ST SPYRIDON COLLEGE

2013
Year 12
HSC Assessment Task 2
Half Yearly Examination
Monday, 8 April

Mathematics

Weighting: 35%

Reading Time: 5 minutes

Working time: 2 hours

Total marks: 70

Topics examined:
Geometrical applications of differentiation
Integration
Logarithmic and exponential functions
Coordinate methods in geometry

Outcomes assessed: P4, H3, H5, H6, H8 and H9

General instructions:
- Write using blue or black pen (black preferred) on the writing paper provided
- Board-approved calculators may be used
- All necessary working should be shown in every question
- Questions are not of equal value
- Full marks may not be awarded for careless or badly arranged work
- Questions are not necessarily arranged in order of difficulty
- Diagrams are not necessarily drawn to scale
- Begin Questions 1, 2, 3 and 4 on a new page
- Use the multiple-choice answer sheet for Questions 5 – 14

Section I
60 marks
Attempt Questions 1 – 4
Allow about 100 minutes for this Section
The questions are of equal value
Answer each question on the writing paper provided.

Marks

Question 1 (15 marks)

(a) Differentiate

(i) $y=e^x\left(e^x+1\right)$ 2

(ii) $y=\dfrac{x}{e^x}$ 2

(iii) $y=x^2\log_e x$. 2

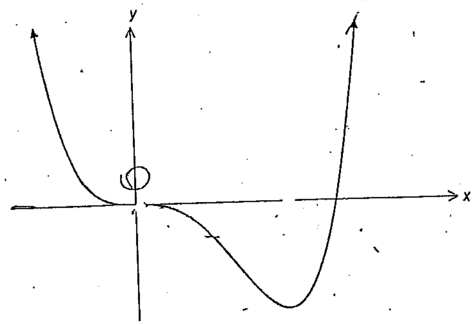
(b) Consider the function $f(x)=e^x-3$.

(i) Find $f(1)$. 1

(ii) Find $f'(1)$. 2

(iii) Hence find the equation of the **tangent** at the point where $x=1$. 2

(c) The diagram shows the graph of $y=g(x)$.



Copy this graph onto your page.

(ii) On the **same** set of axes, draw a sketch of $y=g'(x)$. 4

Question 2 (15 marks) Begin on a new page. **Marks**

(a) Show that $\int_1^6 x(x+1)\,dx = 89\frac{1}{6}$. **3**

(b) Evaluate $\int_1^e \frac{1}{x}\,dx$. **3**

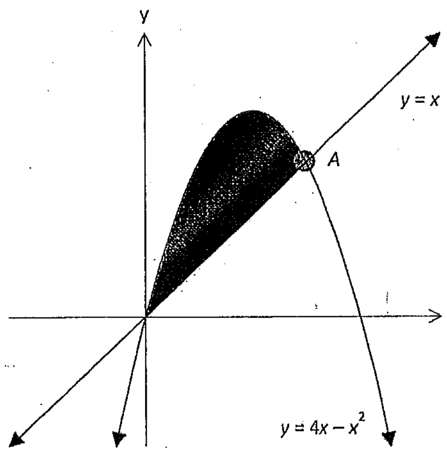
(c) Find

(i) $\int (4x-3)^7\,dx$ **1**

(ii) $\int \frac{x}{x^2+8}\,dx$ **2**

(iii) $\int \frac{1}{e^x}\,dx$. **1**

(d) The diagram shows the graphs of $y = 4x - x^2$ and $y = x$.

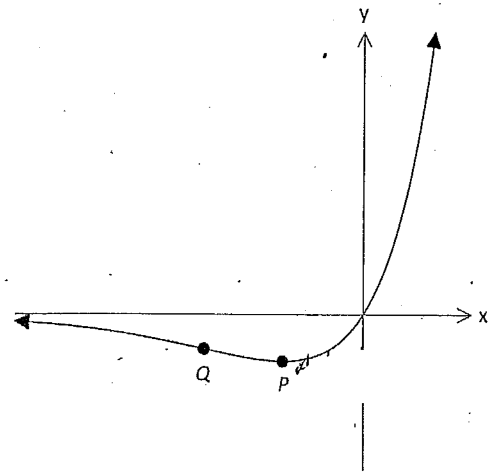


(i) Find the coordinates of the point of intersection, A. **2**

(ii) Calculate the **area** of the shaded region. **3**

Question 3 (15 marks) Begin on a new page. **Marks**

(a) The graph of $y = xe^x$ is shown below. The point P is a minimum turning point and the point Q is a point of inflection.



