SAINT IGNATIUS' COLLEGE

Trial Higher School Certificate

2002

MATHEMATICS

EXTENSION 2

9:00am – 12:05 pm
Thursday 29th August 2002

General Instructions

- Reading time : 5 minutes
- Working time: 3 hours
- Write using blue or black pen
- *Write your name on each answer booklet*
- Board approved calculators may be used
- A table of standard integrals is provided

- Total Marks (120)
- Attempt Questions 1 – 8
- All questions are of equal value

Students are reminded that this is a trial examination only and cannot in any way guarantee the content or the format of the 2002 Mathematics Extension 2 Higher School Certificate examination

Question 1 (15 marks) Use a SEPARATE writing booklet. Marks

a) The complex number u is given by $(-1 + i\sqrt{3})$.

i) Show that $u^2 = 2\bar{u}$. 2

ii) Evaluate $|u|$ and $\arg u$. 2

iii) Show that u is a root of the equation $u^3 - 8 = 0$. 1

b) If $z = x + iy$ sketch, on separate axes, the locus of z satisfying

i) $\text{Re}(z) = |z|$. 2

ii) Both $\text{Im}(z) \geq 2$ and $|z - 1| \leq 3$. 3

c) Given that both c and d are real numbers, find their values such that 2

$$\frac{c}{1+i} - \frac{d}{1+2i} = 1.$$

d) The points P, Q, R and S on an Argand diagram represent the complex numbers a, b, c and d respectively. 3
If $a + c = b + d$ and $a - c = i(b - d)$, find what type of quadrilateral $PQRS$ is.

Question 2 (15 marks) Use a SEPARATE writing booklet. Marks

a) Sketch the following, showing all essential features.

(i) $y = \ln x^2$ 2

(ii) $\sin(x + y) = 1$ 2

(iii) $y = e^x - e^{-x}$ 2

b) (i) Draw (without using the Calculus) a neat sketch of the curve 2

$y = x^3 - c^2x$; where c is a positive constant.

Mark clearly any intercepts.

(ii) Use your graph in part (i) to draw neat sketches, on separate number planes, of:

(α) $y = \dfrac{1}{x^3 - c^2x}$ 2

(β) $y = \left|\dfrac{1}{x^3 - c^2x}\right|$ 2

(γ) $y^2 = \dfrac{1}{x^3 - c^2x}$ 3

Question 3 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) Find the following indefinite integrals.

(i) $\displaystyle\int 2^{2x}\,dx$ 2

(ii) $\displaystyle\int x\,e^{x}\,dx$ 2

(iii) $\displaystyle\int \frac{2x}{(x+1)(x+3)}\,dx$ 3

b) By using the substitution $u = t - 4$ evaluate $\displaystyle\int_{4}^{4\cdot5} \frac{dt}{(t-3)(5-t)}$. 3

c) (i) If $u_n = \displaystyle\int_0^{\frac{\pi}{2}} x^n .\sin x\ dx \quad , n \geq 2$, 3

prove that $u_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)\,u_{n-2}$

(ii) Hence evaluate $\displaystyle\int_0^{\frac{\pi}{2}} x^2 \sin x\ dx$. 2

Question 4 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) For the hyperbola $\frac{x^2}{20} - \frac{y^2}{5} = 1$, find

(i) the co-ordinates of the two foci, 2

(ii) the equations of the asymptotes 2

b) Explain why $\frac{x^2}{h-19} + \frac{y^2}{3-h} = 1$ cannot represent the equation of an ellipse. 2

c) Tangents to the ellipse with the equation $x^2 + 4y^2 = 4$ at the points $A\ (2\cos\theta,\ \sin\theta)$ and $B\ (2\cos\alpha,\ \sin\alpha)$ are at right angles to each other. Show that: $4\tan\theta.\tan\alpha = -1$. 3

d) A and B are variable points on the rectangular hyperbola $xy = c^2$

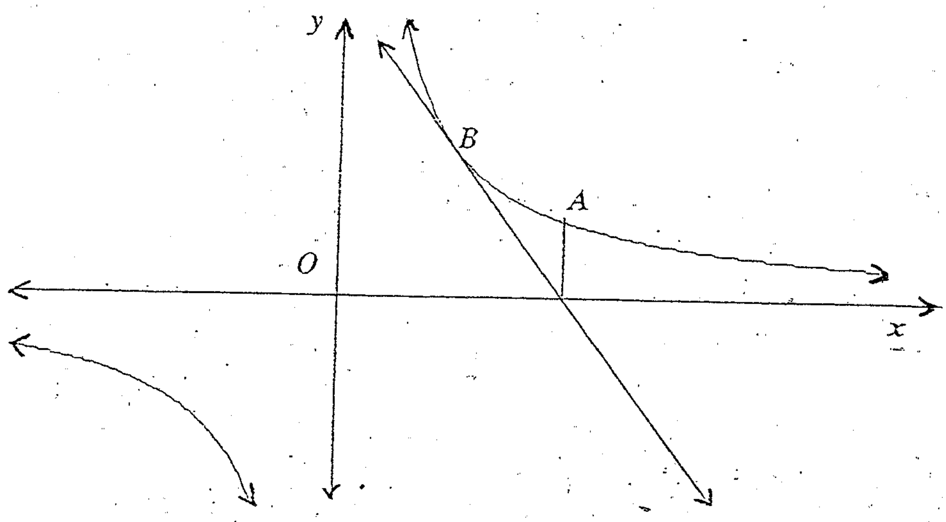


Diagram not to scale.

(i) The tangent at B passes through the foot of the ordinate of A. If A and B have parameters t_1 and t_2, show that $t_1 = 2t_2$ 4

(ii) Hence prove that the locus of the midpoint of AB is a rectangular hyperbola. 2

Question 5 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) Prove that both 1 and -1 are zeroes of multiplicity 2 of the polynomial

$$P(x) = x^6 - 3x^2 + 2.$$

Hence express $P(x)$ as a product of irreducible factors over the field of

(i) real numbers **4**

(ii) complex numbers. **1**

b) (i) Assuming the result $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$ and using the substitution $x = \cos\theta$ solve the equation $8x^3 - 6x + 1 = 0$. **3**

(ii) Hence prove that :

(α) $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9} = 0$ **2**

(β) $\sec\frac{2\pi}{9} + \sec\frac{4\pi}{9} + \sec\frac{8\pi}{9} = 6$. **2**

c) If α and $-\alpha$ are both roots of $x^3 + mx^2 + nx + h = 0$, show that $mn - h = 0$. **3**

Question 6 (15 marks) Use a SEPARATE writing booklet. **Marks**

a) The base of a solid is a right-angled triangle on the horizontal x–y plane; bounded by the lines $y = 0$, $x = 4$ and $y = x$. Vertical cross-sections of the solid, parallel to the y-axis, are semicircles with their diameter on the base of the solid as shown in the diagram below.
Find the volume of the solid. 5

