SYDNEY TECHNICAL HIGH SCHOOL

HIGHER SCHOOL CERTIFICATE
ASSESSMENT TASK 3

JUNE 2015

Mathematics Extension 2

General Instuctions

- Working time - 90 minutes
- Write using black or blue pen
- Approved calculators may be used
- All necessary working should be shown in questions 6 to 9
- Full marks may not be awarded for careless work or illegible writing
- Start each question on a new page
- All answers are to be in the writing booklet provided
- Marks for each question are indicated on the question
- A table of standard integrals is provided at the back of this paper

Total marks - 55

Section 1 - 5 marks

Attempt Questions 1 – 5.
Allow about 10 minutes for this section.

Section 2 - 50 marks

Attempt Questions 6 – 9.
Allow about 80 minutes for this section.

Name : ____________________

Teacher : ____________________

STANDARD INTEGRALS

$\int x^n dx$	$= \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$
$\int \frac{1}{x} dx$	$= \ln x,\ x > 0$
$\int e^{ax} dx$	$= \frac{1}{a}e^{ax},\ a \neq 0$
$\int \cos ax\, dx$	$= \frac{1}{a}\sin ax,\ a \neq 0$
$\int \sin ax\, dx$	$= -\frac{1}{a}\cos ax,\ a \neq 0$
$\int \sec^2 ax\, dx$	$= \frac{1}{a}\tan ax,\ a \neq 0$
$\int \sec ax \tan ax\, dx$	$= \frac{1}{a}\sec ax,\ a \neq 0$
$\int \frac{1}{a^2+x^2} dx$	$= \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$
$\int \frac{1}{\sqrt{a^2-x^2}} dx$	$= \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$
$\int \frac{1}{\sqrt{x^2-a^2}} dx$	$= \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$
$\int \frac{1}{\sqrt{x^2+a^2}} dx$	$= \ln\left(x+\sqrt{x^2+a^2}\right)$

NOTE: $\ln x = \log_e x,\ x > 0$

Section 1 (5 marks)

Attempt Questions 1 – 5

Use the multiple-choice answer sheet in your answer booklet for Questions 1 – 5.
Do not remove the multiple-choice answer sheet from your answer booklet.

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1. The polynomial $P(x)$ of degree 4 has real coefficients.
$P(x)$ has roots α, β, γ and δ and it is known that $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = -10$.

Which of the following must be true ?

(A) $P(x)$ has all its roots real.

(B) $P(x)$ has one real and three imaginary roots.

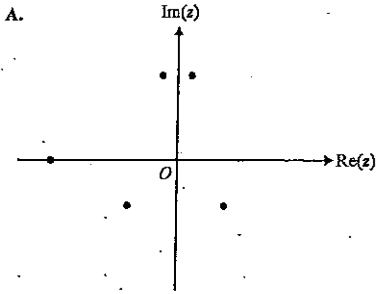
(C) $P(x)$ has two real and two imaginary roots.

(D) $P(x)$ has at least two imaginary roots.

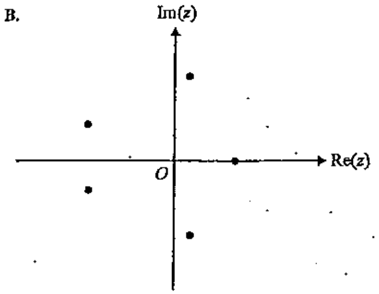
2. Which one of the following diagrams could represent the location of the roots of

$z^5 + z^2 - z + c = 0$ in the complex plane, given that c is real ?

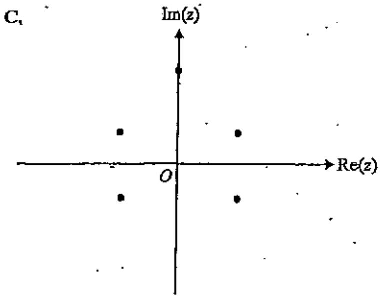
A.