(i) Differentiate $y = xe^x$. **2**

(ii) Show that P has coordinates $\left(-1, -\frac{1}{e}\right)$. **3**

(iii) Show that $\frac{d^2y}{dx^2} = e^x(x+2)$. **2**

(iv) Find the coordinates of Q. **2**

(v) Determine the **range** of $y = xe^x$. **1**

(vi) For what value(s) of x is this function **decreasing**? **1**

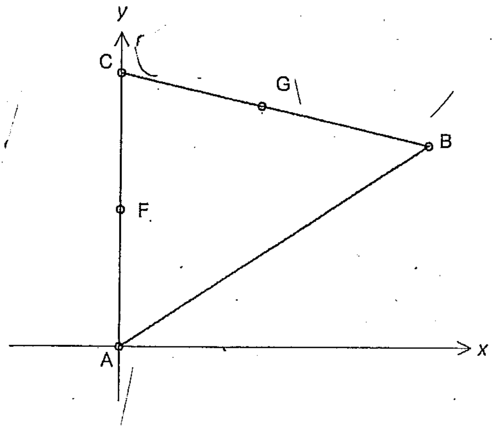
(vii) For what value(s) of x is this function **concave up**? **1**

(viii) Copy and complete the table of values below and then use **one** application of Simpson's rule to find an approximation to the **area** bounded by the curve, the x-axis and the line $x = 1$. Give your answer correct to 2 decimal places. **3**

x	0	0.5	1
xe^x			

Question 4 (15 marks) Begin on a new page **Marks**

(a) The points A (0, 0), B ($2p$, $2q$) and C (0, $2r$) form the triangle ABC, as shown. The point F is the mid-point of AC and the point G is the mid-point of BC.

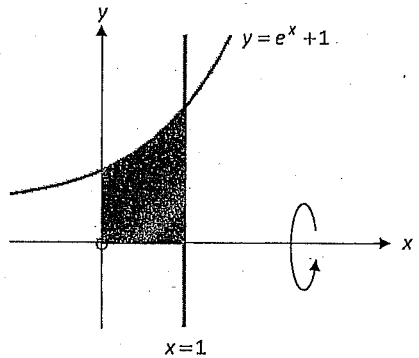


(i) Find the coordinates of the points F and G. **2**

(ii) Show that the intervals FG and AB are **parallel**. **2**

(iii) Show that $FG = \frac{1}{2}AB$. **2**

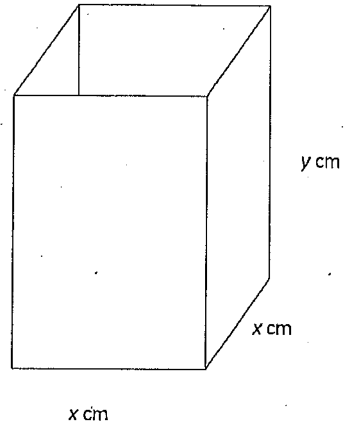
(b) In the diagram below, the **shaded** region is bounded by the curve $y = e^x + 1$, the x-axis, the y-axis and the line $x = 1$.



Find the exact **volume** of the solid formed when the region is rotated about the **x-axis**. **4**

Question (continued) **Marks**

(c) The base of an **open** box is a square of side length x cm. The height of the box is y cm, as shown in the diagram.



(i) The volume of the box is 32 cm^3. 2 Show that $y = \frac{32}{x^2}$. **1**

(ii) Show that the **surface area (S)** of the box is given by $S = x^2 + \frac{128}{x}$. **1**

(iii) Find the value of x that **minimises** the surface area. **3**

End of Section I

Section II

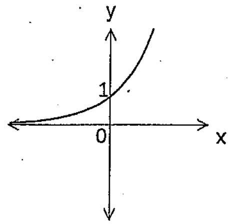
10 marks
Attempt Questions 5 – 14
Allow about 20 minutes for this Section

Use the multiple-choice Answer Sheet for Questions 5 – 14.