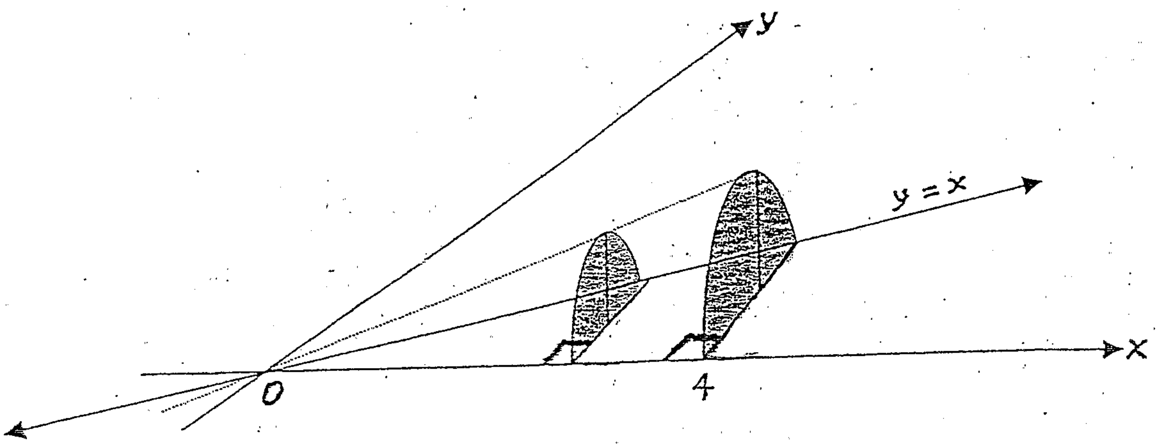


Diagram not to scale

(b) The area bounded by the line $y = 4 - 2x$, the x-axis and the y-axis, is rotated about the line $x = 4$.
By using the method of cylindrical shells find the volume formed. 5

c) Given that for a particular value of x that $\sin^{-1}x$, $\cos^{-1}x$ and $\sin^{-1}(1 - x)$ are acute: 5

(i) Show that: $\sin(\sin^{-1}x - \cos^{-1}x) = 2x^2 - 1$.

(ii) Solve the equation: $\sin^{-1}x - \cos^{-1}x = \sin^{-1}(1 - x)$.

Question 7 (15 marks) Use a SEPARATE writing booklet. Marks

a) A particle is attached to one end of a light string. The other end is fixed. The particle moves in a horizontal circle (below the fixed point) with a speed of 2 $m\ sec^{-1}$ and the string makes an angle of size $tan^{-1}(\frac{5}{12})$ with the vertical. **4**

Show that the length of the string is approximately 2·5 metres.

Take g as 10 $m\ sec^{-2}$.

b) A particle of unit mass moves in a straight line with variable acceleration $(\frac{16}{v} - v)\ m\ sec^{-2}$, where $v\ m\ sec^{-1}$ is the velocity at time t and $v > 0$, and x is the displacement.

If when $t = 0$, $x = 0$ and $v = 2m\ sec^{-1}$,

(i) Find an expression for the velocity of the particle at time t sec. **4**

(ii) Find the limiting velocity of the particle. **2**

(iii) Find the displacement of the particle when $v = 3m\ sec^{-1}$. **5**

Question 8 (15 marks) Use a SEPARATE writing booklet. Marks

a)

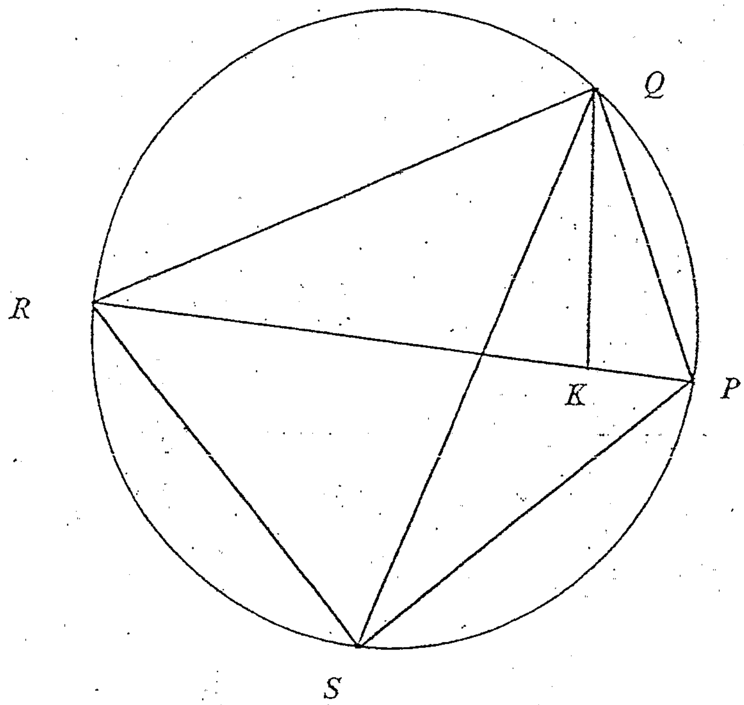


Diagram not to scale.

The above figure is a cyclic quadrilateral. K is the point on RP such that angle PQK is equal to angle SQR.
Let angle $SQR = x^0$ and

(i) Show that triangle PQS is similiar to triangle KQR and that the triangle PQK is similiar to triangle SQR. 6

(ii) Hence show that $PR \, . \, SQ = PQ \, . \, SR + PS \, . \, QR$. 2

b) Prove by Mathematical Induction that: 5

$$\sum_{r=1}^{n} \sin((2r-1)\theta) = \frac{\sin^2 n\theta}{\sin\theta}, \text{ where } n \text{ is a positive integer.}$$

c) For the following statement answer <u>true</u> or <u>false</u> giving a reason for your answer. 2

For $n = 1, 2, 3, \ldots \quad \int_0^1 \frac{dx}{1+x^n} \leq \int_0^1 \frac{dx}{1+x^{n+1}}$.

SAINT IGNATIUS' COLLEGE

Trial Higher School Certificate

2002

MATHEMATICS

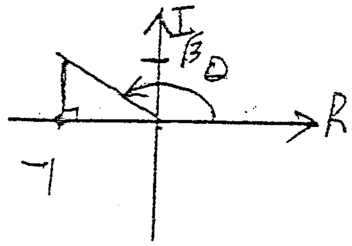
EXTENSION 2

Aids To Solutions

Question 1

(a) $u = -1 + i\sqrt{3}$

(ii) $\arg u = \tan^{-1}\frac{\sqrt{3}}{-1}$



$= \frac{2\pi}{3}$

$|u| = \sqrt{3+1} = 2.$

(i) $u^2 = \left[2\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]^2$

$= 4\left[\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right]$

$= 4\left(-\cos\left(\frac{\pi}{3}\right) \quad i\sin\left(\frac{\pi}{3}\right)\right)$

$= 4\left(-\frac{1}{2} - i\frac{\sqrt{3}}{2}\right)$

$= 2(-1 - i\sqrt{3})$

$= 2\overline{u}$

Note
This could have been done without mod-arg form.

