Name: .... Maths Class: ....

SYDNEY TECHNICAL HIGH SCHOOL

Year 12

Mathematics Extension 1

HSC Course

Assessment 3

June, 2015

Time allowed: 90 minutes

General Instructions:

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A set of Standard Integrals is provided at the rear of this Question Booklet, and may be removed at any time.

Section 1 Multiple Choice
Questions 1-5
5 Marks

Section II Questions 6-11
55 Marks

STANDARD INTEGRALS

$\int x^n dx$	$= \frac{1}{n+1} x^{n+1}, \ n \neq -1; x \neq 0, \text{if } n < 0$
$\int \frac{1}{x} dx$	$= \ln x, \ x > 0$
$\int e^{ax} dx$	$= \frac{1}{a} e^{ax}, \ a \neq 0$
$\int \cos ax \, dx$	$= \frac{1}{a} \sin ax, \ a \neq 0$
$\int \sin ax \, dx$	$= -\frac{1}{a} \cos ax, \ a \neq 0$
$\int \sec^2 ax \, dx$	$= \frac{1}{a} \tan ax, \ a \neq 0$
$\int \sec ax \tan ax \, dx$	$= \frac{1}{a} \sec ax, \ a \neq 0$
$\int \frac{1}{a^2 + x^2} dx$	$= \frac{1}{a} \tan^{-1} \frac{x}{a}, \ a \neq 0$
$\int \frac{1}{\sqrt{a^2 - x^2}} dx$	$= \sin^{-1} \frac{x}{a}, \ a > 0, \ -a < x < a$
$\int \frac{1}{\sqrt{x^2 - a^2}} dx$	$= \ln\left(x + \sqrt{x^2 - a^2}\right), \ x > a > 0$
$\int \frac{1}{\sqrt{x^2 + a^2}} dx$	$= \ln\left(x + \sqrt{x^2 + a^2}\right)$

NOTE: $\ln x = \log_e x, \ x > 0$

SECTION 1

5 Marks

Attempt questions 1-5

Use multiple choice answer sheet

1. The inverse function of $g(x)$, where $g(x)=\sqrt{2x-4}$ is

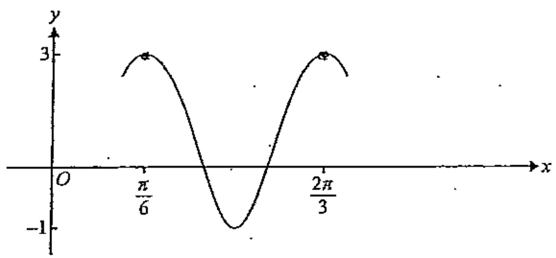
(A) $g^{-1}(x)=\dfrac{x^2+4}{2}$

(B) $g^{-1}(x)=(2x-4)^2$

(C) $g^{-1}(x)=\sqrt{\dfrac{x}{2}+4}$

(D) $g^{-1}(x)=\dfrac{x^2-4}{2}$

2. The graph below could have the equation



(A) $y=2\cos\left(x+\dfrac{\pi}{6}\right)+1$

(B) $y=2\cos 2\left(x+\dfrac{\pi}{6}\right)+1$

(C) $y=2\cos 4\left(x-\dfrac{\pi}{6}\right)+1$

(D) $y=2\cos 4\left(x+\dfrac{2\pi}{3}\right)+1$

3. The domain and range of the function $f(x)$, where $f(x)=3\sin^{-1}(4x-1)$ are respectively.

(A) $0\le x\le\dfrac{1}{2}$ and $-\dfrac{3\pi}{2}\le y\le\dfrac{3\pi}{2}$

(B) $-\dfrac{1}{2}\le x\le 0$ and $-\dfrac{\pi}{2}\le y\le\dfrac{5\pi}{2}$.

(C) $0\le x\le\dfrac{1}{2}$ and $-\dfrac{\pi}{2}\le y\le\dfrac{5\pi}{2}$

(D) $-\dfrac{1}{2}\le x\le 0$ and $-\dfrac{3\pi}{2}\le y\le\dfrac{3\pi}{2}$.

