St Ursula's College
KINGSGROVE

MATHEMATICS EXTENSION 2 – YEAR 12, 2012

ASSESSMENT TASK 1

Name / Student No: ______________________

Teacher: Mr Ratcliffe

Date: Tuesday 6th December, 2011

Time Allowed: 1 hour

Weighting: 15 %

Total Marks: ____ / 48

INSTRUCTIONS

- Attempt all questions.
- Start each question in a new answer booklet.
- Board approved calculators may be used.
- Write using blue or black pen.
- Diagrams must be drawn in pencil.
- All necessary working should be shown in each question.

TEST STRUCTURE

3 questions of equal value (16marks)

Total Marks: ____ / 48

Question 1 *(Start a new sheet of paper)* **MARKS**

(a) Evaluate $\dfrac{10i}{5i^3}$. 1

(b) If $z = 3 + 4i$ and $w = 2 - i$, find in the form of $x + iy$...

(a) zw (b) $z - \overline{w}$ (c) $\dfrac{1}{w}$ 3

(c) (a) Express $1 + i$ in modulus-argument form. 2

(b) Show that $(1 + i)^4$ is a real number. 2

(d) Solve $x^2 + 3x + 5 = 0$. 2

(e) Find $\sqrt{8 - 6i}$ 3

(f) Find in mod-arg form, the solutions of $z^9 = 1$ where $0 \le \arg z \le \dfrac{\pi}{2}$. 2

(g) If $1, w$ and w^2 are the three cube roots of unity prove that...

$$(3w^5 + 3w^4 + 8w^3)^2 = 25$$ 1

Question 2 *(Start a new sheet of paper)* **MARKS**

(a) Solve $z^3 = 4 + 4\sqrt{3}\,i$. 3

(b) Given that $z = x + iy$, simplify $(\bar{z} - (-z))^3$. 2

(c) (i) Use the binomial theorem to expand $(\cos\theta + i\sin\theta)^3$. 1

(ii) Use De Moivre's Theorem and your result from part (i) to prove that

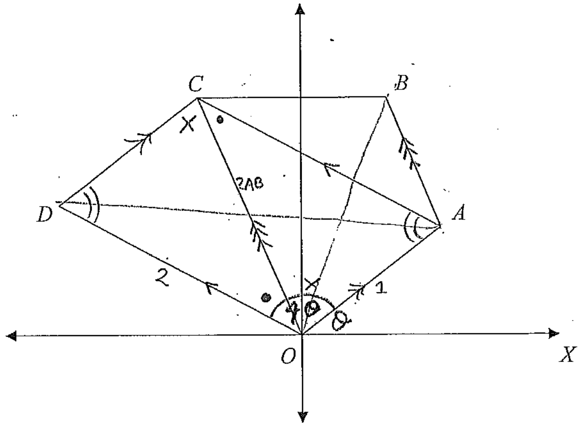
$$\sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$$ 3

(d) Given that $z = x + iy$, $|z| = 10$ and $\arg(z) = \frac{\pi}{6}$, find values for x and y. 2

(e) The point A in the complex plane corresponds to the complex number z.

$OACD$ is a parallelogram ; $OA = 1$; $OD = 2OA$; $OC = 2AB$;

OC is parallel to AB ; $\angle AOX = \theta$; $\angle DOA = 4\theta$



(i) Explain why D corresponds to the complex number $2z^5$. 2

(ii) What complex number in terms of z, corresponds to point C ? 1

(iii) What complex number in terms of z, corresponds to vector AB ? 1

(iv) What complex number in terms of z, corresponds to vector BC ? 1

Question 3 *(Start a new sheet of paper)* **MARKS**

(a) The point P on the Argand diagram represents the complex number $z = x + iy$.
Write down a **complex equation** (i.e. in terms of z) for the locus of P if P represents all points...

(i) which lie 5 units left of the imaginary axis 1

(ii) with $\text{Re}(z) = \text{Im}(z)$ and $\arg(z) > 0$ 1

(iii) that lie above the real axis and on the semi-circle with a diameter that extends from -3 to 3 1

(iv) which are twice as far in distance from $2i$ as they are from 1 1

(v) that lie on the real axis and have a positive argument 1

(b) The point P on the Argand diagram represents the complex number $z = x + iy$.
The locus of z is an arc of the circle with its centre at the origin.
The arc rotates clockwise **between** 2 and $2i$.

(i) Sketch the locus of z. 1

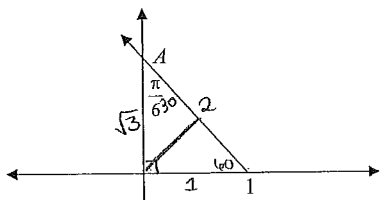
(ii) Determine the size of the angle at the circumference of the circle subtended by the chord that extends from 2 to $2i$. 1

(iii) Write down a **complex equation** (i.e. in terms of z) for the locus of P. 1

(c) Sketch the region in the complex plane where the inequalities

$|z - 2| \le 2$ and $\arg(z - 2 + 2i) \le \frac{\pi}{4}$ hold simultaneously. 3

(d) In the diagram below, the ray extends from 1 on the real axis and crosses the imaginary axis at point A. The angle between the ray and the imaginary axis is $\frac{\pi}{6}$.