(iii) $u^3 - 8 = 0$

$LHS = \left[2\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]^3 - 8$

$= 8\left[\cos 2\pi - i\sin 2\pi\right]^3 - 8$

$= 8(1 - i(0)) - 8$

$= 8 - 8 = 0$

$= RHS.$

(b) $R(z) = |z|$

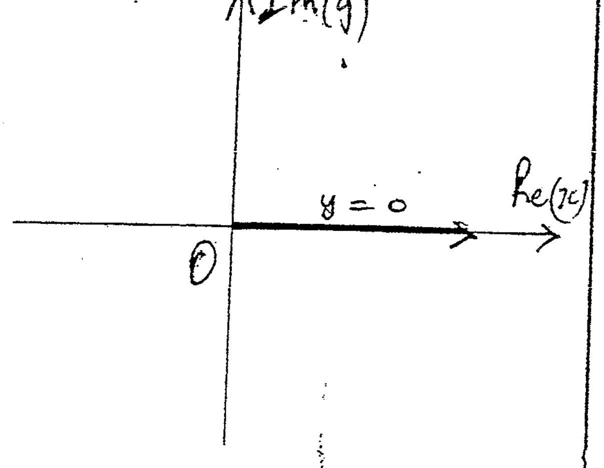
$x = \sqrt{x^2+y^2}$; Note $x \geq 0$

$x^2 = x^2 + y^2$

$y^2 = 0$

$y = 0.$

(∴ locus is the +ive x-axis and zero)



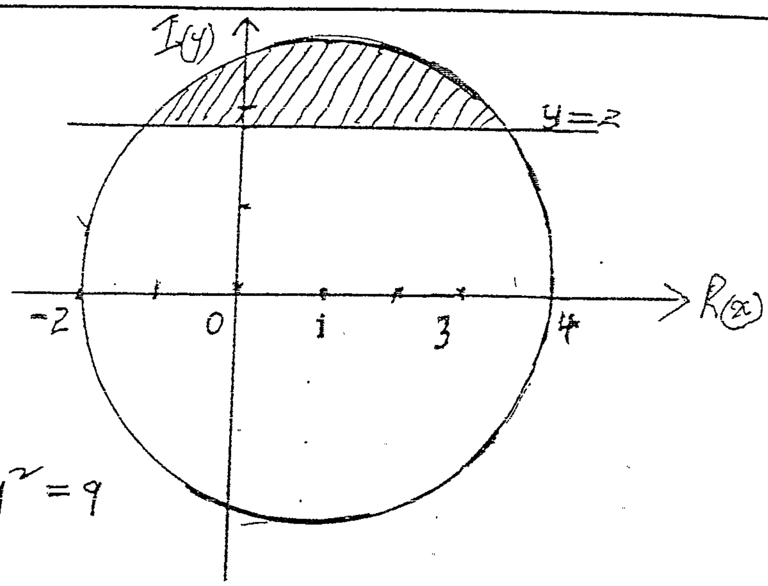
Question 1 (Continued)

b (ii) $Im(z) \geq 2$

ie $y \geq 2$

$|z-1| \leq 3$

circle centre $(1,0)$
radius $= 3$



$(x-1)^2+y^2=9$

(c) $\frac{c}{1+i} - \frac{d}{1+2i} = 1$

$\frac{c(1-i)}{1+1} - \frac{d(1-2i)}{1+4} = 1$

$\frac{c}{2} - \frac{c}{2}i - \frac{d}{5} + \frac{2di}{5} = 1$

$\left(\frac{c}{2} - \frac{d}{5}\right) + i\left(\frac{2d}{5} - \frac{c}{2}\right) = 1 + 0i$

Equate Real & Im parts.

$\frac{c}{2} - \frac{d}{5} = 1 \Rightarrow 5c - 2d = 10$ --- ①

$\frac{2d}{5} - \frac{c}{2} = 0 \Rightarrow 5c - 4d = 0$ --- ②

Solving : $4d - 2d = 10$

$d = 5$

$\therefore c = 4$

(d) If $a + c = b + d$

Then $\frac{a+c}{2} = \frac{b+d}{2}$

$\therefore$ midpoint PR = midpoint QS

$\therefore$ diagonals PR and QS bisect each other

If $a - c = i(b - d)$

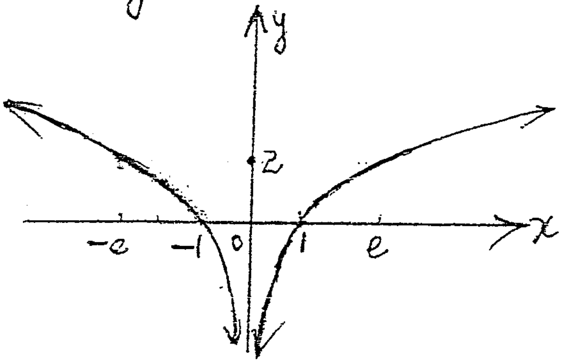
$\therefore$ diagonals PR and QS are equal and perpendicular

$\therefore$ PQRS is a square.

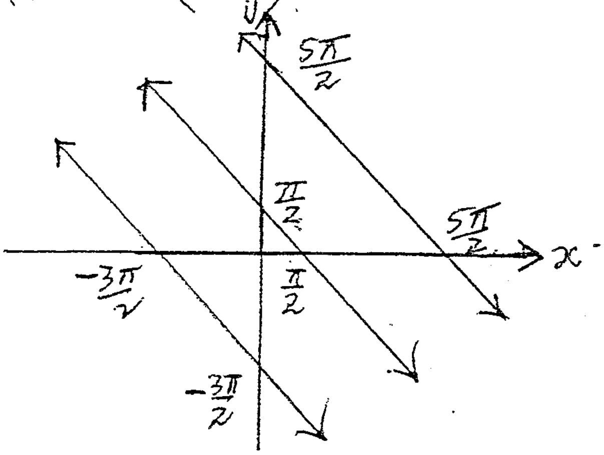
Note A diagram would be useful.

Question 2

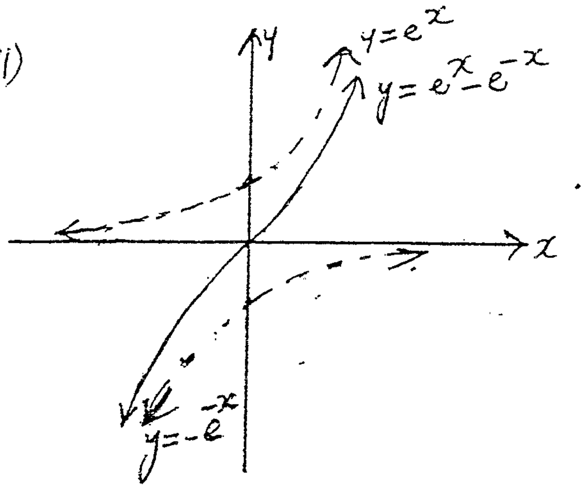
(a) (i) $y = \ln x^2$



(ii) $\sin(x+y) = 1$

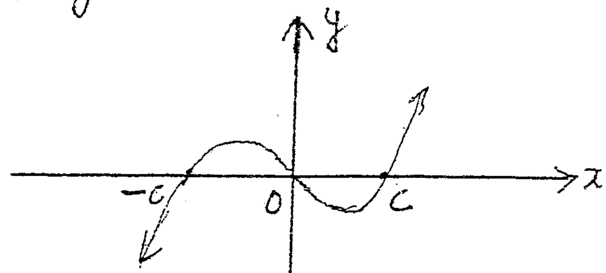


(iii)



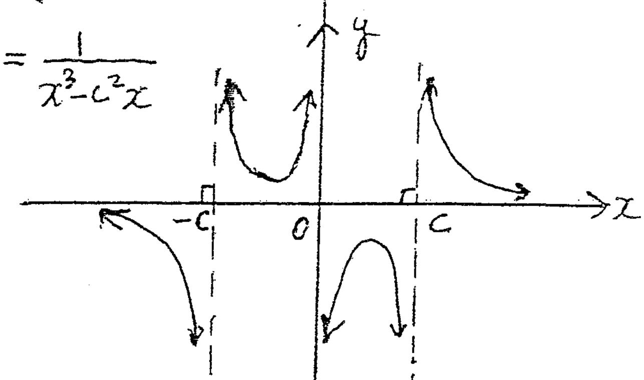
Note. In HSC each of these should be about $\frac{1}{2}$ page.

