St Vincent's College

Assessment Task Three

Mathematics

2009

General Instructions

- Reading time 5 minutes
- Working time $1\frac{1}{2}$ hours
- Write using blue or black pen
- Board approved calculators may be used
- A table of standard integrals is provided
- All necessary working should be shown in all questions
- Weighting 25%

Examination structure

- Total Marks 79
- Attempt all questions 1 – 7
- All questions are of equal value (12 marks) except question 7 which is 7 marks
- Answer in 4 booklets, Book 1 questions 1&2,
  Book 2 questions 3&4,
  Book 3 questions 5&6,
  Book4 question 7 only

QUESTION ONE (12 marks)

i. Calculate, correct to two decimal places the value of $e^{3.4}$ (1)

ii. Write down the exact value of $\cos 150°$ (1)

iii. Convert $\frac{3\pi}{5}$ radians to degrees (1)

iv.

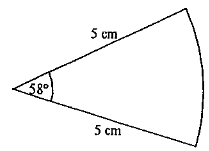
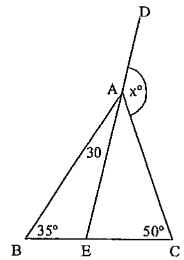


FIGURE NOT TO SCALE.

The diagram shows the sector of a circle.
Find the area of this sector. Give your answer to the nearest square centimetre. (2)

v.



NOT TO SCALE

Find the value of x. (1)

vi. State the domain of the function $\sqrt{2x-4}$ (1)

vii. Sketch the function $y = 4x - x^2$ and then solve $4x - x^2 < 0$ (3)

viii. Find $\int \cos 4x\,dx$ (1)

ix. Factorise $x^2 - 3x + 2$ (1)

QUESTION TWO (12 Marks)

i.

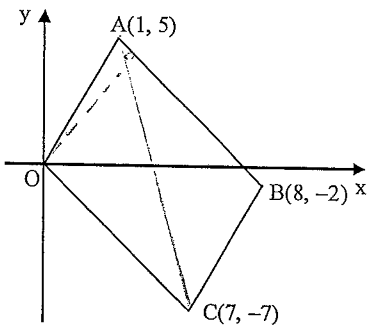


DIAGRAM NOT DRAWN TO SCALE

In the diagram O(0, 0), A(1, 5), B(8, −2) and C(7, −7) are the vertices of quadrilateral OABC.

a. Find the midpoint of the interval joining AC. (1)

b. Find the gradient of AB. (1)

c. Show that the equation of AB is $x + y = 6$. (1)

d. Find the exact length of AB. (1)

e. Show that AB is parallel to OC. (1)

f. Find the exact perpendicular distance from O to AB. (1)

g. Hence find the area of parallelogram OABC. (1)

ii.. For $y = 3\cos 2x$, state the amplitude and period of the function. (2)

iii. Sketch the graph $y = e^x + 2$, and state its range? (2)

QUESTION THREE (12 marks)

i. Differentiate each of the following:

a. $\dfrac{\sin x}{x}$ (2)

b. e^{2x+1} (1)

c. $\tan(4x - 3)$ (1)

d. $(e^x - e^{-x})^2$ (2)

e. $\cos^4 3x$ (2)

f. e^π (1)

ii. Find the equation of the tangent to the curve $y = x\sin x$ at the point $(\pi, 0)$. (3)

QUESTION FOUR (12 marks)

i.

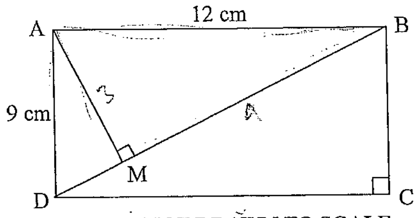


DIAGRAM NOT DRAWN TO SCALE

ABCD is a rectangle with AB = 12 cm, AD = 9 cm and AM is perpendicular to BD.

a. Copy the diagram onto your answer sheet. (1)

b. Find the length of BD. (1)

c. Prove that ΔABM is similar to ΔDBA. (3)