4. If the substitution $u=x^2-1$ is used then the definite integral $\displaystyle\int_0^2\frac{x}{\sqrt{x^2-1}}dx$ can be simplified to

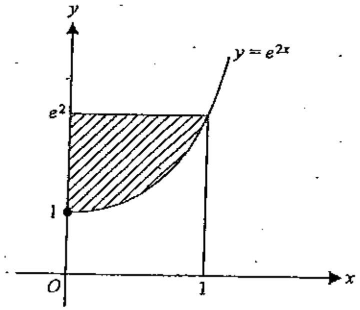
(A) $\dfrac{1}{2}\displaystyle\int_{-1}^{3}u^{-\frac{1}{2}}\,du$

(B) $2\displaystyle\int_{-1}^{3}u^{-\frac{1}{2}}\,du$

(C) $\dfrac{1}{2}\displaystyle\int_{0}^{2}u^{-\frac{1}{2}}\,du$

(D) $2\displaystyle\int_{0}^{2}u^{-\frac{1}{2}}\,du$

5. To find the area of the shaded region in the diagram below, four different students proposed the following calculations.



Student 1: $\displaystyle\int_0^1 e^{2x}\,dx$

Student 2: $e^2-\displaystyle\int_0^1 e^{2x}\,dx$

Student 3: $\displaystyle\int_1^{e^2} e^{2y}\,dy$

Student 4: $\displaystyle\int_1^{e^2}\frac{\log_e y}{2}\,dy$

Which of the following is correct?

(A) Student 2 only.

(B) Students 2 and 3 only.

(C) Students 2 and 4 only.

(D) Students 1 and 4 only.

SECTION II

Question 6 (9 Marks) — Mark

a) Differentiate

i) $e^{\sin x}$ — 1

ii) $\ln(\cos x)$ — 1

iii) $\sin^{-1}\sqrt{x}$ — 2

b) Find the <u>exact</u> values of

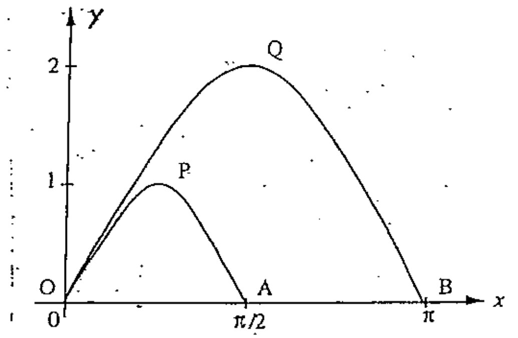
i) $\cos^{-1}(-\frac{\sqrt{3}}{2})$ — 1

ii) $\tan^{-1}(2\cos\frac{5\pi}{6})$ — 2

c) Evaluate $\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$ — 2

Question 7 (9 Marks) (Start a new page)

a) — 3



The diagram shows portions of the graphs of

$$y = 2\sin x \text{ and } y = \sin 2x$$

Calculate the area of the region bounded by the arc OPA, the arc OQB and the interval AB.

Mark

b) i) Find $\frac{d}{dx}(x\ln x)$ — 2

ii) Hence prove $\int_e^{e^2} \frac{1+\ln x}{x\ln x}\,dx = 1+\ln 2$ — 2

c) i) Write $\cos 2x$ in terms of $\sin^2 x$ — 1

ii) Hence or otherwise find — 1

$$\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$$

Question 8 (10 Marks) (Start a new page)

a) If $y = a.10^{bx}$ make x the subject — 2

b) Find the general solution of $\sin x = -\frac{1}{2}$ — 2

c) The gradient function of a curve is given by $\frac{dy}{dx} = \frac{2}{x+1}$. If the curve passes through the point (0, 1), find the equation of the curve. — 2

d) i) Express $\sin^2 x\cos^2 x$ in terms of $\sin 2x$ — 1

ii) Hence find $\int \sin^2 x.\cos^2 x\,dx$ — 3

Question 9 (9 Marks) (Start a new page)

a) i) Sketch $g(x) = (x-2)^2 - 3$ showing and labelling the vertex and y intercept. — 1

ii) What is the largest domain containing $x = 0$ for which $g(x)$ has an inverse? — 1

iii) Find the inverse function $g^{-1}(x)$ and sketch it on your diagram showing where it cuts the x axis. — 2

Label your curve clearly

Mark

b) i) Differentiate $y = \cos^{-1} x + \sin^{-1} x$ 1

ii) Hence sketch $y = \cos^{-1} x + \sin^{-1} x$ 2

(Label both axes and show a suitable scale)

c) Find $\int \sec^2 x . \tan x \, dx$ by using the substitution $u = \tan x$ or otherwise. 2

Question 10 (8 Marks) (Start a new page)

a) i) Sketch $y = 1 - \frac{2}{x}$ (do not use calculus) and indicate on your sketch any asymptotes and where the curve cuts the x axis. 2

ii) The region bounded by the curve and the x axis from $x = 1$ to $x = 2$ is rotated around the x axis 2