(b) (i) $y = x^3 - c^2x$

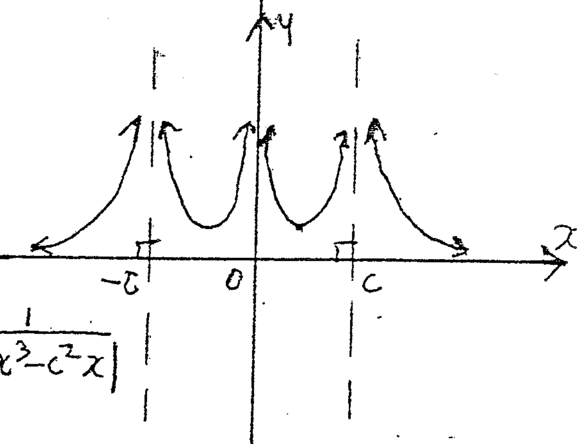


(ii) (α)

$y = \frac{1}{x^3-c^2x}$

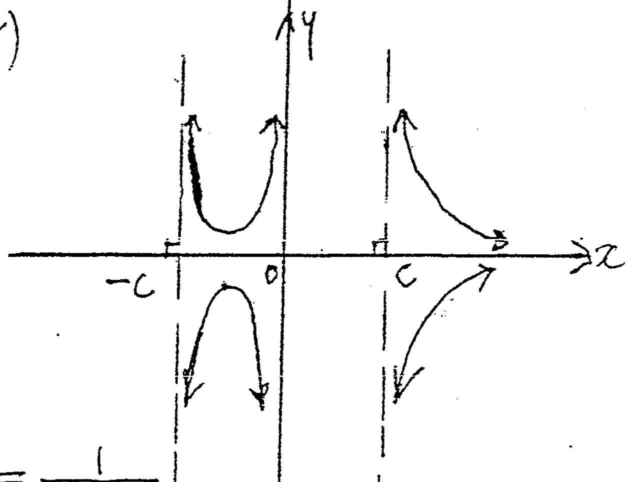


(β)



$y = \frac{1}{|x^3-c^2x|}$

(γ)



$y^2 = \frac{1}{x^3-c^2x}$

Question 3

(a) (i) $\int 2^{2x}\,dx$

$= \frac{1}{\ln 4} 2^{2x} + c$

<u>Note</u> If $y = 2^{2x}$

$\ln y = 2x \ln 2$

$\frac{\frac{dy}{dx}}{y} = 2 \ln 2$

$\frac{dy}{dx} = 2y \ln 2$

$= 2 \cdot 2^{2x} \ln 2$

$= 2^{2x} \ln 4$

(ii) $I = \int x e^x\,dx$

Let $v' = e^x$, $u = x$

$I = xe^x - \int e^x \times 1\,dx$

$= xe^x - e^x + c$.

(iii) Let $\frac{2x}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$.

$\therefore A = -1, \; B = 3$

$I = \int \frac{2x}{(x+1)(x+3)}\,dx$.

$= \int \left(-\frac{1}{x+1} + \frac{3}{x+3} \right) dx$

$= 3 \ln|x+3| - \ln|x+1| + c$

$= \ln\left|\frac{(x+3)^3}{x+1}\right| + c$.

Question 3 (continued)

(b) $I = \int_4^{4.5} \frac{dt}{(t-3)(5-t)}$

$u = t-4$

$du = dt$

$t-3 = u+1$

$5-t = 1-u$

when $t=4$, $u=0$

when $t=4.5$, $u=\frac{1}{2}$

$I = \int_0^{1/2} \frac{du}{1-u^2}$

$= \frac{1}{2}\int_0^{1/2}\left(\frac{1}{1-u} + \frac{1}{1+u}\right)du$

(Note use of partial fractions)

$= \frac{1}{2}\left[\ln(1+u) - \ln(1-u)\right]_0^{1/2}$

$= \frac{1}{2}\left[\ln\left(\frac{1+u}{1-u}\right)\right]_0^{1/2}$

$= \frac{1}{2}\left[\ln\left(\frac{3}{2} \div \frac{1}{2}\right) - \ln 1\right]$

$= \frac{1}{2}\ln 3$

$= \ln\sqrt{3}$

(c) (i) $U_n = \int_0^{\pi/2} x^n \sin x\, dx$

Let $u = x^n$, $v' = \sin x$

$\therefore U_n = \left[-\cos x . x^n\right]_0^{\pi/2} + \int_0^{\pi/2} n x^{n-1}\cos x\, dx$

$U_n = n\int_0^{\pi/2} x^{n-1}\cos x\, dx$

Now if $u = x^{n-1}$, $v' = \cos x$

$U_n = n\left\{\left[x^{n-1}\sin x\right]_0^{\pi/2} - \int_0^{\pi/2}(n-1)x^{n-2}\sin x\, dx\right\}$

$U_n = n\left[\left(\frac{\pi}{2}\right)^{n-1}\right] - n(n-1)\int_0^{\pi/2} x^{n-2}\sin x\, dx$

$U_n = n\left(\frac{\pi}{2}\right)^{n-1} - n(n-1)U_{n-2}$

(ii) $\int_0^{\pi/2} x^2 \sin x\, dx$

$U_2 = 2\left(\frac{\pi}{2}\right)^1 - 2(1)U_0$

$= \pi - 2\int_0^{\pi/2}\sin x\, dx$

$= \pi - 2\left[-\cos x\right]_0^{\pi/2}$

$= \pi + 2\left(\cos\frac{\pi}{2} - \cos 0\right)$

$= \pi + 2(0-1)$

$= \pi - 2$

Question 4

(a) $\frac{x^2}{20} - \frac{y^2}{5} =$

$a = \sqrt{20} = 2\sqrt{5}$
$b = \sqrt{5}$
$b^2 = a^2(e^2 - 1)$
$\therefore e^2 = 1 + \frac{5}{20}$

$e^2 = \frac{5}{4}$
$e = \frac{\sqrt{5}}{2}$

(i) foci : $S = (ae, 0) = (5, 0)$
hence $S' = (-5, 0)$

(ii) asymptotes : let $\frac{x^2}{20} - \frac{y^2}{5} = 0$

$\left(\frac{x}{2\sqrt{5}} + \frac{y}{\sqrt{5}}\right)\left(\frac{x}{2\sqrt{5}} - \frac{y}{\sqrt{5}}\right) = 0$

hence $y = \frac{x}{2}$ or $y = -\frac{x}{2}$

(b) $\frac{x^2}{h-19} + \frac{y^2}{3-h} = 1$

$h - 19 > 0$ and $3 - h > 0$
ie $h > 19$ and $h < 3$.
sets have no intersection hence not possible.

