \left(\dfrac{-3\times 2 + 2\times -1}{-1}, \dfrac{-3\times -1 + 2\times 5}{-1}\right)$

$= (8,-13)$

(d) $RHS = \dfrac{1-\cos 2x}{\sin 2x}$

$= \dfrac{1-(1-2\sin^2 x)}{2\sin x\cos x}$

$= \dfrac{2\sin^2 x}{2\sin x\cos x}$

$= \dfrac{\sin x}{\cos x}$

$= \tan x$

$= LHS$

QUESTION 7

(a) i. $y = \dfrac{3}{x}$

$\dfrac{dy}{dx} = -3x^{-2}$

$= -\dfrac{3}{x^2}$

ii. $y = 5x^{\frac{1}{2}}$

$\dfrac{dy}{dx} = \dfrac{5}{2}x^{-\frac{1}{2}}$

$= \dfrac{5}{2\sqrt{x}}$

iii. $y = (2x^3-1)(x^2+1)^3$

$u = 2x^3-1 \qquad v = (x^2+1)^3$

$u' = 6x^2 \qquad v' = 3(x^2+1)^2 . 2x$

$= 6x(x^2+1)^2$

$\dfrac{dy}{dx} = 6x^2(x^2+1)^3 + (2x^3-1)6x(x^2+1)^2$

$= 6x(x^2+1)^2\left[x(x^2+1)+(2x^3-1)\right]$

$= 6x(x^2+1)^2(3x^3+x-1)$

(b) (i) $m_{BC} = 1 \qquad B(0,3)$

$y-3 = 1(x-0)$

$y = x+3$

(ii) $AC = 6$ units $\qquad OB = 3$ units

$A_{\triangle ABC} = \frac{1}{2}\times 6\times 3$

$= 9\ u^2$

(c) $\tan\beta = \tan[(\alpha+\beta)-\alpha]$

$= \dfrac{\tan(\alpha+\beta) - \tan\alpha}{1+\tan(\alpha+\beta)\tan\alpha}$

$= \dfrac{\frac{2}{x} - \frac{1}{x}}{1+\frac{2}{x^2}}$

$= \dfrac{\frac{1}{x}}{\frac{x^2+2}{x^2}}$

$= \dfrac{1}{x}\times\dfrac{x^2}{x^2+2}$

$= \dfrac{x}{x^2+2}$

QUESTION 8

(a)

$u = x^3 \qquad v = 1+x^2$

$u' = 3x^2 \qquad v' = 2x$

$\dfrac{dy}{dx} = \dfrac{3x^2(1+x^2) - x^3\times 2x}{(1+x^2)^2}$

$= \dfrac{3x^2+x^4}{(1+x^2)^2}$

> 0 since both numerator and denominator are positive for all x, $x\neq 0$

At $x=0$, $\dfrac{dy}{dx}=0$.