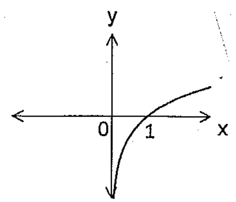
Question 5

Which of the following graphs could represent $y = e^{-x}$?

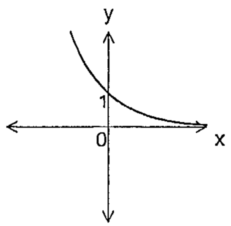
A



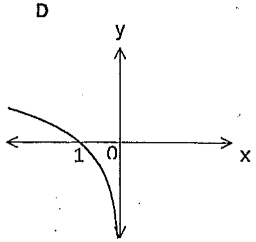
B



C



D



Question 6

The gradient of the **tangent** to $y = \log_e x$ at the point (1, 0) is given by

A 0   B 1   C e   D $\frac{1}{e}$

Question 7

The expression $e^{\ln x} =$

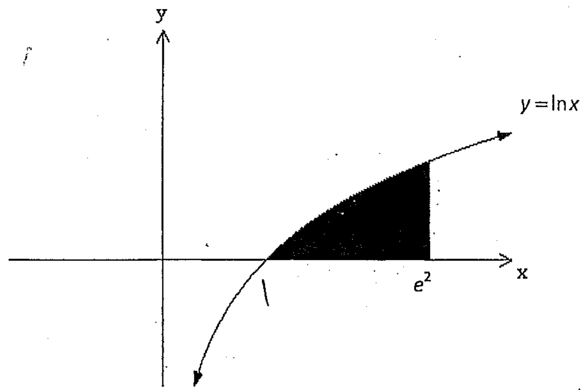
A 1   B e   C x   D $\ln x$

Question 8

The solution to the exponential equation $3^x = 50$ is

A $\frac{\log 50}{\log 3}$ B $\frac{\log 3}{\log 50}$ C log3   D log50

Question 9



The diagram above shows the graph of the function $y = \ln x$. The correct expression for an approximation to the area of the shaded region using one application of the trapezoidal rule is

A $\frac{e^2}{2}[0+1]$ B $\frac{e^2}{2}[0+2]$ C $\frac{e^2-1}{2}[0+1]$ D $\frac{e^2-1}{2}[0+2]$

Question 10

The function $y = e^x - 2$ has range

A $y \ge -2$ B $\cancel{y \ge -1}$ C $y > -2$ D $y > -1$

Question 11

If $\log_a N = y$ then

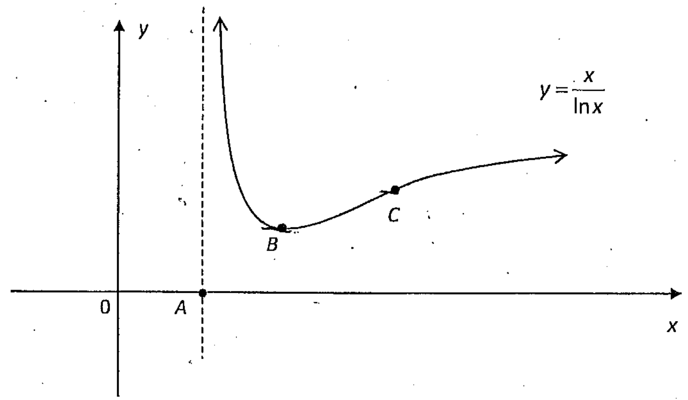
A $a^N = y$ B $a^y = N$ C $N^a = y$ D $y^a = N$

Question 12

$\ln\sqrt{x} =$

A $\frac{1}{2}x$ B $x^{\frac{1}{2}}$ C $\frac{1}{2}\ln x$ D $\frac{1}{2} + x$

Please refer to this diagram for Questions 13 and 14



The diagram shows **part** of the graph of $y = \frac{x}{\ln x}$. A vertical asymptote passes through the point A, the point B is a local minimum and the point C is a point of inflection.