(c) $\frac{x^2}{4} + y^2 = 1$; $a = 2, b = 1$
Equation of tangent is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

At A : $\frac{x\cos\theta}{2} + y\sin\theta = 1$; $\therefore m_1 = -\frac{\cos\theta}{2\sin\theta}$
At B : $\frac{x\cos\alpha}{2} + y\sin\alpha = 1$; $\therefore m_2 = -\frac{\cos\alpha}{2\sin\alpha}$
If tangents are at right angles then $m_1 m_2 = -1$

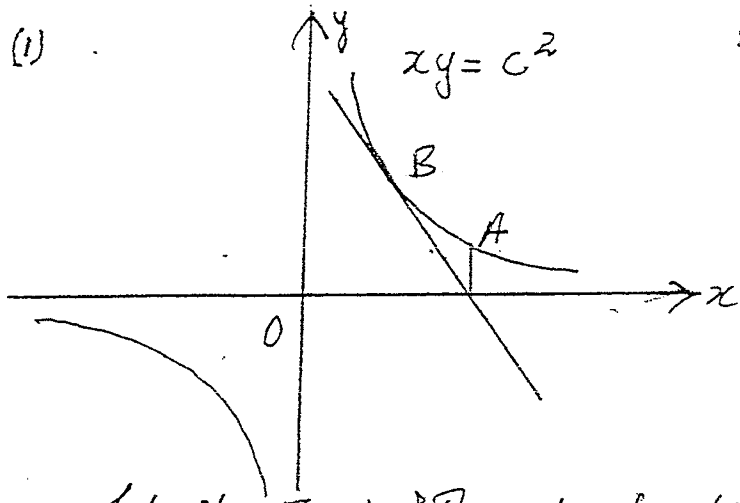
$\left(\frac{-\cos\theta}{2\sin\theta}\right)\left(-\frac{\cos\alpha}{2\sin\alpha}\right) = -1$

$\frac{1}{2\tan\theta} \times \frac{1}{2\tan\alpha} = -1$

ie $4\tan\theta\tan\alpha = -1$

Question 4 (continued)

(d) (i)



Let N = Foot of the ordinate of A.

$B\left(ct_2, \frac{c}{t_2}\right)$; $A\left(ct_1, \frac{c}{t_1}\right)$

tangent at B : $y' = -\frac{c^2}{x^2}$

gradient m at B is $m = -\frac{c^2}{c^2 t_2^2} = -\frac{1}{t_2^2}$

Eqn : $y - \frac{c}{t_2} = -\frac{1}{t_2^2}(x - ct_2)$

$t_2^2 y - ct_2 = -x + ct_2$.

For Co-ordinates of N let $y = 0$

$\therefore\ x = 2ct_2$

But this is equal to the x co-ordinate of A.

$ct_1 = 2ct_2$

$\therefore\ t_1 = 2t_2$ --- ①

(ii) Co-ordinates of the mid point of AB

$x = \frac{c}{2}(t_1 + t_2)$ --- ②

$y = \frac{c}{2}\left(\frac{1}{t_1} + \frac{1}{t_2}\right)$ --- ③

Eliminate parameters.

sub ① in both ② and ③

$x = \frac{c}{2}(3t_2)$ --- ②a

$y = \left(\frac{1}{2t_2} + \frac{1}{t_2}\right)\frac{c}{2}$ ---

$y = \left(\frac{1+2}{2t_2}\right)\frac{c}{2}$

$y = \frac{c}{2}\left(\frac{3}{2t_2}\right)$ --- ③a

②a × ③a

$xy = \frac{c}{2}(3t_2) \times \frac{c}{2}\left(\frac{3}{2t_2}\right)$

$xy = \frac{c^2}{8} \times 9$

$8xy = 9c^2$.

which is a rectangular hyperbola.

Question 5.

(a) $P(x) = x^6 - 3x^2 + 2$

$P'(x) = 6x^5 - 6x$

$P(1) = 1 - 3 + 2 = 0$

$P'(1) = 6 - 6 = 0$

$\therefore$ 1 is a zero of multiplicity 2

$P(-1) = 1 - 3 + 2 = 0$

$P'(-1) = -6 + 6 = 0$

$\therefore (-1)$ is a zero of multiplicity 2.

Now If $x^2 = t$ we have $t^3 - 3t + 2 = Q(t)$

and $Q(-2) = -8 + 6 + 2 = 0$

$\therefore (t+2)$ is a factor of $Q(t)$

Hence (x^2+2) is a factor of $P(x)$

(i) $P(x) = (x+1)^2(x-1)^2(x^2+2)$

(ii) $P(x) = (x+1)^2(x-1)^2(x+i\sqrt{2})(x-i\sqrt{2})$

(b)(i) $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$.

Now $8x^3 - 6x + 1 = 0$ and $x = \cos\theta$.

$8\cos^3\theta - 6\cos\theta + 1 = 0$.

$2(4\cos^3\theta - 3\cos\theta) + 1 = 0$.

$2\cos 3\theta = -1$

$\cos 3\theta = -\frac{1}{2}$

General soln $3\theta = 2n\pi \pm \cos^{-1}(-\frac{1}{2})$

$\therefore 3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \ldots$

$\theta = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \ldots$

$\therefore x = \cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \cos\frac{8\pi}{9}$ 3 unique solutions only.

Question 5 (continued)

(α) $\cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9}$ represents the sum of the roots

$$\text{sum} = -\frac{b}{a} = \frac{0}{8} = 0$$

$$\therefore \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} + \cos\frac{8\pi}{9} = 0.$$

(β) $\sec\frac{2\pi}{9} + \sec\frac{4\pi}{9} + \sec\frac{8\pi}{9}$

$$= \frac{1}{\cos\frac{2\pi}{9}} + \frac{1}{\cos\frac{4\pi}{9}} + \frac{1}{\cos\frac{8\pi}{9}}$$

$$= \frac{\cos\frac{4\pi}{9}\cos\frac{8\pi}{9} + \cos\frac{2\pi}{9}\cos\frac{8\pi}{9} + \cos\frac{2\pi}{9}\cos\frac{4\pi}{9}}{\cos\frac{2\pi}{9}\cdot\cos\frac{4\pi}{9}\cdot\cos\frac{8\pi}{9}}$$

$$= \frac{\text{sum of roots 2 at a time}}{\text{product of roots}}$$

$$= \frac{-\frac{6}{8}}{-\frac{1}{8}} = 6.$$

(c) $x^3 + mx^2 + nx + h = 0$

Let roots be $\alpha, -\alpha, \beta$.

sum $\alpha - \alpha + \beta = -m$ --- (1)

sum 2 at a time $\alpha\times-\alpha + \alpha\times\beta + -\alpha\times\beta = n$ --- (2)

product $-\alpha^2\beta = -h$ --- (3)

$\beta = -m$ --- (1a)

$-\alpha^2 = n$ --- (2a)

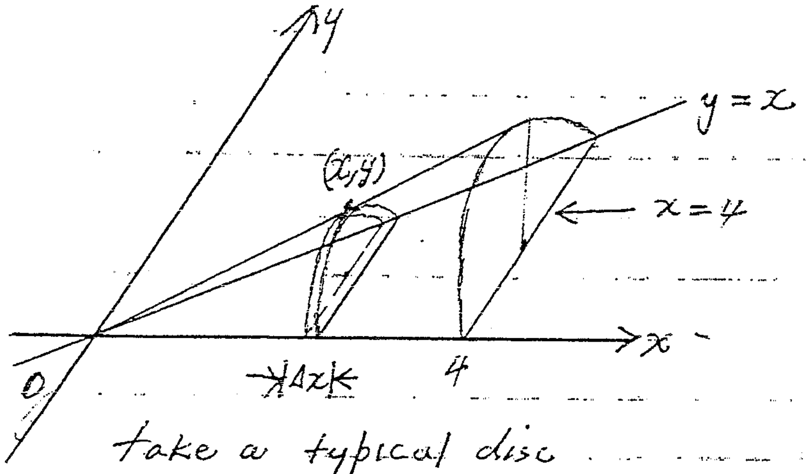
Sub in (3)