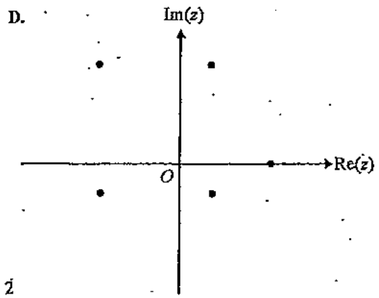
B.



C.



D.



3. With a suitable substitution, $\int_0^{\frac{\pi}{3}} \cos^2 x \sin^3 x \, dx$ can be expressed as

(A) $\int_{0.5}^{1} u^2 - u^4 \, du$

(B) $\int_{1}^{0.5} u^2 - u^4 \, du$

(C) $\int_{0}^{\frac{\pi}{3}} u^2 - u^4 \, du$

(D) $-\int_{0}^{\frac{\sqrt{3}}{2}} u^2 - u^4 \, du$

4. Which one of the following is a primitive function of $\frac{6}{\sqrt{1-4x^2}}$?

(A) $3\sin^{-1}(2x)$

(B) $6\sin^{-1}(2x)$

(C) $12\sin^{-1}(\frac{x}{2})$

(D) $3\sin^{-1}(\frac{x}{2})$

5. $\int_0^{a} (\sin^2(\frac{3x}{2}) - \cos^2(\frac{3x}{2}))dx$ is equal to

(A) $-\frac{4}{3}\sin(\frac{3a}{4})$

(B) $-\frac{1}{3}\sin(3a)$

(C) $\frac{1}{3}\sin(3a)$

(D) $\frac{1}{3}(1 - \sin(3a))$

Section 2 (50 marks)

Attempt Questions 6 – 9
Start each question on a new page

Question 6 (12 marks)

(a) Given the polynomial $P(x) = 3x^4 - 14x^3 + 12x^2 + 24x - 32$ has a triple root, solve the equation $P(x) = 0$. **3**

(b) Find $\int \frac{dx}{x^2-6x+11}$. **3**

(c) Use the substitution $u = -x$ to evaluate $\int_{-1}^{1} \frac{dx}{e^x+1}$. **3**

(d) Using the trigonometric identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, or otherwise, solve the polynomial equation $8x^3 - 6x + 1 = 0$, giving your answers correct to 3 decimal places. **3**

Question 7 (13 marks) (Start a new page in your answer booklet)

(a) Find $\int \frac{dx}{x^2+6x-7}$. **3**

(b) Use the substitution $x = 3\sin\theta$ to evaluate $\int_0^{\frac{3}{\sqrt{2}}} \sqrt{9-x^2}\, dx$. **4**

(c) The polynomial $P(x) = x^3 - 5x^2 + 8x + b$, where b is a constant, has a factor in the form $(x-k)^2$.

(i) Show that the possible values of k are $\frac{4}{3}$ and 2. **3**

(ii) For $k = 2$, find the value of b and hence fully factorise $P(x)$. **3**

Question 8 (13 marks) (Start a new page in your answer booklet)

(a) Evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{5+3\cos x-4\sin x}$ using the substitution $t = \tan\frac{x}{2}$. 4

(b) If α, β and γ are the roots of the equation $x^3+4x^2+3x-3=0$

find the polynomial equation whose roots are

(i) $\frac{\alpha}{2}, \frac{\beta}{2}$ and $\frac{\gamma}{2}$. 2

(ii) $\alpha\beta-1$, $\alpha\gamma-1$ and $\beta\gamma-1$ 3

(c) (i) If $I_n = \int_1^e (1-\ln x)^n\, dx$, $n \geq 0$ 2

show that $I_n = -1 + n\, I_{n-1}$, $n \geq 1$.

(ii) Hence, or otherwise, evaluate $\int_1^e (1-\ln x)^3\, dx$ 2

Question 9 (12 marks) (Start a new page in your answer booklet)

(a) (i) Use the substitution $x = u^2$, $u > 0$, to show that 4

$$\int_4^{16} \frac{\sqrt{x}}{x-1}\, dx = 4 + 2\ln 3 - \ln 5.$$

(ii) Hence use integration by parts to evaluate 2

$$\int_4^{16} \frac{\ln(x-1)}{\sqrt{x}}\, dx$$

(b) (i) Solve the equation $z^5+1=0$ over the complex field, 2

giving the complex roots in the form $r(\cos\theta + i\sin\theta)$.