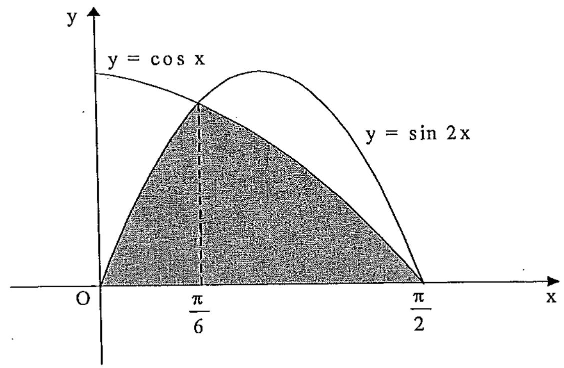
d. Hence find the length of BM. (2)

ii. Evaluate $\int_2^4 e^{2x-4}\,dx$ (2)

iii. The derivative at any point on the curve $y = f(x)$ is given by $\frac{dy}{dx} = e^{-3x}$. Find the equation of the curve, $f(x,)$ given $(0, 2)$ lies on the curve. (3)

QUESTION SIX (12 marks)

i.



The diagram shows the graphs of the functions $y = \cos x$ and $y = \sin 2x$ between $x = 0$ and $x = \frac{\pi}{2}$.

The two graphs intersect at $x = \frac{\pi}{6}$ and $x = \frac{\pi}{2}$.

Calculate the area of the shaded region. (4)

ii. Consider the function $y = \frac{1}{x}e^{-x}$:

a. For what values of x is this function defined? (1)

b. Describe the behaviour of the function as x:

α. approaches zero. (1)

β. increases indefinitely. (1)

c. Find any stationary points and determine their nature. (3)

d. Sketch the curve of this function. (2)

QUESTION FIVE (12 marks)

i. Consider the curve $y = x + e^{-x}$,

Show that the curve is concave up for all values of x. (2)

ii. Find the volume of the solid formed when the area bounded by the lines $x = 0$ and $x = 1$ and the curve $y = e^x$ is rotated about the x axis. Leave your answer in exact form. (3)

iii. Given that $\frac{d}{dx}\left(e^{x^2}\right) = 2xe^{x^2}$ 1, evaluate $\int_0^1 xe^{x^2}\,dx$ (2)

iv. Consider the function given by $y = \sin^2 x$.

a. Copy and complete the following table in your examination booklet. (Note that x is measured in radians.)

x	0	π/4	π/2	3π/4	π
y	0				

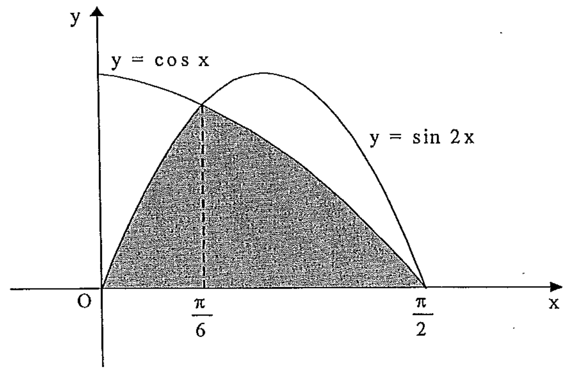
(1)

b. Apply Simpson's rule with five function values to find an approximation to $\int_0^{\pi} \sin^2 x\,dx$ (2)

v. If $y = e^{2x} + e^{4x}$, show that $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 8y = 0$ (2)

QUESTION SIX (12 marks)

i.



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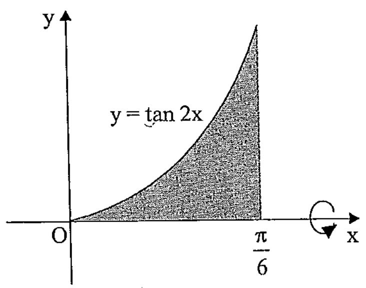
d. Sketch the curve of this function. (2)

QUESTION SEVEN (7 Marks)

i. By expressing $\sec\theta$ and $\tan\theta$ in terms of $\sin\theta$ and $\cos\theta$, show that $\sec^2\theta - \tan^2\theta = 1$. (2)

ii.



The diagram shows part of the graph of the function $y = \tan 2x$.

The shaded region is bounded by the curve, the x axis, and the line $x = \frac{\pi}{6}$.

The region is rotated about the x axis to form a solid.

a. Show that the volume of the solid is given by $V = \pi\int_0^{\frac{\pi}{6}}\left(\sec^2 2x - 1\right)dx$.