$n\times -m = -h$

$\therefore h - mn = 0$

Question 6

(a)



take a typical disc

Area of cross-section of semi ⊖

$$\Delta A = \frac{1}{2}\pi\left(\frac{y}{2}\right)^2$$

$$= \frac{\pi y^2}{8}$$

Volume of slice :

$$\Delta V \doteq \frac{\pi y^2}{8} \cdot \Delta x$$

$$V = \lim_{\Delta x \to 0} \sum_{x=0}^{4} \frac{\pi y^2}{8} \Delta x$$

$$V = \int_0^4 \frac{\pi y^2}{8} dx$$

$$V = \frac{\pi}{8}\int_0^4 x^2 dx \qquad \text{Note } y = x$$

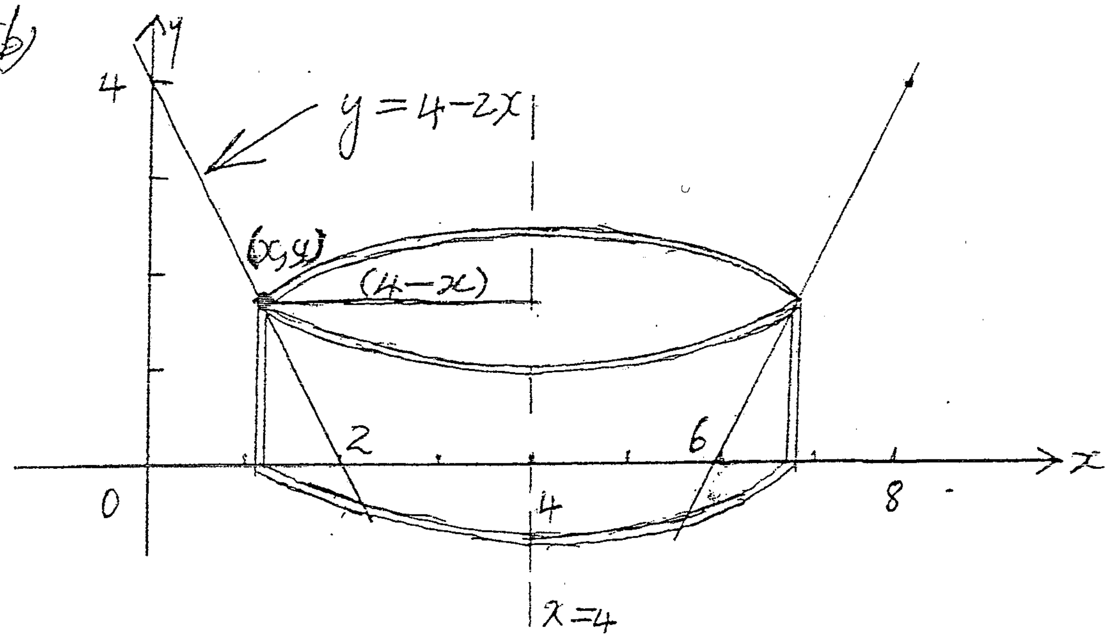
$$V = \frac{\pi}{8}\left[\frac{x^3}{3}\right]_0^4$$

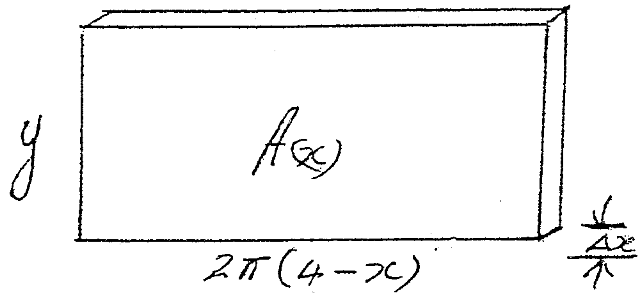
$$V = \frac{\pi}{8} \times \frac{64}{3}$$

$$\text{Volume} = \frac{8\pi}{3} \text{ units}^3$$

Question 6 (Continued)

(b)





$$A(x) = 2\pi(4-x)y$$

$$\Delta V \doteqdot 2\pi(4-x)(4-2x)\,\Delta x$$

$$V = \lim_{x \to 0} \sum_{x=0}^{2} 2\pi(4-x)(4-2x)\,\Delta x$$

$$V = 2\pi \int_0^2 (16 - 12x + 2x^2)\,dx$$

$$V = 2\pi \left[16x - 6x^2 + \frac{2x^3}{3}\right]_0^2$$

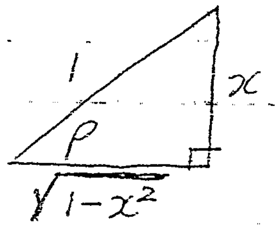
$$V = 2\pi\left[32 - 24 + \frac{16}{3}\right]$$

$$V = 2\pi\left[\frac{24}{3} + \frac{16}{3}\right]$$

$$V = \frac{8\pi}{3} \text{ units}^3$$

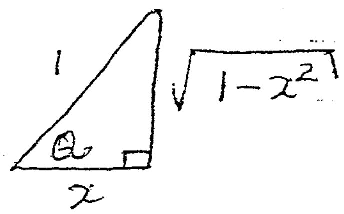
(c)(i) Let $P = \sin^{-1}x$

$\therefore x = \sin P$



Let $Q = \cos^{-1}x$

$\therefore x = \cos Q$



Hence $\cos P = \sqrt{1-x^2}$

and $\sin Q = \sqrt{1-x^2}$

$$LHS = \sin(\sin^{-1}x - \cos^{-1}x)$$
$$= \sin(P - Q)$$
$$= \sin P \cos Q + \cos P \sin Q$$
$$= x \times x - \sqrt{1-x^2} \times \sqrt{1-x^2}$$
$$= x^2 - (1-x^2)$$
$$= 2x^2 - 1$$
$$= RHS.$$

(ii) $\sin^{-1}x - \cos^{-1}x = \sin^{-1}(1-x)$

$\sin(\sin^{-1}x - \cos^{-1}x) = 1-x.$

$2x^2 - 1 = 1 - x$ using result of (i)

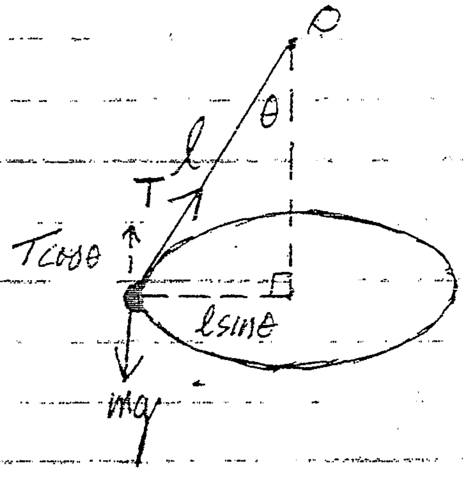
$2x^2 + x - 2 = 0$

$$x = \frac{-1 \pm \sqrt{1-4(2)(-2)}}{2(2)}$$

$$x = \frac{-1 \pm \sqrt{17}}{4}$$

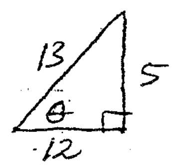
Question 6

(a)