(ii) If α is the complex root of $z^5+1=0$ with smallest positive argument, 2

show that the other complex roots can be expressed as $-\alpha^2$, α^3 and $-\alpha^4$.

(iii) If α is the complex root of $z^5+1=0$ with smallest positive argument, 2

form the quadratic equation with roots $\alpha-\alpha^4$ and $\alpha^3-\alpha^2$,

giving your answer in the form $ax^2+bx+c=0$.

Extension 2 Solutions - June 2015

1. D 2. B 3. A 4. A 5. B

6. a) $P(x) = 3x^4 - 14x^3 + 12x^2 + 24x - 32$

$P'(x) = 12x^3 - 42x^2 + 24x + 24$

$P''(x) = 36x^2 - 84x + 24$

Solving $36x^2 - 84x + 24 = 0$

$3x^2 - 8x + 2 = 0$

$(3x-1)(x-2) = 0$

$x = \frac{1}{3}, 2$

$P'(\frac{1}{3}) \neq 0$, $P'(2) = 0$

$\therefore$ triple root is $x = 2$

$\therefore$ $2 + 2 + 2 + \alpha = \frac{14}{3}$

$\alpha = -1\frac{1}{3}$

$\therefore$ solutions $2, 2, 2, -1\frac{1}{3}$

b) $\int \frac{dx}{x^2 - 6x + 11}$

$= \int \frac{dx}{(x-3)^2 + 2}$

$= \frac{1}{\sqrt{2}} \tan^{-1}\left(\frac{x-3}{\sqrt{2}}\right) + c$

c) $\int_{-1}^{1} \frac{dx}{e^x + 1}$ $u = -x$, $du = -dx$

$= \int_{1}^{-1} \frac{-du}{e^{-u} + 1}$

$= \int_{-1}^{1} \frac{du}{\frac{1}{e^u} + 1}$

$= \int_{-1}^{1} \frac{e^u \, du}{1 + e^u}$

$= \ln(1 + e^u) \Big]_{-1}^{1}$

$= \ln(1 + e) - \ln(1 + e^{-1})$

$= \ln\left(\frac{1+e}{1+e^{-1}}\right)$

$= \ln\left(\frac{1+e}{\frac{1+e}{e}}\right)$

$= \ln e$

$= 1$

d) $8x^3 - 6x = -1$

$2(4x^3 - 3x) = -1$ let $x = \cos\theta$

$2(4\cos^3\theta - 3\cos\theta) = -1$

$\cos 3\theta = -\frac{1}{2}$

$3\theta = 120^\circ, 240^\circ, 480^\circ$

$\theta = 40^\circ, 80^\circ, 160^\circ$

$\therefore$ $x = \cos 40^\circ, \cos 80^\circ, \cos 160^\circ$

$= 0.766, 0.174, -0.940$

Q7

a) $\displaystyle\int \frac{dx}{x^2+6x-7}$

$= \displaystyle\int \frac{dx}{(x+7)(x-1)}$ $\qquad \dfrac{1}{(x+7)(x-1)} = \dfrac{A}{x+7} + \dfrac{B}{x-1}$

$=$

$\therefore\ 1 = A(x-1) + B(x+7)$

$x=1:\quad 1 = 8B$

$B = \frac{1}{8}$

$x=-7:\quad 1 = -8A$

$A = -\frac{1}{8}$

$= \displaystyle\int \frac{-\frac{1}{8}}{x+7} + \frac{\frac{1}{8}}{x-1}\,dx$

$= -\frac{1}{8}\ln(x+7) + \frac{1}{8}\ln(x-1)$

$= \frac{1}{8}\ln\left(\frac{x-1}{x+7}\right) + c$

b) $\displaystyle\int_0^{\frac{3}{\sqrt{2}}} \sqrt{9-x^2}\,dx$ $\qquad x = 3\sin\theta$