(i) What is the equation of the locus of P ($z = x + iy$) if P lies above the real axis on the ray. 1

(ii) What is the imaginary part of the complex number that represents point A ? 1

(iii) Find the minimum value of $|z|$. 1

(iv) If $\text{Re}(z) \ge 0$, find the maximum value of the $\arg(z)$. 1

(v) If $\text{Re}(z) \ge 0$, find the maximum value of the $|z - 1|$. 1

THE END OF THE ASSESSMENT TASK

(1) (a) $\frac{10i}{5i^3} = \frac{2}{i^2} = -2$ — 1mk for correct result

(b) (i) $zw = (3+4i)(2-i) = 6 - 3i + 8i - 4i^2 = 10 + 5i$ — 1mk for correct answer

(ii) $z - \overline{w} = 3 + 4i - (2 + i) = 1 + 3i$ — 1mk for correct answer

(iii) $\frac{1}{w} = \frac{1}{2-i} \times \frac{2+i}{2+i} = \frac{2+i}{4-i^2} = \frac{2}{5} + \frac{i}{5}$ — 1mk for correct answer

(c) (i) $|1+i| = \sqrt{2}$ and $\arg(1+i) = \frac{\pi}{4}$ — 1mk for correct modulus

$1 + i = \sqrt{2}\, cis\left(\frac{\pi}{4}\right)$ — 1mk for correct argument

(ii) $(1+i)^4 = (\sqrt{2})^4 cis\left(\frac{4\pi}{4}\right) = 4(\cos\pi + i\sin\pi)$ — 1mk for working towards

$= 4(-1+0) = -4$ which is purely real — 2mks for correct answer

(d) $x^2 + 3x + 5 = 0 \Rightarrow x = \frac{-3 \pm \sqrt{-11}}{2} \quad \therefore x = -\frac{3}{2} \pm \frac{\sqrt{11}}{2} i$ — 1mk for working towards; 2mks for correct roots

(e) Let $(x + iy)^2 = 8 - 6i \qquad \therefore x^2 - y^2 + 2ixy = 8 - 6i$

$x^2 - y^2 = 8$..........(1)

$2xy = -6$..........(2)

Now, substituting $y = -\frac{3}{x}$ from (2) into (1) gives

$x^2 - \left(-\frac{3}{x}\right)^2 = 8 \Rightarrow x^4 - 8x^2 - 9 = 0 \Rightarrow (x^2-9)(x^2+1) = 0$ — 1mk for $x^2 - \left(-\frac{3}{x}\right)^2 = 8$

$\therefore x^2 = 9$ or $x^2 = -1$, but since we require real x and y we discard $x^2 = -1 \quad \therefore x = \pm 3$ — 1mk for $x = \pm 3$

Substituting $x = \pm 3$ into (2) gives $y = \pm 1$

Therefore, the square roots of $8 - 6i$ are $\pm(3 + i)$ — 1mk for $\pm(3+i)$

(f) $z^9 = 1$ roots are $1,\ cis\left(\frac{2\pi}{9}\right)\ cis\left(\frac{4\pi}{9}\right)$ — 1 mk working towards; 2 mks 3 correct roots

(g) $(3w^5 + 3w^4 + 8w^3)^2 = [3w^3(w^2 + w + 1) + 5w^3]^2 = (0+5)^2 = 25$ — 1mk for correct showing

(2) (a) $z^3 = 4 + 4\sqrt{3}\, i$

Let $z = r(\cos\theta + i\sin\theta) \quad \therefore z^3 = r^3(\cos 3\theta + i\sin 3\theta)$

Now $4 + 4\sqrt{3}\, i = 8\text{cis}\frac{\pi}{3}$

$\therefore z^3 = 8 \Rightarrow z = 2$ and

$3\theta = \frac{\pi}{3} + 2k\pi \quad (k = 0,1,2) \quad \Rightarrow \theta = \frac{\pi}{9} + \frac{2k\pi}{3}$ — 1mk for significant progress

for $k = 0$, $z_1 = 2\text{ cis }\frac{\pi}{9}$ — 2mks for 1 correct root

for $k = 1$, $z_2 = 2\text{ cis }(\frac{\pi}{9} + \frac{2\pi}{3}) = 2\text{ cis }\frac{7\pi}{9}$ — 3mks for 3 correct roots

for $k = 2$, $z_3 = 2\text{ cis }(\frac{\pi}{9} + \frac{4\pi}{3}) = 2\text{ cis }(-\frac{5\pi}{9})$

(b) $(\overline{z} - (-z))^3 = [x - iy - (-x - iy)]^3$ — 1 mk for working towards

$= (2x)^3 = 8x^3$ — 2mks for correct answer

(c) (i) $(\cos\theta + i\sin\theta)^3 = \cos^3\theta + 3\cos^2\theta\, i\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$ — 1mk for correct expansion

(ii) $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin\theta$ (De Moivre's Theorem)

Equating imaginary parts gives...