$\tan\theta = \frac{5}{12}$

$\therefore \sin\theta = \frac{5}{13}$ using



Resolving Vertically : $T\cos\theta = mg$ - - - ①

Resolving Horizontally : $T\sin\theta = \frac{mv^2}{r}$ — — — ②

$T\sin\theta = \frac{m \times 2^2}{l\sin\theta}$

$T\sin\theta = \frac{4m}{l\sin\theta}$ - - - - (2a)

Now (2a) ÷ ①

$\frac{T\sin\theta}{T\cos\theta} = \frac{4m}{l\sin\theta} \times \frac{1}{mg}$

$\tan\theta = \frac{4}{lg\sin\theta}$

$l = \frac{4}{(\tan\theta)g\sin\theta}$

$l = \frac{4}{\frac{5}{12} \times 10 \times \frac{5}{13}}$

$l = \frac{4 \times 12 \times 13}{5 \times 10 \times 5}$

$l \doteqdot 2.49$ ie approx $2.5m$

Question (7 continued)

(b) $\ddot{x} = \frac{16}{v} - v$.

$\frac{dv}{dt} = \frac{16}{v} - v$.

$= \frac{16 - v^2}{v}$.

$\frac{dt}{dv} = \frac{v}{16-v^2}$

$t = -\frac{1}{2}\ln(16-v^2) + c$

when $t=0$, $v = 2$.

$0 = -\frac{1}{2}\ln(16-4) + c$

$0 = -\frac{1}{2}\ln 12 + c$

$c = \frac{1}{2}\ln 12$.

$t = \frac{1}{2}\ln 12 - \frac{1}{2}\ln(16-v^2)$

$t = \frac{1}{2}\ln\frac{12}{16-v^2}$

$2t = \ln\frac{12}{16-v^2}$

$\frac{12}{16-v^2} = e^{2t}$.

$\frac{16-v^2}{12} = e^{-2t}$.

$16 - v^2 = 12e^{-2t}$.

$v^2 = 16 - 12e^{-2t}$.

$v^2 = 16\left(1 - \frac{12}{16}e^{-2t}\right)$

$V^2 = 16\left(1 - \frac{3}{4}e^{-2t}\right)$

$V = 4\left(1 - \frac{3}{4}e^{-2t}\right)^{1/2}$, $v>0$.

$\lim_{t\to\infty}\left(1 - \frac{3}{4}\cdot\frac{1}{e^{2t}}\right) = 1$

$\therefore \lim_{t\to\infty}\left(1 - \frac{3}{4}\cdot\frac{1}{e^{2t}}\right)^{\frac{1}{2}} = 1$

$\therefore \lim_{t\to\infty}(V) = 4$

$\therefore$ Limiting velocity $= 4\,m\,sec^{-1}$

considering $v\frac{dv}{dx} = \frac{16}{v} - v$

$\frac{dv}{dx} = \frac{16}{v^2} - 1$

$\frac{dx}{dv} = \frac{v^2}{16-v^2}$.

$$\begin{array}{r} -1 \\ -v^2+16 \,\overline{)\, v^2 + 0} \\ -v^2 - 16 \\ \hline 16 \end{array}$$

Note.

Transform to partial fractions

$\frac{dx}{dv} = \frac{16}{16-v^2} - 1$

$\frac{dx}{dv} = 16\left[\frac{1}{8(4+v)} + \frac{1}{8(4-v)}\right] - 1$

$x = 2\ln\left|\frac{4+v}{4-v}\right| - v + c$.

when $x=0$, $v=2$.

$0 = 2\ln\left(\frac{6}{2}\right) - 2 + c$.

$c = 2 - 2\ln 3$

$x = 2\ln\left|\frac{4+v}{4-v}\right| - v + 2 - 2\ln 3$

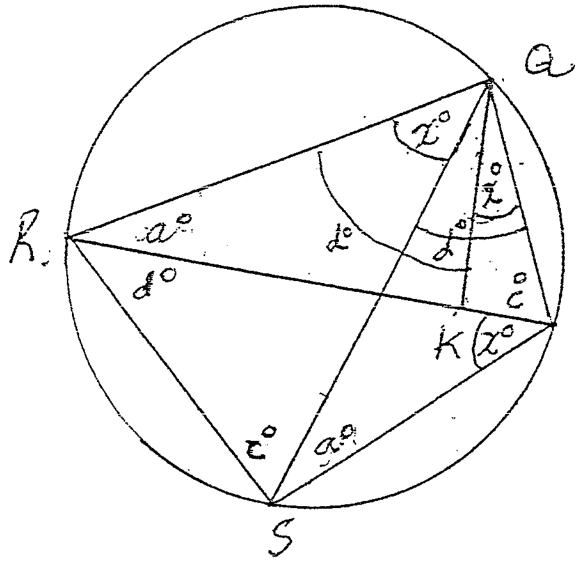
$x = 2\ln\left|\frac{4+v}{3(4-v)}\right| + 2 - v$

When $v=3$

$x = \left[2\ln\left(\frac{7}{3}\right) - 1\right]$ metres.

Question 8

(a) (i)



Now $\hat{PQK} = \hat{SQR} = x^\circ$ (given)

and $\hat{QSP} = \hat{QRP} = a^\circ$ (say) ($\wedge$'s subtended at the circumference of the circle by a common arc (QP) are equal)

similarly $\hat{RPS} = \hat{RQS} = x^\circ$

similarly $\hat{RSQ} = \hat{QPR} = c^\circ$ (say)

similarly $\hat{SQP} = \hat{SRP} = d^\circ$ (say)

$\therefore \hat{SQP} = d^\circ - x^\circ$

$\therefore \hat{RQK} = d^\circ$

Now in Δ's PQS and KQR

$\hat{PQS} = \hat{KQR} = d^\circ$

$\hat{QSP} = \hat{KRQ} = a^\circ$

$\hat{QPS} = \hat{QKR}$ (remaining $\wedge$'s of Δ are $=$, because $\wedge$ sum of $\Delta \cong 180^\circ$) *

So Δs are equiangular

$\therefore \Delta PQS \,|||\, \Delta KQR$

Also in Δ's PQK and SQR.

$\hat{PQK} = \hat{SQR} = x^\circ$

$\hat{QPK} = \hat{QSR} = c^\circ$

$\therefore \hat{PKQ} = \hat{SRQ}$ (same reason as *)

So Δs are equiangular

$\therefore \Delta PQK \,|||\, \Delta SQR$.

Question 8 (continued)

(a) (ii) As $\Delta PQS \parallel\!\mid \Delta KQR$

$$\frac{PQ}{KQ} = \frac{PS}{KR} = \frac{QS}{QR}$$ (In similar Δ's ratios of corresponding sides are equal.)

$\therefore PS.QR = QS.KR$ ---- ①

Also as $\Delta PQK \parallel\!\mid \Delta SQR$

$$\frac{PQ}{SQ} = \frac{PK}{SR} = \frac{QK}{QR}$$ (In similar Δ's ratios of corresponding sides are equal.)