(b) i. $\tan 15° = \tan(45°-30°)$

$= \dfrac{\tan 45° - \tan 30°}{1+\tan 45°\tan 30°}$

$= \dfrac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}$

$= \dfrac{\sqrt{3}-1}{\sqrt{3}+1}$

$= 2-\sqrt{3}$

ii. $\cot 15° + \tan 15°$

$= \dfrac{1}{2-\sqrt{3}} + 2-\sqrt{3}$

$= \dfrac{2+\sqrt{3}}{1} + 2-\sqrt{3}$

$= 4$

(c) $2\sin^2 x + \cos x - 2 = 0$

$2(1-\cos^2 x) + \cos x - 2 = 0$

$2 - 2\cos^2 x + \cos x - 2 = 0$

$2\cos^2 x - \cos x = 0$

$\cos x(2\cos x - 1) = 0$

$\cos x = 0$ or $\cos x = \frac{1}{2}$

$x = 90°, 270° \qquad x = 60°, 300°$

(d) $A(4,0) \quad B(-4,0)$

i. $m_{PA} = \dfrac{b}{a-4}$

ii. $m_{PB} = \dfrac{b}{a+4}$

$m_{PA}\times m_{PB} = \dfrac{b^2}{a^2-16}$

and since $a^2+b^2=16$

$a^2-16 = -b^2$

$\therefore m_{PA}\times m_{PB} = \dfrac{b^2}{-b^2}$

$= -1$

$\therefore$ PA is perpendicular to PB

QUESTION 9

(a) i. $f(x) = \dfrac{2}{\sqrt{x-4}}$

$= 2(x-4)^{-\frac{1}{2}}$

$f'(x) = 2\times -\frac{1}{2}(x-4)^{-\frac{3}{2}}$

$= \dfrac{-1}{(x-4)^{3/2}}$

$f'(8) = \dfrac{-1}{4^{3/2}}$

$= -\dfrac{1}{8}$

ii. $m_{normal} = 8 \qquad (8, 1)$

$y - 1 = 8(x - 8)$

$\therefore y = 8x - 63$

(b) $\dfrac{dy}{dx} = \lim_{h \to 0} \dfrac{f(x+h) - f(x)}{h}$

$= \lim_{h \to 0} \dfrac{(x+h)^2 + (x+h) - (x^2 + x)}{h}$

$= \lim_{h \to 0} \dfrac{x^2 + 2xh + h^2 + x + h - x^2 - x}{h}$

$= \lim_{h \to 0} \dfrac{2xh + h^2 + h}{h}$

$= \lim_{h \to 0} 2x + h + 1$

$= 2x + 1$

(c) $m_1 = m$ and $m_2 = \frac{1}{2}$

$\tan\theta = \left| \dfrac{m_1 - m_2}{1 + m_1 m_2} \right|$

$\dfrac{1}{\sqrt{3}} = \dfrac{m - \frac{1}{2}}{1 + \frac{1}{2}m}$

$\sqrt{3}m - \dfrac{\sqrt{3}}{2} = 1 + \dfrac{1}{2}m$

$m(2\sqrt{3} - 1) = 2 + \sqrt{3}$

$m = \dfrac{2 + \sqrt{3}}{2\sqrt{3} - 1}$

$= \dfrac{8 + 5\sqrt{3}}{11}$

QUESTION 10

(a) i. $R = \sqrt{2}$

Using $\cos(\theta + \alpha) = \cos\theta\cos\alpha - \sin\theta\sin\alpha$

$\cos\theta - \sin\theta = \sqrt{2}\left(\frac{1}{\sqrt{2}}\cos\alpha - \frac{1}{\sqrt{2}}\sin\alpha\right)$

$\cos\alpha = \frac{1}{\sqrt{2}}$

$\alpha = 45°$

$\therefore \cos\theta - \sin\theta = \sqrt{2}\cos(\theta + 45°)$

ii. $\cos\theta - \sin\theta = 1$

$\sqrt{2}\cos(\theta + 45°) = 1$

$\cos(\theta + 45°) = \frac{1}{\sqrt{2}}$

$\therefore \theta + 45° = 45°$ or $315°$ or $405°$

$\therefore \theta = 0°, 270°, 360°$

iii. $\cos A \le 1$ for all A

$\therefore \cos\theta - \sin\theta \le \sqrt{2}$

(b) i. let $AD = x$ and $BD = y$

$\frac{h}{x} = \tan 55°$ & $\frac{h}{y} = \tan 40°$

$x = \dfrac{h}{\tan 55°} \qquad y = \dfrac{h}{\tan 40°}$

since $x^2 + y^2 = 100^2$

$\dfrac{h^2}{\tan^2 55°} + \dfrac{h^2}{\tan^2 40°} = 100^2$

$\dfrac{h^2\tan^2 40° + h^2\tan^2 55°}{\tan^2 55° \tan^2 40} = 100^2$

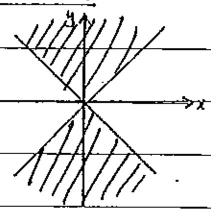
$h^2(\tan^2 40° + \tan^2 55°) = 10\,000 \tan^2 55 \; ta$

$\therefore h^2 = \dfrac{10\,000 \tan^2 55° \tan^2 40°}{\tan^2 40° + \tan^2 55°}$

ii. $h = 72\text{m}$

QUESTION 11

(a)



(b) i. Prove $\cos 3A = 4\cos^3 A - 3\cos A$

$LHS = \cos(2A + A)$

$= \cos 2A \cos A - \sin 2A \sin A$

$= (2\cos^2 A - 1)\cos A - 2\sin A \cos A \sin A$

$= 2\cos^3 A - \cos A - 2\sin^2 A \cos A$

$= 2\cos^3 A - \cos A - 2(1 - \cos^2 A)\cos A$

$= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A$

$= 4\cos^3 A - 3\cos A$

$= RHS$

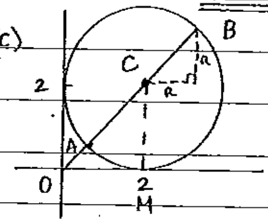
ii. $4\cos^3 A - 3\cos A = 1 \qquad 0° \le A \le 360°$

$\cos 3A = 1 \qquad 0° \le 3A \le 1080°$

$3A = 0°, 360°, 720°, 1080°$

$\therefore A = 0°, 120°, 240°, 360°$

(c)



since $BC = 2$

then $OB = 2\sqrt{2} + 2$.

O, A, C and B are in a straight line

$OC^2 = OM^2 + MC^2$

$= 4 + 4$

$= 8$

$\therefore OC = 2\sqrt{2}$

For B:

$x^2 + x^2 = (2\sqrt{2} + 2)^2$

$2x^2 = 12 + 8\sqrt{2}$

$x^2 = 6 + 4\sqrt{2}$

$x = \sqrt{6 + 4\sqrt{2}}$

OR $a^2 + a^2 = 4$

$2a^2 = 4$

$a^2 = 2$

$a = \sqrt{2}$

$\therefore x$ value of B is $2 + \sqrt{2}$