You may use the result of part (i) (2)

b. Find the exact volume of the solid.
(You may like to use the Table of Standard Integrals) (3)

Year 12 2009 Ass 3 Quint Mathematics (Solutions)

Question 1

(i) $e^{3.4} \doteqdot 29.96$

(ii) $\cos 150^\circ = -\cos 30^\circ$
$= -\frac{\sqrt{3}}{2}$

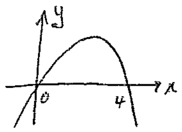
(iii) $\frac{3\pi}{5} = \frac{3\pi}{5} \times \frac{180}{\pi}$
$= 108^\circ$

(iv) $A = \frac{58}{360} \times \pi \times 5^2$
$= 12.65$
$\doteqdot 13$

(v) 115°

(vi) $\sqrt{2x-4} \geq 0$
$\therefore 2x-4 \geq 0$
$x \geq 2.$

(vii)



$\therefore 4x - x^2 < 0$
when $x < 0$ or $x > 4$

(viii) $\frac{1}{4} \sin 4x + C$

(ix) $(x-2)(x-1)$.

Question 2

(a) $(4,-1)$

(b) -1

(c) $y - 5 = -1(x-1)$
$y - 5 = -x + 1$
$y = -x + 6$
$x + y = 6$

(d) $\sqrt{98}$

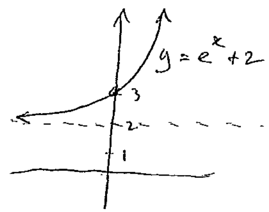
(e) $m = -1$ $\therefore$ parallel

(f) $d = \frac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$
$= \frac{|0 + 0 - 6|}{\sqrt{1^2+1^2}}$
$= \frac{6}{\sqrt{2}}$ or $3\sqrt{2}$

(g) $A = bh$ (area of parallelogram)
$= \sqrt{98} \times \frac{6}{\sqrt{2}}$
$= \sqrt{49} \times 6$
$= 42.$

(h) Amplitude 3, Period $\frac{2\pi}{2} = \pi$

(i)



Range $y > 2$.

Solutions Q3

i) a) $\frac{d}{dx}\left(\frac{\sin x}{x}\right) = \boxed{\frac{x\cos x - \sin x}{x^2}}$ ✓✓

b) $\frac{d}{dx}\left(e^{2x+1}\right) = \boxed{2e^{2x+1}}$ ✓

c) $\frac{d}{dx} \tan(4x-3) = \boxed{4\sec^2(4x-3)}$ ✓

d) $\frac{d}{dx}\left(e^x - e^{-x}\right)^2 = 2(e^{+x} - e^{-x})(e^x + e^{-x})$
$= \boxed{2(e^{2x} - e^{-2x})}$ ✓✓

e) $\frac{d}{dx}\left(\cos^4 3x\right) = 4\cos^3 3x \cdot -\sin 3x \cdot 3$
$= \boxed{-12\cos^3 3x \cdot \sin 3x}$ ✓✓

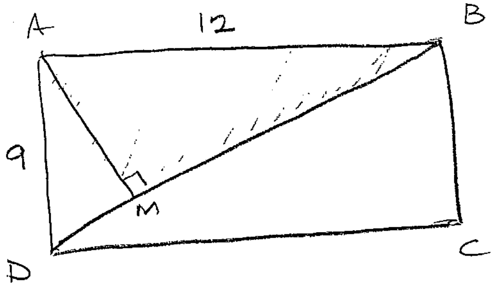
f) $\frac{d}{dx}\left(e^\pi\right) = \boxed{0}$ ✓

ii) $\frac{d}{dx}(x \sin x) = \sin x + x\cos x$

$y'(\pi) = \sin\pi + \pi\cos\pi$
$= 0 + \pi(-1)$
$= -\pi$ ✓

$\therefore y = mx + b$
$y = -\pi x + b$
when $x = \pi$, $y = 0$
$\therefore 0 = -\pi(\pi) + b$ ✓
$\therefore b = +\pi^2$ ✓
$\therefore$ tangent is $y = -\pi x + \pi^2$

Q4. i)



✓

(ii) $BD^2 = 12^2 + 9^2 = 144 + 81$
$= 225$

$\therefore BD = 15$ ✓

(iii) In Δ's ABM, DBA ✓

$\angle ADM \equiv \angle ADB$ (Common)
$\angle AMD \equiv \angle DA$ (Rt $\angle$) ✓