$PQ.SR = SQ.PK$ ------- ②

Now $PR.SQ = (PK + KR)SQ$

$= PK.SQ + KR.SQ.$

$PR.SQ = PQ.SR + PS.QR$ (using ① and ②)

Question 8 (continued)

(b) $\sum_{r=1}^{n} \sin(2r-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}$

Let $T_n = \sin(2n-1)\theta$

Let $S_n = \frac{\sin^2 n\theta}{\sin\theta}$

$\sum_{r=1}^{n} \sin(2r-1)\theta = \sin\theta + \sin 3\theta + \sin 5\theta + \dots + \sin(2n-1)\theta$

when $n=1$

$\frac{\sin^2 n\theta}{\sin\theta} = \frac{\sin^2\theta}{\sin\theta} = \sin\theta$ which is true.

Let $n=k$ be true.

$\therefore \quad S_k = \frac{\sin^2 k\theta}{\sin\theta}$

Now $S_k + T_{k+1} = \frac{\sin^2 k\theta}{\sin\theta} + \sin[2(k+1)-1]\theta$

$$= \frac{\sin^2 k\theta}{\sin\theta} + \sin(2k+1)\theta$$

$$= \frac{2\sin^2 k\theta + 2\sin(2k+1)\theta\sin\theta}{2\sin\theta}$$

$$= \frac{1-\cos 2k\theta + 2\sin\theta[\sin 2k\theta\cos\theta + \cos 2k\theta\sin\theta]}{2\sin\theta}$$

$$= \frac{1-\cos 2k\theta + \sin 2k\theta\sin 2\theta + 2\sin^2\theta\cos 2k\theta}{2\sin\theta}$$

$$= \frac{1-\cos 2k\theta + \sin 2k\theta\sin 2\theta + 2\times\frac{1}{2}(1-\cos 2\theta)\cos 2k\theta}{2\sin\theta}$$

$$= \frac{1-\cos 2k\theta + \sin 2k\theta\sin 2\theta + \cos 2k\theta - \cos 2k\theta\cos 2\theta}{2\sin\theta}$$

$$= \frac{1-\cos(2k\theta+2\theta)}{2\sin\theta}$$

$$= \frac{1-\cos 2(k\theta+\theta)}{2\sin\theta}$$

$S_k + T_{k+1} = \frac{2\sin^2(k\theta+\theta)}{2\sin\theta} = \frac{\sin^2(k+1)\theta}{\sin\theta}$

Question 8 (continued)

(b) This result is the required form ie S_k with $(k+1)$ in place of k.

The result is true for $n=k+1$ if it is true for $n=k$ ie if it is true for one integer then it is true for the next consecutive integer.

$\therefore$ true when $n=1$, also true when $n=2$

" ✓ ✓ $n=2$, ✓ ✓ ✓ ✓ $n=3$

" ✓ ✓ $n=3$, ✓ ✓ ✓ ✓ $n=4$ etc

Hence $\displaystyle\sum_{r=1}^{n} \sin(2r-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}$ true for all positive integers n.

(c) for $0 \le x \le 1$

and $n = 1, 2, 3, \ldots$ $\quad x^n \ge x^{n+1}$

$$\therefore\ 1+x^n \ge 1+x^{n+1}$$

$$\therefore\ \frac{1}{1+x^n} \le \frac{1}{1+x^{n+1}}$$

$$\therefore\ \int_0^1 \frac{dx}{1+x^n} \le \int_0^1 \frac{dx}{1+x^{n+1}}$$

$\therefore$ The statement is true.

Question 8 (continued)

(a) (ii) As $\Delta PQS \,|||\, \Delta KQR$

$$\frac{PQ}{KQ} = \frac{PS}{KR} = \frac{QS}{QR}$$ (In similar Δ's ratios of corresponding sides are equal.)

$$\therefore\ PS.QR = QS.KR \quad ----①$$

Also as $\Delta PQK \,|||\, \Delta SQR$

$$\frac{PQ}{SQ} = \frac{PK}{SR} = \frac{QK}{QR}$$ (In similar Δ's ratios of corresponding sides are equal.)

$$PQ.SR = SQ.PK \quad ------②$$

Now $PR.SQ = (PK + KR)SQ$

$= PK.SQ + KR.SQ.$

$PR.SQ = PQ.SR + PS.QR$ (using ① and ②)

Question 8 (continued)

(b) $\displaystyle\sum_{r=1}^{n} \sin(2r-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}$

Let $T_n = \sin(2n-1)\theta$

Let $S_n = \dfrac{\sin^2 n\theta}{\sin\theta}$

$\displaystyle\sum_{r=1}^{n} \sin(2r-1)\theta = \sin\theta + \sin 3\theta + \sin 5\theta + \dots + \sin(2n-1)\theta$

when $n=1$

$\dfrac{\sin^2 n\theta}{\sin\theta} = \dfrac{\sin^2\theta}{\sin\theta} = \sin\theta$ which is true.

Let $n=k$ be true

$\therefore\ S_k = \dfrac{\sin^2 k\theta}{\sin\theta}$

Now $S_k + T_{k+1} = \dfrac{\sin^2 k\theta}{\sin\theta} + \sin[2(k+1)-1]\theta$

$= \dfrac{\sin^2 k\theta}{\sin\theta} + \sin(2k+1)\theta$

$= \dfrac{2\sin^2 k\theta + 2\sin(2k+1)\theta\sin\theta}{2\sin\theta}$

$= \dfrac{1-\cos 2k\theta + 2\sin\theta[\sin 2k\theta\cos\theta + \cos 2k\theta\sin\theta]}{2\sin\theta}$

$= \dfrac{1-\cos 2k\theta + \sin 2k\theta\sin 2\theta + 2\sin^2\theta\cos 2k\theta}{2\sin\theta}$

$= \dfrac{1-\cos 2k\theta + \sin 2k\theta\sin 2\theta + 2\times\frac{1}{2}(1-\cos 2\theta)\cos 2k\theta}{2\sin\theta}$

$= \dfrac{1-\cos 2k\theta + \sin 2k\theta\sin 2\theta + \cos 2k\theta - \cos 2k\theta\cos 2\theta}{2\sin\theta}$

$= \dfrac{1-\cos(2k\theta+2\theta)}{2\sin\theta}$

$= \dfrac{1-\cos 2(k\theta+\theta)}{2\sin\theta}$

$S_k + T_{k+1} = \dfrac{2\sin^2(k\theta+\theta)}{2\sin\theta} = \dfrac{\sin^2(k+1)\theta}{\sin\theta}$

Question 8 (continued)

(b) This result is the required form ie S_k with $(k+1)$ in place of k.

The result is true for $n=k+1$ if it is true for $n=k$ ie if it is true for one integer then it is true for the next consecutive integer.

$\therefore$ true when $n=1$, also true when $n=2$

" ✓ ✓ $n=2$, ✓ ✓ ✓ ✓ $n=3$

" ✓ ✓ $n=3$, ✓ ✓ ✓ ✓ $n=4$ etc

Hence $\sum_{r=1}^{n} \sin(2r-1)\theta = \frac{\sin^2 n\theta}{\sin\theta}$ true for all positive integers n.

(c) for $0 \le x \le 1$

and $n = 1, 2, 3, \ldots \quad x^n \ge x^{n+1}$

$$\therefore \ 1 + x^n \ge 1 + x^{n+1}$$

$$\therefore \ \frac{1}{1+x^n} \le \frac{1}{1+x^{n+1}}$$

$$\therefore \ \int_0^1 \frac{dx}{1+x^n} \le \int_0^1 \frac{dx}{1+x^{n+1}}$$

$\therefore$ The statement is true.