J.M.J.Ch.

MARCELLIN COLLEGE RANDWICK

YEAR 12

MATHEMATICS

HSC ASSESSMENT TASK # 4

2007

Weighting: 10% of HSC Assessment Mark.

STUDENT NAME: ____________________ **MARK:** **/ 30**

Time Allowed: **50 minutes.**

Directions:
- Answer all questions on separate lined paper.
- Show all necessary working.
- Marks may not be awarded for careless or badly arranged work.
- Begin your answers to each new question on a new answer page.

Structure: 2 questions each worth 15 marks – Total 30 marks.

OUTCOMES TO BE ASSESSED:

- H1 – seeks to apply mathematical techniques to problems in a wide range of practical contexts
- H2 – constructs arguments to prove and justify results
- H4 – expresses practical problems in mathematical terms based on simple given models
- H5 – applies appropriate techniques from the study of calculus, trigonometry and series to solve problems
- H8 – uses techniques of integration to calculate areas and volumes

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1

a. Differentiate $5x^2 - \tan 3x$ 1

b. Find the primitive of $\sin\frac{x}{5}$. 1

c. i. Differentiate $y = \log_e(\cos x)$. 1

ii. Hence evaluate $\int_0^{\pi/3} 2\tan x\,dx$ correct to 3 decimal places. 2

iii. Use Simpson's rule with 3 function values to estimate $\int_0^{\pi/3} 2\tan x\,dx$. 3

d. i. Sketch the curves of $y = \sin 2x$ and $y = \cos 2x$ on the same number plane for $0 \le x \le \pi$. 2

ii. Find the points where the curves intersect each other in the given domain. 2

iii. Calculate the exact area enclosed by the curves for $0 \le x \le \pi$. 3

Question 2

a. In order to study the history of the Earth's climate, a team of scientists drilled an "ice core" in the Antarctic ice sheet. They drilled 5 metres on the first day, a further 7 metres on the second day, a further 9 metres on the third day and so on.

i. Find how many metres they drilled on the 40^{th} day. 1

ii. Find how deep they had drilled after 40 days. 1

iii. Find how many days it took to drill to a depth of 480 metres. 2

b. For the series $\cos^4\vartheta + \cos^4\vartheta\sin^2\vartheta + \cos^4\vartheta\sin^4\vartheta + \ldots$

i. Find the simplest expression for the limiting sum of the series, assuming it exists. 2

ii. For what values of ϑ in the interval $0 \le \vartheta \le 2\pi$ does the limiting sum exist? 2

c. A business owner borrows \$100000 to pay for renovations. The interest is calculated monthly at the rate of 2% per month, and is compounded each month.

The business man intends to repay the loan with interest in two annual instalments \$$M$ at the end of the first and second years.

i. How much does the business man owe at the end of the first month? 1

ii. Write down an expression involving M for the total amount owed by the business man after 12 months, just after the first instalment of \$$M$ has been paid. 2

iii. Find an expression for the amount owed at the end of the second year and deduce that 3

$$M = \frac{100000 \times (1.02)^{24}}{(1.02)^{12} + 1}$$

iv. What is the total interest over the two year period? 1

Year 12 HSC assessment task #4 2007

Question 1

$5x^2 - \tan 3x$

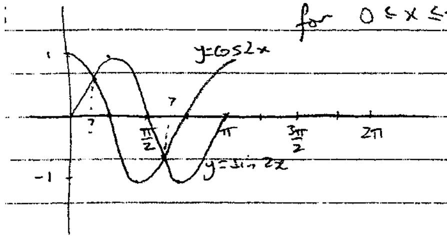
$\frac{d}{dx} = 10x - 3\sec^2 3x$

$\int \sin \frac{1}{5}x \, dx$

$= -\frac{1}{1/5}\cos \frac{1}{5}x + C$

$= -5\cos \frac{1}{5}x + C$

for $0 \le x \le \pi$



$y = \sin 2x \qquad y = \cos 2x$

$\sin 2x = \cos 2x$

$\frac{\sin 2x}{\cos 2x} = 1$

$\therefore \tan 2x = 1$

$\therefore 2x = \frac{\pi}{4}, \frac{5\pi}{4}$

$\therefore x = \frac{\pi}{8}, \frac{5\pi}{8}$

$\therefore \left(\frac{\pi}{8}, \frac{1}{\sqrt2}\right)$ and $\left(\frac{5\pi}{8}, -\frac{1}{\sqrt2}\right)$

$\int_{\pi/8}^{5\pi/8} \sin 2x \, dx - \int_{\pi/8}^{5\pi/8} \cos 2x \, dx$

$= \left[-\frac{1}{2}\cos 2x\right]_{\pi/8}^{5\pi/8} - \left[\frac{1}{2}\sin 2x\right]_{\pi/8}^{5\pi/8}$