$\therefore \Delta AMB \,|||\, \Delta DAB$ ✓ (AA)

(iv) $\therefore$ $\frac{AM}{DA} = \frac{MB}{AB} = \frac{AB}{DB}$ ✓✓

$\frac{MB}{12} = \frac{12}{15}$ → $15 MB = 144$

$MB = \frac{144}{15} = \frac{48}{5}$
$= 9.6$

ii) $\int_2^4 e^{2x-4}\,dx = \left[\frac{e^{2x-4}}{2}\right]_2^4$ ✓

$= \frac{e^4}{2} - \frac{e^0}{2} = \frac{1}{2}\left[e^4 - 1\right]$ ✓ (26.79907)

iii) $y' = e^{-3x}$

$\therefore y = \int e^{-3x}\,dx = \frac{e^{-3x}}{-3} + c$ ✓

But when $x = 0, y = 2$ ✓ $\therefore y = \frac{-e^{-3x}}{3} + \frac{1}{3}$ ✓

$\therefore 2 = \frac{-e^0}{-3} + c$ → $c = \frac{7}{3}$ $y = \frac{1}{3}[7 - e^{-3x}$

5) i) $y = x + e^{-x}$

$\frac{dy}{dx} = 1 - e^{-x}$

$\frac{d^2y}{dx^2} = e^{-x}$

$e^{-x} > 0$ for all x, hence $\frac{d^2y}{dx^2} > 0$

$\therefore$ the curve is concave up for all x. ✓✓

ii) $V = \pi \int y^2\,dx = \pi \int_0^1 e^{2x}\,dx$ ✓

$= \frac{\pi}{2} \int_0^1 2e^{2x}\,dx$

$= \frac{\pi}{2} \left[e^{2x}\right]_0^1$ ✓

$= \frac{\pi}{2}(e^2 - 1)$ ✓

iii) Given $\frac{d}{dx}(e^{x^2}) = 2x \cdot e^{x^2}$

$\int_0^1 x \cdot e^{x^2}\,dx = \frac{1}{2}\int_0^1 2x \cdot e^{x^2}\,dx$

$= \frac{1}{2}\left[e^{x^2}\right]_0^1$ ✓

$= \frac{1}{2}(e - 1)$ ✓

iv) a)

x	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{3\pi}{4}$	π
y	0	$\frac{1}{2}$	1	$\frac{1}{2}$	0

✓ $y = \sin^2 x$.

$\int_0^{\pi} \sin^2 x\,dx = \frac{\frac{\pi}{2} - 0}{6}\left[f(0) + 4f(\tfrac{\pi}{4}) + f(\tfrac{\pi}{2})\right]$
$+ \frac{\pi - \frac{\pi}{2}}{6}\left[f(\tfrac{\pi}{2}) + 4f(\tfrac{3\pi}{4}) + f(\pi)\right]$

$= \frac{\pi}{12}\left[0 + 4(\tfrac{1}{2}) + 1\right] + \frac{\pi}{12}\left[1 + 4(\tfrac{1}{2}) + 0\right]$ ✓

$= \frac{\pi}{12}(3 + 3) = \frac{\pi}{2} u^2$ ✓

v) $y = e^{2x} + e^{4x}$

$\frac{dy}{dx} = 2 \cdot e^{2x} + 4e^{4x}$

$\frac{d^2y}{dx^2} = 4e^{2x} + 16e^{4x}$

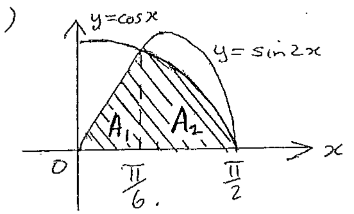
$y'' - 6y' + 8y = 4e^{2x} + 16e^{4x} - 6(2e^{2x} + 4e^{4x}) + 8(e^{2x} + e^{4x})$ ✓

$= e^{4x}(16 - 24 + 8) + e^{2x}(4 - 12 + 8)$

$= 0 \cdot e^{4x} + 0 \cdot e^{2x}$

$= 0$ ✓

6) (i)



$A = \int_0^{\frac{\pi}{6}} \sin 2x\,dx + \int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \cos x\,dx$ ✓

