Name: _

Mathematics Teacher: _

St Mary's Cathedral College

Year 12 Mathematics, 2004
Week 6, Term 1
Mathematics Extension I

Time allowed: **45 minutes** Topic Test 2: **Trigonometric Functions II**

Weighting: **7.5% (15%)**

Total Mark: **43**

Outcomes to be assessed:
A student:
PE1 appreciates the role of mathematics in the solution of practical problems.
PE2 uses multi-step deductive reasoning in a variety of contexts.
HE1 appreciates interrelationships between ideas drawn from different areas of mathematics.
HE6 determines integrals by reduction to a standard form through a given substitution.

General Instructions:
- Write using blue or black pen. Use pencil for diagrams and graphs.
- Board-approved calculators may be used.
- All necessary working should be shown for every question.
- Write your answers on A4 paper, writing on one side only.

1. Find the simplest possible expression (one term) for

(a) $\cos(A+B)\cos(C-B)-\sin(A+B)\sin(C-B)$ **2 marks**

(b) $(\cos\theta+\sin\theta)^2\ -1$ **3 marks**

2. Given that $\frac{\pi}{6}+\frac{\pi}{4}=\frac{5\pi}{12}$, prove that $\tan\frac{5\pi}{12}=\frac{\sqrt{2}+\sqrt{3}}{\sqrt{6}-1}$. **3 marks**

3. Evaluate $\int_0^{\frac{\pi}{6}}\sin^2 x\,dx$ and answer in exact form. **4 marks**

4. Find the solution to $\sin x+\cos x-1=0$, using the "t-method" on $0\le x\le 2\pi$. **4 marks**

5. Solve the following equations on the domain given.

(a) $\cos 3x=\frac{1}{2}$ on $0\le x\le 2\pi$ **3 marks**

(b) $3\tan^2 x-1=0$ on $-\pi\le x\le\pi$ **3 marks**

(c) $\sin 2x=\cos x$ on $0\le x\le 2\pi$. **4 marks**

6. Write the following as a single algebraic fraction in terms of t where $t=\tan\frac{\theta}{2}$

(a) $2\sin\theta+\cos\theta$ **2 marks**

(b) $\cos 2\theta$ **3 marks**

7. Find the acute angle between the lines $y=x+1$ and $x-2y+1=0$. Answer to the nearest minute. **3 marks**

8. Prove that $\dfrac{2\cos x}{\operatorname{cosec} x - 2\sin x} = \tan 2x$ (assuming $\sin x \neq 0$). **3 marks**

9. Consider the equation $\cos 2x - \sqrt{3}\sin 2x = \sqrt{2}$.

   (a) Write $\cos 2x - \sqrt{3}\sin 2x$ in the form $A\cos(2x+\alpha)$ where α is an acute angle. **3 marks**

   (b) Using your result from part (a), solve the equation $\cos 2x - \sqrt{3}\sin 2x = \sqrt{2}$, $0 \leqslant x \leqslant 2\pi$. **3 marks**

END OF ASSESSMENT TASK

1

① (a) $\cos\left[(A+B)+(C-B)\right] = \cos(A+C)$ *1 mark formula form
*1 mark final form

(b) $(\cos^2\theta + 2\cos\theta\sin\theta + \sin^2\theta) - 1$ *1 mark expansion

$= (\cos^2\theta + \sin^2\theta) + 2\cos\theta\sin\theta - 1$ *1 mark use of pythagorean identity

$= 1 + \sin 2\theta - 1$

$= \sin 2\theta$ *1 mark for single term.

② $\tan\frac{5\pi}{12} = \tan\left(\frac{\pi}{6}+\frac{\pi}{4}\right)$ *1 mark for understanding the need to use the $\tan(\alpha+\beta)$ result

$= \dfrac{\tan\frac{\pi}{6}+\tan\frac{\pi}{4}}{1-\tan\frac{\pi}{6}\tan\frac{\pi}{4}}$ *1 mark for correct use

$= \dfrac{\frac{1}{\sqrt3}+1}{1-\frac{1}{\sqrt3}\times 1}$ *1 mark for getting to here or any equivalent.

$= \dfrac{\sqrt3+1}{\sqrt3-1}$ (no simplification penalties!)