Question 13

The x-coordinate of A is

A 0 B 1 C e D undefined

Question 14

Between the points B and C, the graph of $y = \frac{x}{\ln x}$ is

A Stationary and concave up

B Stationary and concave down

C Increasing and concave up

D Increasing and concave down

End of Examination

Suggested Solutions

Mathematics (2u) HSC T2 – 2013

Question 1

(a) (i) $y = e^x(e^x+1)$

$= e^{2x} + e^x$

$\frac{dy}{dx} = 2e^{2x} + e^x$

(ii) $y = \frac{x}{e^x}$

$\frac{dy}{dx} = \frac{e^x.1 - x.e^x}{e^{2x}}$

$= \frac{e^x(1-x)}{e^{2x}}$

$= \frac{1-x}{e^x}$

(iii) $y = x^2\log_e x$

$\frac{dy}{dx} = 2x.\log_e x + x^2.\frac{1}{x}$

$= 2x\log_e x + x$

(b) Let $f(x) = e^x - 3$

(i) $f(1) = e^1 - 3 = e-3$

(ii) $f'(x) = e^x$

$f'(1) = e^1 = e$

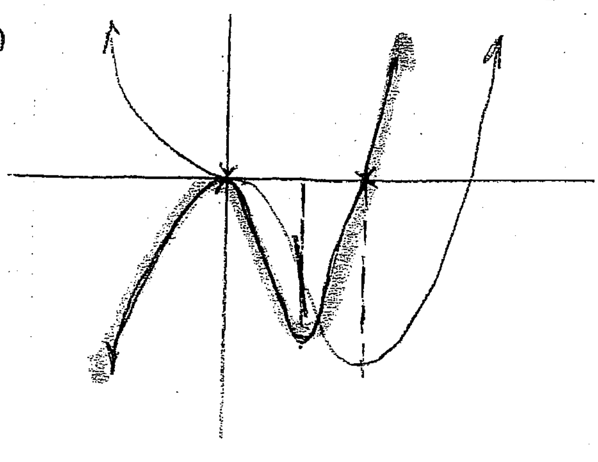
(iii) $y - y_1 = m(x - x_1)$ $m = e$

$y - (e-3) = e(x-1)$ $x_1 = 1$

$y - e + 3 = ex - e$ $y_1 = e-3$

$y = ex - 3$ OR $ex - y - 3 = 0$

(c)



Question 2

(a) $\int_1^6 x(x+1)dx = \int_1^6 (x^2+x)dx$

$= \left[\frac{x^3}{3} + \frac{x^2}{2}\right]_1^6$

$= \frac{216}{3} + \frac{36}{2} - \left(\frac{1}{3} + \frac{1}{2}\right)$

$= \frac{432+108-2-3}{6}$ OR $72+18-\frac{5}{6}$

$= 89\frac{1}{6}$

(b) $\int_1^e \frac{1}{x}dx = [\ln x]_1^e$

$= \ln e - \ln 1$

$= 1 - 0$

$= 1$

(c) (i) $\int (4x-3)^7 dx = \frac{(4x-3)^8}{4\times 8} + C$

$= \frac{1}{32}(4x-8)^8 + C$

(ii) $\int \frac{x}{x^2+8}dx = \frac{1}{2}\int \frac{2x}{x^2+8}dx$

$= \frac{1}{2}\log_e(x^2+8) + C$

(iii) $\int \frac{1}{e^x}dx = \int e^{-x}dx = -e^{-x} + C$

$= \frac{-1}{e^x} + C$

(d) (i) At A $4x - x^2 = x \Rightarrow x(3-x) = 0$

$\Rightarrow x = 0$ or $x = 3$

At A, from the diagram $x=3$, $y=3$

i.e. A is the point (3,3).

(ii) Area $= \int^3 (4x - x^2 - x)\,dx$

$= \int_0^3 (3x - x^2)dx = \left[\frac{3x^2}{2} - \frac{x^3}{3}\right]_0^3 = \frac{27}{2} - \frac{27}{3}$