$= \left[-\frac{\cos 2x}{2}\right]_0^{\frac{\pi}{6}} + \left[\sin x\right]_{\frac{\pi}{6}}^{\frac{\pi}{2}}$ ✓

$= \left[-\tfrac{1}{2}(\cos\tfrac{\pi}{3}) - -\tfrac{1}{2}(\cos 0)\right] + \left[\sin\tfrac{\pi}{2} - \sin\tfrac{\pi}{6}\right]$

$= \left[-\tfrac{1}{2}(\tfrac{1}{2}) - -\tfrac{1}{2}(1)\right] + \left[1 - \tfrac{1}{2}\right]$ ✓

$= -\tfrac{1}{4} + \tfrac{1}{2} + \tfrac{1}{2}$

$= \frac{3}{4} u^2$ ✓

(ii) $y = \frac{1}{x} \cdot e^{-x}$

a) All real $x \neq 0$ ✓

b) α) $x \to \pm\infty$ ✓

β) $x \to 0$ ✓

c) $y = x^{-1} \cdot e^{-x}$

$\frac{dy}{dx} = x^{-1} \cdot -e^{-x} + e^{-x} \cdot -x^{-2}$

$= -e^{-x}\left(\frac{1}{x} + \frac{1}{x^2}\right)$

For stationary pts let $\frac{dy}{dx} = 0$.

$-e^{-x} \neq 0$, $\frac{1}{x} + \frac{1}{x^2} = 0$

$\frac{x+1}{x^2} = 0 \therefore x = -1$ ✓

$y = f(-1) = (-1)^{-1} \cdot e^{-(-1)} = -e$

$\therefore$ Stat pt at $(-1, -e)$ ✓

Test

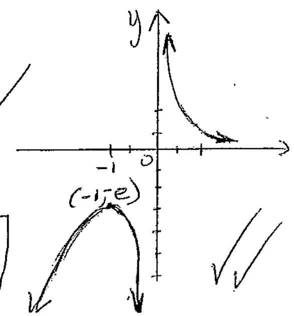
x	-2	-1	$-\frac{1}{2}$
y'	>0	0	<0

$\therefore$ Max at $x = -1$ ✓

Other Pts:

x	$\frac{1}{2}$	1
y	0.6	0.4

As $x \to \infty$ $y \to 0$



✓✓

Question 7

i) $L.H.S = \sec^2\theta - \tan^2\theta$

$= \frac{1}{\cos^2\theta} - \frac{\sin^2\theta}{\cos^2\theta}$

$= \frac{1-\sin^2\theta}{\cos^2\theta}$

$= \frac{\cos^2\theta}{\cos^2\theta}$

$= 1$ ✓✓

$= R.H.S.$

ii) a) $\text{Volume} = \pi \int_0^{\frac{\pi}{6}} (\tan 2x)^2 \, dx$ ✓

using (i).

$\sec^2\theta - \tan^2\theta = 1$

$\sec^2\theta - 1 = \tan^2\theta$

$\therefore \sec^2 2\theta - 1 = \tan^2 2\theta$

$= \pi \int_0^{\frac{\pi}{6}} (\tan^2 2x) \, dx$

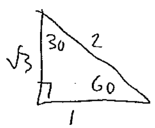
$= \pi \int_0^{\frac{\pi}{6}} (\sec^2 2x - 1) \, dx$ ✓

b) $\text{Volume} = \pi \int_0^{\frac{\pi}{6}} \sec^2 2x - 1 \, dx$

$= \pi \left[\frac{1}{2}\tan 2x - x \right]_0^{\frac{\pi}{6}}$

$= \pi \left[\left(\frac{1}{2}\tan 2 \times \frac{\pi}{6} - \frac{\pi}{6} \right) - \left(\frac{1}{2}\tan 0 - 0 \right) \right]$

$= \pi \left[\frac{1}{2}\tan\frac{\pi}{3} - \frac{\pi}{6} - 0 \right]$



$= \pi \left[\frac{1}{2} \times \sqrt{3} - \frac{\pi}{6} \right]$

$= \pi \left[\frac{\sqrt{3}}{2} - \frac{\pi}{6} \right] \text{ units}^3$

$= \pi \left[\frac{3\sqrt{3} - \pi}{6} \right] \text{ units}^3$ ✓✓✓

$= \pi \left[\sqrt{3} \quad \pi \right]$