Show the volume generated is π (3-4 ln2) units^3

b) i) Find the co-ordinates of the stationary point on the graph of $y = \frac{e^x}{x^2+1}$ and prove it is neither a maximum nor a minimum. 2

ii) Sketch $y = \frac{e^x}{x^2+1}$ showing the stationary point and where the curve cuts the y axis and any asymptotes. 2

Question 11 (10 Marks) (Start a new page)

a) i) Prove $\frac{1}{x-2} - \frac{1}{x+2} = \frac{4}{x^2-4}$ 1

ii) Hence find $\int_3^6 \frac{1}{x^2-4} \, dx$ in exact form 2

b) i) If $\tan^{-1} x = \alpha$ and $\tan^{-1} y = \beta$ prove that 2

$$\tan^{-1} x + \tan^{-1} y = \tan^{-1}\left[\frac{x+y}{1-xy}\right]$$

ii) Hence evaluate $\tan^{-1}(\frac{1}{2}) + \tan^{-1}(\frac{1}{3})$ 1

Mark

c) i) Prove that $\frac{d}{dx}(x^2 \tan^{-1} x)$ may be written as

$2x \tan^{-1} x + 1 - \frac{1}{x^2+1}$ 2

ii) Hence find $\int_0^{\sqrt{3}} x . \tan^{-1} x \, dx$ in exact form 2

$\therefore \int \sec^2 x \cdot \tan x \, dx$

$= \int \sec^2 x \cdot u \cdot \frac{1}{\sec^2 x} \, du$

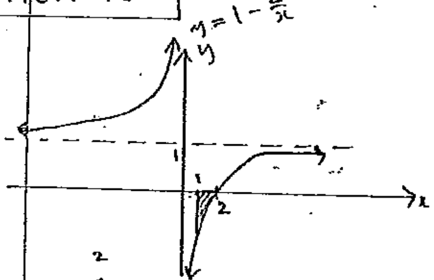
$= \int u \, du$

$= \frac{u^2}{2} + c$

$= \frac{\tan^2 x}{2} + c$

Question 10

a) i)



$V = \pi \int_1^2 \left(1 - \frac{2}{x}\right)^2 dx$

$= \pi \int_1^2 \left(1 - \frac{4}{x} + \frac{4}{x^2}\right) dx$

$= \pi \left[x - 4\ln x - \frac{4}{x} \right]_1^2$

$= \pi \left[(2 - 4\ln 2 - 2) - (1 - 4\ln 1 - 4) \right]$

$= \pi \left[-4\ln 2 + 3 \right]$

$= \pi \left[3 - 4\ln 2 \right]$

b) i) $y = \frac{e^x}{x^2+1}$

$u = e^x \qquad v = x^2 + 1$

$u' = e^x \qquad v' = 2x$

$\frac{dy}{dx} = \frac{e^x(x^2+1) - 2x e^x}{(x^2+1)^2}$

$= \frac{e^x(x^2 + 1 - 2x)}{(x^2+1)^2}$

st pt if $\frac{dy}{dx} = 0$

$\therefore x^2 - 2x + 1 = 0$

$(x-1)^2 = 0$

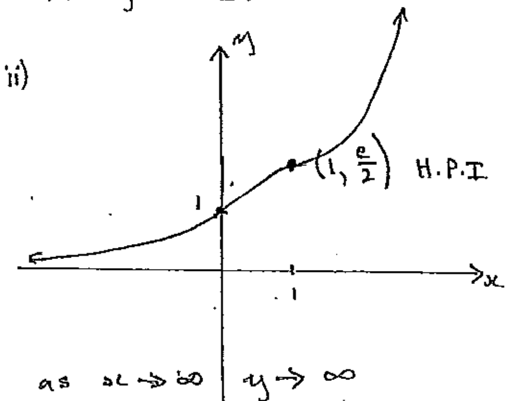
$x = 1$

test max/min

x	0	1	2
y'	+	0	+

$\therefore$ gradient +ve on either side of $x = 1$ $\therefore$ H.P.I on a rising curve.

ii)



as $x \to \infty \quad y \to \infty$

as $x \to -\infty \quad y \to {}^{+}0$

Question 11

a) i) LHS $= \frac{1}{x-2} - \frac{1}{x+2}$

$= \frac{(x+2) - (x-2)}{(x-2)(x+2)}$

$= \frac{4}{x^2 - 4}$

$=$ RHS

ii) $\int_3^6 \frac{1}{x^2-4} \, dx = \frac{1}{4} \int_3^6 \frac{4}{x^2-4} \, dx$