③ $\int_0^{\frac{\pi}{6}} \sin^2 x\,dx = \int_0^{\frac{\pi}{6}} \frac12\left[1-\cos 2x\right]dx$ *1 mark $\sin^2 x$ result

$= \frac12\left[x - \frac12\sin 2x\right]_0^{\frac{\pi}{6}}$ *1 mark primitive

$= \frac12\left(\left(\frac{\pi}{6} - \frac12\sin\left(2\times\frac{\pi}{6}\right)\right) - \left(0 - \frac12\sin(2\times 0)\right)\right)$

$= \frac12\left[\frac{\pi}{6} - \frac12\sin\frac{\pi}{3}\right]$ *1 mark substitution

$= \frac{\pi}{12} - \frac14\times\frac{\sqrt3}{2}$

$= \frac{\pi}{12} - \frac{\sqrt3}{8}$ or equivalent. *1 mark any exact value expression

2

④ $\sin x + \cos x - 1 = 0$

$\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} - 1 = 0$ where $t = \tan\frac{x}{2}$

$\dfrac{2t+1-t^2-(1+t^2)}{1+t^2} = 0$ *1 mark substitution

$2t + \cancel{1} - t^2 \cancel{-1} - t^2 = 0$

$2t - 2t^2 = 0$

$2t^2 - 2t = 0$

$2t(t-1) = 0$

$t = 0$ and $t = 1$ *1 mark substantial simplification

If $t=0$ then $\tan\frac{x}{2} = 0$

$\therefore \frac{x}{2} = 0 + 2k\pi,\ \pi + 2k\pi$

$x = 4k\pi,\ (4k+2)\pi$ where $k\in\mathbb{Z}$

If $t=1$ then $\tan\frac{x}{2} = 1$

$\frac{x}{2} = \frac{\pi}{4}+2k\pi,\ \frac{5\pi}{4}+2k\pi$

$= \frac{(8k+1)\pi}{4},\ \frac{(8k+5)\pi}{4}$

$x = \frac{(8k+1)\pi}{2},\ \frac{(8k+5)\pi}{2}$ *1 mark for explicitly expressed values of $\tan\frac{x}{2}$ (not just "t").

On $0\le x\le 2\pi$

Test $x=\pi \Rightarrow$ LHS $= \sin\pi + \cos\pi - 1$

$= -1 + 0 - 1$

$= -2$

$\neq 0$

$\therefore x = 0,\ 2\pi,\ \frac{\pi}{2},$ ($k=0$ in 1st three formulae.)

*Award full marks if only a minor error in this last section (*1 mark for answer

3

⑤ (a) $\cos 3x = \frac12$ on $0\le x\le 2\pi$

$3x = \frac{\pi}{3}+2k\pi,\ \frac{5\pi}{3}+2k\pi$ *1 mark $3x$ the subject

$3x = \frac{(6k+1)\pi}{3},\ \frac{(6k+5)\pi}{3}$

$x = \frac{(6k+1)\pi}{9},\ \frac{(6k+5)\pi}{9}$ for $k\in\mathbb{Z}$ *1 mark x the subject

On $0\le x\le 2\pi$

$x = \frac{\pi}{9},\ \frac{5\pi}{9},\ \frac{7\pi}{9},\ \frac{11\pi}{9},\ \frac{5\pi}{9};\ \frac{13\pi}{9},\ \frac{17\pi}{9}$ *1 mark all valid solutions.

(b) $3\tan^2 x - 1 = 0$

$3\tan^2 x = 1$

$\tan^2 x = \frac13$

$\tan x = \pm\frac{1}{\sqrt3}$ *1 mark $\tan x$ the subject.

$x = \frac{\pi}{6}+2k\pi,\ \frac{5\pi}{6}+2k\pi,\ \frac{7\pi}{6}+2k\pi,\ \frac{11\pi}{6}+2k\pi$

$x = \frac{(12k+1)\pi}{6},\ \frac{(12k+5)\pi}{6},\ \frac{(12k+7)\pi}{6},\ \frac{(12k+11)\pi}{6}$ for $k\in\mathbb{Z}$.

On $-\pi\le x\le\pi$ *1 mark x the subject.

$x = -\frac{5\pi}{6},\ -\frac{\pi}{6},\ \frac{\pi}{6},\ \frac{5\pi}{6}$

*1 mark all valid solutions.

4

(c) $\sin 2x = \cos x$

$2\sin x\cos x = \cos x$ *1 mark for use of result

$2\sin x\cos x - \cos x = 0$

$\cos x(2\sin x - 1) = 0$

$\cos x = 0$ or $\sin x = \frac12$ *1 mark for both statements.

If $\cos x = 0$ $x = \frac{\pi}{2},\ \frac{3\pi}{2}$

If $\sin x = \frac12$ $x = \frac{\pi}{6},\ \frac{5\pi}{6}$ *2 marks (1 for each properly followed through

$\therefore x = \frac{\pi}{6},\ \frac{\pi}{2},\ \frac{5\pi}{6},\ \frac{3\pi}{2}.$

(Hence if $\cos x = 0$ solution is lost through cancelling there are only 2 marks available.)