$dx = 3\cos\theta\, d\theta$

$= \displaystyle\int_0^{\frac{\pi}{4}} \sqrt{9-9\sin^2\theta}\cdot 3\cos\theta\,d\theta$

$= 9\displaystyle\int_0^{\frac{\pi}{4}} \sqrt{1-\sin^2\theta}\cdot\cos\theta\,d\theta$

$= 9\displaystyle\int_0^{\frac{\pi}{4}} \cos^2\theta\,d\theta$

$= \frac{9}{2}\displaystyle\int_0^{\frac{\pi}{4}} 1+\cos 2\theta\,d\theta$

$= \frac{9}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{4}}$

$= \frac{9}{2}\left(\frac{\pi}{4} + \frac{1}{2}\right)$

c) i) $(x-k)^2$ is a factor $\Rightarrow$ k is a double root

$\therefore\ P'(x) = 3x^2 - 10x + 8$

$= (3x-4)(x-2)$

$\therefore$ double root is $\frac{4}{3}$ or 2

$\therefore\ k = \frac{4}{3}$ or 2

ii) when $k=2$

$2^3 - 5(2)^2 + 8(2) + b = 0$

$b = -4$

$\therefore$ Sum of roots $\quad 2+2+\alpha = 5$

$\alpha = 1$

$\therefore\ P(x) = (x-2)^2(x-1)$

Q8

a) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{5+3\cos x - 4\sin x}$

$= \displaystyle\int_0^1 \frac{\frac{2\,dt}{1+t^2}}{5 + 3\left(\frac{1-t^2}{1+t^2}\right) - 4\left(\frac{2t}{1+t^2}\right)}$

$= \displaystyle\int_0^1 \frac{2\,dt}{5(1+t^2) + 3(1-t^2) - 4(2t)}$

$= \displaystyle\int_0^1 \frac{2\,dt}{8 - 8t + 2t^2}$

$= \displaystyle\int_0^1 (t-2)^{-2}\,dt$

$= \left[-(t-2)^{-1}\right]_0^1$

$= \frac{1}{2}$

b) i) $y = \frac{x}{2} \Rightarrow x = 2y$

$\therefore P(2y) = 0$ is required polynomial equation

$(2y)^3 + 4(2y)^2 + 3(2y) - 3 = 0$

$8y^3 + 16y^2 + 6y - 3 = 0$.

ii) $\alpha\beta - 1 = \frac{\alpha\beta\gamma}{\gamma} - 1$ $\quad \alpha\gamma - 1 = \frac{3}{\beta} - 1$

$= \frac{3}{\gamma} - 1$ $\quad \beta\gamma - 1 = \frac{3}{\alpha} - 1$

$\therefore y = \frac{3}{x} - 1$

$\frac{3}{x} = y + 1$

$x = \frac{3}{y+1}$

$\therefore P\left(\frac{3}{y+1}\right) = 0$ is required polynomial equation

$\left(\frac{3}{y+1}\right)^3 + 4\left(\frac{3}{y+1}\right)^2 + 3\left(\frac{3}{y+1}\right) - 3 = 0$

$27 + 36(y+1) + 9(y+1)^2 - 3(y+1)^3 = 0$

$27 + 36y + 36 + 9y^2 + 18y + 9 - 3y^3 - 9y^2 - 9y - 3 = 0$

$69 + 45y - 3y^3 = 0$

$y^3 - 15y - 23 = 0$

c) i) $I_n = \int_1^e (1 - \ln x)^n \, dx$

$u = (1 - \ln x)^n$ $\quad v = x$

$u' = n(1 - \ln x)^{n-1}\left(\frac{-1}{x}\right)$ $\quad v' = 1$

$\therefore I_n = \left. x(1 - \ln x)^n \right]_1^e + \int_1^e n(1 - \ln x)^{n-1}\left(\frac{1}{x}\right) dx$