$= \frac{1}{4} \int_3^6 \frac{1}{x-2} - \frac{1}{x+2} \, dx$

$= \frac{1}{4} \left[\ln(x-2) - \ln(x+2) \right]_3^6$

$= \frac{1}{4} \left[\ln\left(\frac{x-2}{x+2}\right) \right]_3^6$

$= \frac{1}{4} \left[\ln 4/8 - \ln 1/5 \right]$

$= \frac{1}{4} \ln 5/2$

b) i) $\alpha = \tan^{-1} x \qquad \beta = \tan^{-1} y$

$\tan\alpha = x \qquad \tan\beta = y$

$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \cdot \tan\beta}$

$= \frac{x + y}{1 - xy}$

$\therefore \alpha + \beta = \tan^{-1}\left[\frac{x+y}{1-xy}\right]$

$\therefore \tan^{-1} x + \tan^{-1} y = \tan^{-1}\left[\frac{x+y}{1-xy}\right]$

ii $\therefore \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right)$

$= \tan^{-1}\left[\frac{1/2 + 1/3}{1 - \frac{1}{2} \cdot \frac{1}{3}}\right]$

$= \tan^{-1}\left[\frac{5/6}{5/6}\right]$

$= \tan^{-1} 1$

$= \pi/4$

c) $u = x^2 \qquad v = \tan^{-1} x$

i) $u' = 2x \qquad v' = \frac{1}{1+x^2}$

$\therefore \frac{d}{dx}\left(x^2 \cdot \tan^{-1} x\right) = 2x \cdot \tan^{-1} x + \frac{x^2}{1+x^2}$

$= 2x \cdot \tan^{-1} x + \frac{1+x^2}{1+x^2} - \frac{1}{1+x^2}$

$= 2x \cdot \tan^{-1} x + 1 - \frac{1}{1+x^2}$

ii) $\left[\frac{d}{dx}\left(x^2 \cdot \tan^{-1} x\right)\right] - 1 + \frac{1}{1+x^2} = 2x \tan^{-1} x$

$\therefore \int_0^{\sqrt{3}} x \tan^{-1} x \, dx = \frac{1}{2}\left\{ \left[x^2 \cdot \tan^{-1} x\right]_0^{\sqrt{3}} - \int_0^{\sqrt{3}} 1 - \frac{1}{1+x^2} \right\}$

$= \frac{1}{2}\left\{ \left[3\tan^{-1}\sqrt{3}\right] - \left[x - \tan^{-1} x\right]_0^{\sqrt{3}} \right\}$

$= \frac{1}{2}\left\{ 3 \cdot \frac{\pi}{3} - \left(\sqrt{3} - \tan^{-1}\sqrt{3}\right) \right\}$

$= \frac{1}{2}\left\{ \pi - \sqrt{3} + \frac{\pi}{3} \right\}$

$= \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$

Q	
1	A
2	C
3	A
4	A
5	C

Question 6

a) i) $\frac{d}{dx}(e^{\sin x}) = \cos x . e^{\sin x}$

ii) $\frac{d}{dx}(\ln(\cos x)) = \frac{-\sin x}{\cos x}$ OR $= -\tan x$

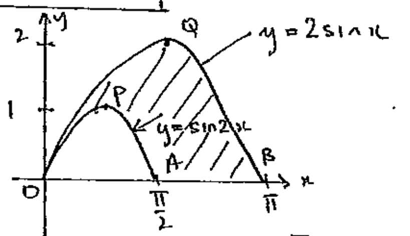
iii) $\frac{d}{dx}(\sin^{-1} x^{1/2}) = \frac{1/2\, x^{-1/2}}{\sqrt{1-x}}$ OR $= \frac{1}{2\sqrt{x}\sqrt{1-x}}$

b) i) $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$
$= \pi - \cos^{-1}\frac{\sqrt{3}}{2}$
$= \pi - \frac{\pi}{6}$
$= \frac{5\pi}{6}$

ii) $\tan^{-1}\left(2\cos\frac{5\pi}{6}\right)$
$\cos\frac{5\pi}{6} = -\cos\frac{\pi}{6}$
$= -\sqrt{3}/2$
$\therefore \tan^{-1}\left(2 \cdot -\frac{\sqrt{3}}{2}\right)$
$\tan^{-1}(-\sqrt{3})$
$-\tan^{-1}\sqrt{3}$
$-\pi/3$

c) $\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx = \left[\sin^{-1}\frac{x}{2}\right]_0^1$
$= \sin^{-1}1/2 = \pi/6$