$= 4\frac{1}{2}$ units^2

Question 3

i) $y = xe^x$

$y' = xe^x + 1.e^x$

$= e^x(x+1)$

ii) At P, $y' = 0$

solve $e^x(x+1) = 0$

$e^x \neq 0$, $x = -1$

when $x = -1$, $y = -1.e^{-1} = \frac{-1}{e}$

$\Rightarrow$ P has coordinates $\left(-1, -\frac{1}{e}\right)$

iii) $y' = e^x(x+1)$

$y'' = e^x(x+1) + e^x$

$= e^x(x+2)$

iv) at Q, $y'' = 0$

solve $e^x(x+2) = 0$

$e^x \neq 0$ $x = -2$

when $x = -2$, $y = -2.e^{-2} = \frac{-2}{e^2}$

$\Rightarrow$ Q has coordinates $\left(-2, -\frac{2}{e^2}\right)$

(v) Range: $y \geq -\frac{1}{e}$

(vi) Solve $y' < 0$ and/or look at graph (and answers above)

This f. is decreasing when $x < -1$

(vii) This f. is concave up when $x > -2$

(viii)

x	0	0.5	1
xe^x	0	$\frac{1}{2}\sqrt{e}$	e

$A = \int_0^1 xe^x\,dx \approx \frac{0.5}{3}\left[0 + 4\left(\frac{1}{2}\sqrt{e}\right) + e\right]$

$= 1.00262\ldots$

$= 1.00$ (2d.pl.)

Question 4

(a)(i) $F = \left(\frac{0+0}{2}, \frac{0+2r}{2}\right) = (0, r)$

$G = \left(\frac{2p+0}{2}, \frac{2q+2r}{2}\right) = (p, q+r)$

(ii) $m_{FG} = \frac{q+r-r}{p-0} = \frac{q}{p}$

$m_{AB} = \frac{2q-0}{2p-0} = \frac{2q}{2p} = \frac{q}{p} = m_{FG}$

$\Rightarrow FG \parallel AB$

(iii) Method 1 $\triangle CFG \,|||\, \triangle CAB$ (equiangular)

(since $\angle FGC = \angle ACB$ common

and $\angle CFG = \angle CAB$, corresp. angles on parallel lines)

$\Rightarrow \frac{FG}{AB} = \frac{CF}{CA} = \frac{1}{2}$ (matching sides in proportion)

$\Rightarrow FG = \frac{1}{2}AB$

method 2

$FG^2 = (p-0)^2 + (q+r-r)^2 = p^2 + q^2$

$AB^2 = (2p-0)^2 + (2q-0)^2 = 4p^2 + 4q^2$

$\Rightarrow 4FG^2 = AB^2$

$2FG = AB$

$FG = \frac{1}{2}AB$

(b) $V = \pi\int_0^1 y^2\,dx$ $\quad y = e^x + 1$

$y^2 = e^{2x} + 2e^x + 1$

$= \pi\int_0^1 (e^{2x} + 2e^x + 1)\,dx$

$= \pi\left[\frac{1}{2}e^{2x} + 2e^x + x\right]_0^1$

$= \pi\left(\frac{1}{2}e^2 + 2e + 1 - \left(\frac{1}{2}e^0 + 2e^0 + 0\right)\right)$

$= \pi\left(\frac{1}{2}e^2 + 2e + 1 - 2\frac{1}{2}\right)$

$= \pi\left(\frac{1}{2}e^2 + 2e - 1\frac{1}{2}\right)$ u^3

Question 4 (continued)

(c) (i) $V = x.x.y = x^2y$

and $V = 32$

$\Rightarrow x^2y = 32$

$\Rightarrow y = \frac{32}{x^2}$

(ii) $S = x.x + 4.x.y$

$= x^2 + 4xy$

$= x^2 + 4x.\frac{32}{x^2}$

$= x^2 + \frac{128}{x}$

(iii) $S = x^2 + 128.x^{-1}$

$\frac{dS}{dx} = 2x - 128.x^{-2}$

$= 2x - \frac{128}{x^2}$

Solve $\frac{dS}{dx} = 0$

$2x - \frac{128}{x^2} = 0$

$2x^3 = 128$

$x^3 = 64$

$x = 4$

$\frac{d^2S}{dx^2} = 2 + 256x^{-3}$

$= 2 + \frac{256}{x^3}$

When $x = 4$, $\frac{d^2S}{dx^2} > 0$

[Note: $\frac{d^2S}{dx^2} > 0$ for all $x > 0$]

And so when $x = 4$, the surface area is minimised

Q5 C

Q6 B

Q7 C

Q8 A

Q9 D

Q10 C

Q11 B

Q12 C

Q13 B

Q14 C