$= e(1 - \ln e) - 1(1 - \ln 1) + n\int_1^e (1 - \ln x)^{n-1} dx$

$= -1 + n I_{n-1}$

ii) $I_3 = -1 + 3 I_2$

$= -1 + 3\left[-1 + 2 I_1\right]$

$= -4 + 6 I_1$

$= -4 + 6\left[-1 + I_0\right]$ $\quad I_0 = \int_1^e (1 - \ln x)^0 dx$

$= -10 + 6(e - 1)$ $\quad = \left[x\right]_1^e$

$= 6e - 16$ $\quad = e - 1$

Q9

a) i) $\int_4^{16} \frac{\sqrt{x}}{x-1}\,dx$

$x = u^2$

$dx = 2u\,du$

$= \int_2^4 \frac{u}{u^2-1} \cdot 2u\,du$

$= \int_2^4 \frac{2u^2\,du}{u^2-1}$

$= 2\int_2^4 \frac{u^2-1+1}{u^2-1}\,du$

$= 2\int_2^4 1 + \frac{1}{u^2-1}\,du$

$\frac{1}{u^2-1} = \frac{A}{u-1} + \frac{B}{u+1}$

$\therefore 1 = A(u+1) + B(u-1)$

$A = \frac{1}{2}, \; B = -\frac{1}{2}$

$= 2\int_2^4 1 + \frac{\frac{1}{2}}{u-1} - \frac{\frac{1}{2}}{u+1}\,du$

$= \left[2u + \ln(u-1) - \ln(u+1)\right]_2^4$

$= (8 + \ln 3 - \ln 5) - (4 + \ln 1 - \ln 3)$

$= 4 + 2\ln 3 - \ln 5$

ii) $\int_4^{16} \frac{\ln(x-1)}{\sqrt{x}}\,dx$

$u = \ln(x-1)$ $\quad v = 2x^{\frac{1}{2}}$

$u' = \frac{1}{x-1}$ $\quad v' = x^{-\frac{1}{2}}$

$= 2\sqrt{x}\ln(x-1)\Big]_4^{16} - 2\int_4^{16} \frac{\sqrt{x}}{x-1}\,dx$

$= (8\ln 15 - 4\ln 3) - 2(4 + 2\ln 3 - \ln 5)$

$= 8\ln 15 - 8\ln 3 + 2\ln 5 - 8$

$= 10\ln 5 - 8$

b) i) $z_1 = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$

$z_2 = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$

$z_3 = -1$

$z_4 = \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}$ or $\cos\frac{3\pi}{5} - i\sin\frac{3\pi}{5}$

$z_5 = \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}$ or $\cos\frac{\pi}{5} - i\sin\frac{\pi}{5}$

ii) $\alpha = \cos\frac{\pi}{5} + i\sin\frac{\pi}{5}$

$z_2 = \cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}$

$= \left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^3$

$= \alpha^3$

$z_4 = \cos\frac{7\pi}{5} + i\sin\frac{7\pi}{5}$

$= \left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^7$

$= \alpha^7$

$= \alpha^5 \cdot \alpha^2$

$= -1 \times \alpha^2$

$= -\alpha^2$

$z_5 = \cos\frac{9\pi}{5} + i\sin\frac{9\pi}{5}$

$= \left(\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}\right)^9$

$= \alpha^9$

$= \alpha^5 \times \alpha^4$

$= -1 \times \alpha^4$

$= -\alpha^4$

iii) quadratic with roots $\alpha-\alpha^4$ and $\alpha^3-\alpha^2$

Sum $= \alpha-\alpha^4+\alpha^3-\alpha^2$

$= 1$ as $-1+\alpha+\alpha^3-\alpha^4-\alpha^2=0$

sum of roots of $z^5+1=0$

product $= (\alpha-\alpha^4)(\alpha^3-\alpha^2)$

$= \alpha^4-\alpha^3-\alpha^7+\alpha^6$

$= \alpha^4-\alpha^3-\alpha^5\cdot\alpha^2+\alpha^5\cdot\alpha$ $(\alpha^5=-1)$

$= \alpha^4-\alpha^3+\alpha^2-\alpha$

$= -(\alpha-\alpha^2+\alpha^3-\alpha^4)$

$= -1$

$\therefore$ required quadratic is $x^2-(\text{sum of roots})x+\text{product}=0$

$x^2-x-1=0$