Question 7



$A = \int_0^{\pi} 2\sin x\,dx - \int_0^{\pi/2} \sin 2x\,dx$
$= \left[-2\cos x\right]_0^{\pi} - \left[-\frac{1}{2}\cos 2x\right]_0^{\pi/2}$
$= (2 - -2) - \left(\frac{1}{2} - -\frac{1}{2}\right)$
$= 3 \text{ unit}^2$

b) $u = x \quad v = \ln x$

i) $u' = 1 \quad v' = \frac{1}{x}$
$\therefore \frac{d}{dx}(x\ln x) = \ln x + 1$

ii) $\int_e^{e^2} \frac{1+\ln x}{x\ln x} = \left[\ln(x\ln x)\right]_e^{e^2}$
$= \ln(e^2\ln e^2) - \ln(e\ln e)$
$= \ln(e^2.2\ln e) - 1$
$= \ln 2e^2 - 1$
$= \ln 2 + \ln e^2 - 1$
$= \ln 2 + 2\ln e - 1$
$= \ln 2 + 1$

c) i) $\cos 2x = \cos^2 x - \sin^2 x$
$= 1 - 2\sin^2 x$

ii) $\lim_{x\to 0} \frac{1-\cos 2x}{x^2} = \frac{1-(1-2\sin^2 x)}{x^2}$
$= \lim_{x\to 0} \frac{2\sin^2 x}{x^2} = 2$

Question 8

a) $y = a.10^{bx}$
$\log_{10} y = \log_{10}(a.10^{bx})$
$\log_{10} y = \log_{10} a + \log_{10} 10^{bx}$
$\log_{10} y = \log_{10} a + bx\log_{10}10$
$bx = \log_{10} y - \log_{10} a$
$bx = \log_{10}\left(\frac{y}{a}\right)$
$\therefore x = \frac{1}{b}\log_{10}\left(\frac{y}{a}\right)$

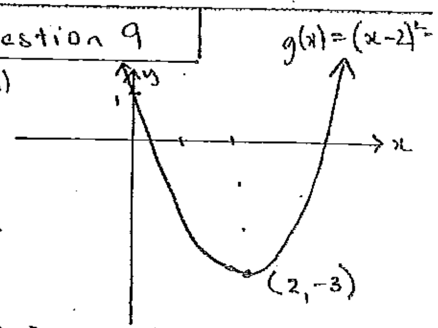
b) $\sin x = -\frac{1}{2} \therefore x = n\pi + (-1)^n \sin^{-1}\left(-\frac{1}{2}\right)$
$\therefore x = n\pi - (-1)^n\frac{\pi}{6}$
where n is an integer

c) $\frac{dy}{dx} = \frac{2}{x+1}$
$y = 2\ln(x+1) + c$
sub $(0, 1)$ $\therefore 1 = 2\ln 1 + c$
$\therefore c = 1$
curve $y = 2\ln(x+1) + 1$

d). $\sin 2x = 2\sin x.\cos x$
$\therefore \sin x.\cos x = \frac{1}{2}\sin 2x$
$\sin^2 x.\cos^2 x = \frac{1}{4}\sin^2 2x$
$\therefore \int \sin^2 x\cos^2 x\,dx = \frac{1}{4}\int \sin^2 2x\,dx$
$= \frac{1}{4}\int \frac{1}{2}(1-\cos 4x)\,dx$
$= \frac{1}{8}\left[x - \frac{1}{4}\sin 4x\right] + c$

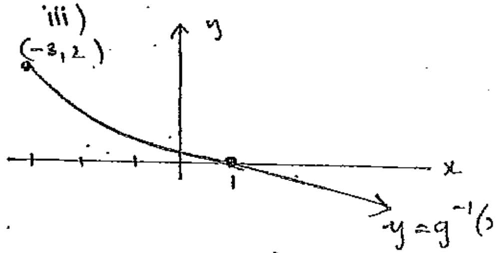
Question 9

a) i)



ii) D: $x \leq 2$

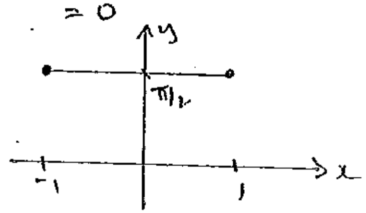
iii)



$x = (y-2)^2 - 3$
$x + 3 = (y-2)^2$
$-\sqrt{x+3} + 2 = y$
$\therefore g^{-1}(x) = -\sqrt{x+3} + 2$

b) i) $\frac{d}{dx}(\cos^{-1}x + \sin^{-1}x)$
$= \frac{-1}{\sqrt{1-x^2}} + \frac{1}{\sqrt{1-x^2}}$
$= 0$

ii)



c) $u = \tan x$
$\frac{du}{dx} = \sec^2 x$
$\therefore dx = \frac{1}{\sec^2 x}$