⑥ (a) $2\sin\theta + \cos\theta$

$= 2\left(\frac{2t}{1+t^2}\right) + \frac{1-t^2}{1+t^2}$ *1 mark for substitution

$= \dfrac{4t+1-t^2}{1+t^2}$ *1 mark for single fraction

(b) $\cos 2\theta = \cos^2\theta - \sin^2\theta$ *1 mark use of result

$= \left(\frac{1-t^2}{1+t^2}\right)^2 - \left(\frac{2t}{1+t^2}\right)^2$ *1 mark substitution

$= \dfrac{1-2t^2+t^4-4t^2}{1+2t^2+t^4}$

$= \dfrac{1+t^4-6t^2}{1+2t^2+t^4}$ *1 mark any correct simplification to one fraction.

(7) $y = x+1 \Rightarrow m_1 = 1$ (5)

$x - 2y + 1 = 0$ * 1 mark gradients

$x + 1 = 2y$

$y = \frac{1}{2}x + \frac{1}{2} \Rightarrow m_2 = \frac{1}{2}$

$\tan\theta = \left|\dfrac{m_2 - m_1}{1 + m_1 m_2}\right|$

$= \left|\dfrac{\frac{1}{2} - 1}{1 + \frac{1}{2}\times 1}\right|$ * 1 mark substitution and some simplification done correctly.

$= \left|\dfrac{\left(-\frac{1}{2}\right)}{\left(\frac{3}{2}\right)}\right|$

$= \left|-\frac{1}{3}\right|$

$= \frac{1}{3}$

$\theta = \tan^{-1}\left(\frac{1}{3}\right)$ * 1 mark answer.

$= 18^\circ 26'$ (nearest minute.)

(8) $LHS = \dfrac{2\cos x}{\frac{1}{\sin x} - 2\sin x}$ * 1 mark replacement with $\sin x$ (reduction to $\sin x/\cos x$ problem).

$= \dfrac{2\cos x \sin x}{1 - 2\sin^2 x}$

$= \dfrac{\sin 2x}{\cos 2x}$ * 1 mark use of at least one "$2x$" result

$= \tan 2x$

$= RHS.$ * 1 mark follow through correctly to end.

(9). Let $\cos 2x - \sqrt{3}\sin 2x = A\cos(2x + \alpha)$ (6)

$\cos 2x - \sqrt{3}\sin 2x = A\cos 2x \cos\alpha - A\sin 2x \sin\alpha$

$(1)\cos 2x - (\sqrt{3})\sin 2x = (A\cos\alpha)\cos 2x - (A\sin\alpha)\sin 2x$

We need $A\cos\alpha = 1$ — (1)

$A\sin\alpha = \sqrt{3}$ — (2)

* 1 mark substantial beginning of any correct method

Squaring and adding

$(A\sin\alpha)^2 + (A\cos\alpha)^2 = (\sqrt{3})^2 + 1^2$

$A^2(\sin^2\alpha + \cos^2\alpha) = 3 + 1$

$A^2 = 4$

$A = \pm 2$

$A = 2$ will do.

* 1 mark for value of A or α correctly justified.

Also (2) ÷ (1)

$\dfrac{\cancel{A}\sin\alpha}{\cancel{A}\cos\alpha} = \dfrac{\sqrt{3}}{1}$

S	A
T	C

$\tan\alpha = \sqrt{3}$

$\alpha = \frac{\pi}{3} + 2k\pi,\ \frac{4\pi}{3} + 2k\pi \quad k \in \mathbb{Z}$

$\alpha = \frac{\pi}{3}$ will do.

$\therefore \cos 2x - \sqrt{3}\sin 2x = 2\cos\left(2x + \frac{\pi}{3}\right)$

* 1 mark for final form

(b) Hence (7)

$2\cos\left(2x + \frac{\pi}{3}\right) = \sqrt{2}$ * 1 mark for substitution made the subject

$\cos\left(2x + \frac{\pi}{3}\right) = \frac{\sqrt{2}}{2}$

$\cos\left(2x + \frac{\pi}{3}\right) = \frac{1}{\sqrt{2}}$

$2x + \frac{\pi}{3} = \frac{\pi}{4} + 2k\pi,\ \frac{7\pi}{4} + 2k\pi$

* 1 mark $2x + \frac{\pi}{3}$ made the subject.

$2x = -\frac{\pi}{12} + 2k\pi,\ \frac{17\pi}{12} + 2k\pi$

$2x = \dfrac{(24k-1)\pi}{12},\ \dfrac{(24k+17)\pi}{12}$

$x = \dfrac{(24k-1)\pi}{24},\ \dfrac{(24k+17)\pi}{24} \quad k \in \mathbb{Z}$

On $0 \le x \le 2\pi$

$x = \frac{17\pi}{24},\ \frac{23\pi}{24},\ \frac{41\pi}{24},\ \frac{47\pi}{24}$

* 1 mark correct solutions.