$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$ — 1mk correct use DeMoivre

$= \sin\theta[3(1 - \sin^2\theta) - \sin^2\theta]$

$= \sin\theta(3 - 4\sin^2\theta)$

$= 3\sin\theta - 4\sin^3\theta$ — 2mks for significant progress

$\therefore \sin 3\theta = 3\sin\theta - 4\sin^3\theta \Rightarrow 4\sin^3\theta = 3\sin\theta - \sin 3\theta$

$\therefore \sin^3\theta = \frac{3}{4}\sin\theta - \frac{1}{4}\sin 3\theta$ — 3mks for correct proof

(d) Using trigonometry, $\sin\frac{\pi}{6} = \frac{y}{10} \quad \therefore y = 5$ — 1mk for correct y

Applying Pythagoras, $10^2 = x^2 + 5^2 \quad \therefore x = 5\sqrt{3}$ — 1mk for correct x

(e) (i) Argument of OD is 5 times the argument of OA $(\angle DOA = 4\angle AOX)$

Modulus of OD is twice the modulus of OA $(OD = 2OA)$

If OA represents $z = \text{cis }\theta$, then OD represents $2\text{ cis }(5\theta) = 2z^5$

(ii) $\vec{OC} = \vec{OD} + \vec{DC} = 2z^5 + z$ — 1mk for correct answer

(iii) $\vec{AB} = \frac{1}{2}\vec{OC} \quad \therefore \vec{AB} = z^5 + \frac{1}{2}z$ — 1mk for correct answer

(iv) $\vec{OA} + \vec{AB} + \vec{BC} = \vec{OC} \quad \therefore \vec{BC} = \vec{OC} - \vec{OA} - \vec{AB}$

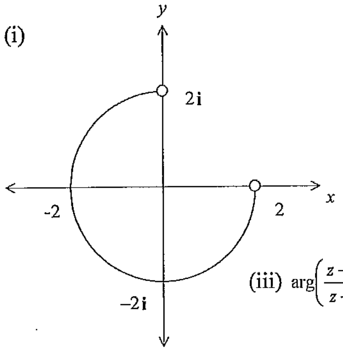
$\therefore \vec{BC} = 2z^5 + z - z - z^5 - \frac{1}{2}z = z^5 - \frac{1}{2}z$ — 1mk for correct answer

(3)(a) (i) $\text{Re}(z) = -5$ (ii) $\arg(z) = \frac{\pi}{4}$ (iii) $\arg\left(\frac{z-3}{z+3i}\right) = \frac{\pi}{2}$

(iv) $|z-2i| = 2|z-1|$ (v) $\arg(z) = \pi$

1mk for each correct answer

(b) (i)



1mk for correct sketch including open circles

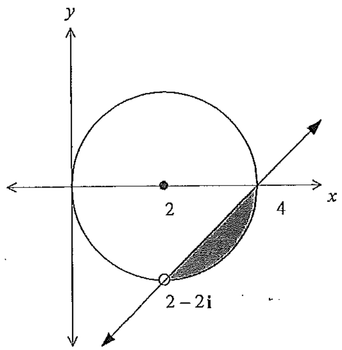
(ii) $\frac{\pi}{4}$

1mk for correct angle

(iii) $\arg\left(\frac{z-2i}{z-2}\right) = \frac{\pi}{4}$ OR $\arg\left(\frac{z-2}{z-2i}\right) = -\frac{\pi}{4}$

1mk for correct equation

(c)



1mk for correct circle

1mk for correct region

1mk for open circle at (2,−2)

(d) (i) Applying the exterior angle of a triangle $\Rightarrow$ $\arg(z-1) = \frac{2\pi}{3}$

1mk for correct answer

(ii) $\tan\frac{\pi}{6} = \frac{1}{OA} \Rightarrow OA = \sqrt{3} \Rightarrow \text{Im}(A) = \sqrt{3}$

1mk for correct answer

(iii) The minimum value of $|z|$ occurs when z is at the foot of the perpendicular drawn from the origin to the ray.
The triangle thus formed is right-angled with angles of $\frac{\pi}{6}$ *and* $\frac{\pi}{3}$.

Applying trigonometry, $\sin\frac{\pi}{3} = \frac{|z|}{1}$.

$\therefore$ The minimum value of $|z| = \frac{\sqrt{3}}{2}$.

1mk for correct answer

(iv) As z moves towards A from 1, the argument of z increases.
Since $\text{Re}(z) \geq 0$, the maximum value of the $\arg(z)$ occurs when z is at A.

$\therefore$ The maximum value of $\arg(z) = \frac{\pi}{2}$.

1mk for correct answer

(v) As z moves towards A from 1, the modulus of z from 1 increases.
Since $\text{Re}(z) \geq 0$, the maximum value of the $|z-1|$ occurs when z is at A.

Applying Pythagoras, $|z-1| = \sqrt{(1)^2 + (\sqrt{3})^2} = 2$.

$\therefore$ The maximum value of $|z-1| = 2$

1mk for correct answer