$= \left[-\frac{1}{2}\cos\frac{5\pi}{4} + \frac{1}{2}\cos\frac{\pi}{4}\right] - \left[\frac{1}{2}\sin\frac{5\pi}{4} - \frac{1}{2}\sin\frac{\pi}{4}\right]$

$= \frac{1}{2}\times\frac{1}{\sqrt2} + \frac{1}{2}\times\frac{1}{\sqrt2} - \left[-\frac{1}{2}\times\frac{1}{\sqrt2} - \frac{1}{2}\times\frac{1}{\sqrt2}\right]$

$= \frac{1}{2\sqrt2} + \frac{1}{2\sqrt2} - \left[-\frac{1}{2\sqrt2} - \frac{1}{2\sqrt2}\right]$

$= \frac{2}{2\sqrt2} - \left[-\frac{2}{2\sqrt2}\right]$

$= \frac{1}{\sqrt2} + \frac{1}{\sqrt2}$

$= \frac{2}{\sqrt2}$

c i $y = \log_e(\cos x)$

$\frac{dy}{dx} = \frac{-\sin x}{\cos x}$ or $-\tan x$

ii $\int_0^{\pi/3} 2\tan x \, dx$

$= 2\int_0^{\pi/3} \tan x \, dx$

$= -2\int_0^{\pi/3} -\tan x \, dx$

$= \left[-2\log_e \cos x\right]_0^{\pi/3}$

$= -2\log_e \cos \frac{\pi}{3} + 2\log_e \cos 0$

$= -2\log_e \frac{1}{2} + 0$

$= -2(-0.693)$

$= 1.386$ (to 3 dec. pl.)

iii

x	f(x)
0	0
$\frac{\pi}{6}$	1.15
$\frac{\pi}{3}$	3.46

$A \doteqdot \frac{\pi/6}{3}\{0 + 3.46 + 4(1.15)\}$

$A \doteqdot \frac{\pi}{18}\{3.46 + 4.6\}$

$\doteqdot \frac{\pi}{18} \times 8.06$

$A \doteqdot 1.407$ (to 3 dec pl)

a $5 + 7 + 9 + \ldots$

i $T_n = a + (n-1)d$

$T_{40} = 5 + (39)2$

$T_{40} = 5 + 78$

$T_{40} = 83$ metres

ii $S_{40} = \frac{40}{2}(5 + 83)$

$= 20(88)$

$= 1760$

iii $S_n = 480$

$480 = \frac{n}{2}[2(5) + (n-1)2]$

$960 = n[10 + 2n - 2]$

$960 = n[8 + 2n]$

$960 = 8n + 2n^2$

$0 = 2n^2 + 8n - 960$

$0 = n^2 + 4n - 480$

$0 = (n - 20)(n + 24)$

$\therefore n = 20 \quad n \ne -24$

$\therefore$ It took 20 days to drill 480m

$A_1 = 100000(1.02)$

$\therefore$ \$102000

$A_1 = 100000(1.02)$

$A_2 = 100000(1.02)(1.02)$

$A_3 = 100000(1.02)(1.02)(1.02)$

$\therefore A_{12} = 100000(1.02)^{12} - M$

$A_{24} = [100000(1.02)^{12} - M]1.02^{12} - M$

$A_{24} = 100000(1.02)^{24} - (1.02)^{12}M - M$

$= 100000(1.02)^{24} - M[1 + (1.02)^{12}]$

After 24 months, \$0 is owed

$\therefore$ Let $A_{24} = 0$

$0 = 10000(1.02)^{24} - M[1 + (1.02)^{12}]$

$M[1 + (1.02)^{12}] = 100000(1.02)^{24}$

$M = \frac{100000(1.02)^{24}}{1 + (1.02)^{12}}$

iv $M = \frac{100000(1.02)^{24}}{1 + (1.02)^{12}}$

$M = 70911.19$

$\therefore$ 2 repayments of \$70911.19

$= 141822.38$

$\therefore$ Total interest $= 141822.38 - 100000$

b i $S_\infty = \frac{a}{1-r}$

$a = \cos^4\theta \quad r = \sin^2\theta$

$S_\infty = \frac{\cos^4\theta}{1 - \sin^2\theta}$

$= \frac{\cos^4\theta}{\cos^2\theta}$

$= \cos^2\theta$

ii $1 - \sin^2\theta \ne 0$

or $\cos^2\theta \ne 0$

$\cos^2\theta = 0$

when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$\therefore$ a limiting sum does not exist at $\frac{\pi}{2}$ and $\frac{3\pi}{2}$