St Vincents College **Extension 1 Mathematics**

May 11$^{\text{Th}}$ 2009 Assessment Task

Weighting 25%

All working must be shown.

1) [1,1 mark]

a) Find $\lim\limits_{x\to 0} \dfrac{\sin 2x}{x}$ b) Find $\lim\limits_{x\to 0} \dfrac{6\sin x\cos x}{x}$

2) [1,3 marks]

a) Find $\dfrac{d}{dx}(1+x^2)^3$.

b) Hence evaluate $\int_0^1 5x(1+x^2)^2\,dx$

3) [1,2 marks]

a. Use $\cos 2x = 2\cos^2 x - 1$, or otherwise, to find a similar expression for $\cos(2x+2x)$

b. Use the result above to find $\int \cos^2 2x\,dx$

4) [4 marks]

Find the volume of the solid formed when the region bounded by the x-axis and the curve $y = x(8-x^3)^4$ between $x = 0$ and $x = 2$ is rotated about the x-axis. (You may need to use the substitution $u = 8 - x^3$ to evaluate the integral involved.)

5) [2,2 marks]

a. Sketch carefully the graph of $y = 3\sin 2x$

b. By sketching the graph of $y = x + \dfrac{1}{2}$ on the graph above find the number (do not solve) of distinct solutions to $2x = -1 + 6\sin 2x$

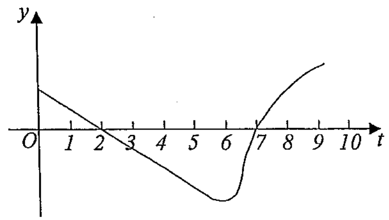
6) [1,3 marks]

The rate at which a body cools in air is assumed to be proportional to the difference between its temperature T and the constant temperature S of the surrounding air. This can be expressed by the differential equation $\dfrac{dT}{dt} = k(T-S)$ where t is the time in hours and k is a constant.

a. Show that $T = S + Be^{kt}$, where B is a constant, is a solution of the differential equation.

b. A heated body cools from 80 °C to 40 °C in 2 hours. The air temperature S around the body is 20 °C. Find the temperature of the body after one further hour has elapsed. Give your answer correct to the nearest degree.

7) [2,2, 2 marks]



A particle moves along the x axis. At time $t = 0$, the particle is at $x = 0$. Its velocity y at time t is shown on the graph. Trace or copy this graph into your Writing Booklet.

i. At what time is the acceleration greatest? Explain your answer.

ii. At what time does the particle first return to $x = 0$? Explain your answer.

iii. Sketch the displacement graph for the particle from $t = 0$ to $t = 9$.

8) [1,3 marks]

a) Show $\dfrac{k}{(2k+1)} + \dfrac{1}{(2k+1)(2k+3)} = \dfrac{k+1}{2k+3}$

b) Hence use mathematical induction to prove that $\dfrac{1}{1\times 3} + \dfrac{1}{3\times 5} + \dfrac{1}{5\times 7} + \ldots + \dfrac{1}{(2n-1)(2n+1)} = \dfrac{n}{2n+1}$ for all positive integers n.

9) [1,2 marks]

The displacement of a particle moving in a straight line is given by $x = \cos(2t + \frac{\pi}{2})$.

a. What is its initial displacement?

b. By drawing a quick sketch or otherwise determine how far does it travel in the first 3π seconds?

10) [2,4 marks]

A particle moves in a straight line so that its acceleration is given by $\frac{dv}{dt} = x - 1$ where v is its velocity and x is its displacement from the origin. Initially, the particle is at the origin and has velocity $v = 1$.

a. Show that $v^2 = (x-1)^2$.

b. By finding an expression for $\frac{dt}{dx}$, or otherwise, find x as a function of t.

11) [1,2,2 marks]

Air is being pumped into a spherical balloon at the rate of 20 cubic centimetres per second.

a) State $\frac{dV}{dt}$, V being volume

b) Find $\frac{dr}{dt}$, where r is the radius

c) Find the rate of increase of the surface area of the balloon when the radius is 5 cm.

(Volume of sphere $= \frac{4}{3}\pi r^3$ and Surface area of a sphere $= 4\pi r^2$)

Extension 1 solutions May 2009

1. a) $\lim_{x \to 0} \frac{\sin 2x}{x}$

$\to \lim_{x \to 0} \frac{2\sin x \cos x}{x} = 2$ ✓

b) $\lim_{x \to 0} \frac{6\sin x \cos x}{x} = 6$ ✓

2. a) $\frac{d}{dx}(1+x^2)^3$

$= 3(1+x^2)^2 \cdot 2x = 6x(1+x^2)^2$ ✓

b) $\int_0^1 5x(1+x^2)^2 dx$

$= \frac{5}{6}\int 6x(1+x^2)^2 dx = \frac{5}{6}\left[(1+x^2)^3\right]_0^1$ ✓

$= \frac{5}{6}[8-7] = \frac{35}{6}$ ✓

3. a) $\cos(2x+2x)$

$= \cos 4x = 2\cos^2 2x - 1$ ✓

b) $2\cos^2 2x = 1 + \cos 4x$

$\cos^2 2x = \frac{1}{2} + \frac{\cos 4x}{2}$ ✓

$\therefore \int \cos^2 2x \, dx$

$= \int \frac{1}{2} + \frac{\cos 4x}{2} dx$

$= \frac{x}{2} + \frac{\sin 4x}{8} + c$ ✓

4. $V = \pi\int_0^2 y^2 dx$

$= \pi\int_0^2 (x(8-x^3)^4)^2 dx$

$= \pi\int_0^2 x^2(8-x^3)^8 dx$ ✓

4. cont.

$u = 8 - x^3 \to \frac{du}{dx} = -3x^2$

$du = -3x^2 dx$ ✓

Also $x = 0$ $u = 8$

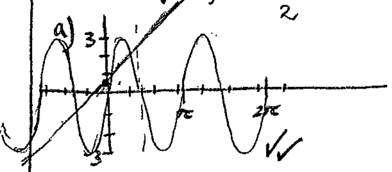
$x = 2$ $u = 0$.

$\therefore -\frac{\pi}{3}\int_0^2 -3x^2(8-x^3)^8 dx$ ✓

$\to -\frac{\pi}{3}\int_8^0 u^8 du = -\frac{\pi}{3}\left[\frac{u^9}{9}\right]_8^0$

$= -\frac{\pi}{3}\left[\frac{-8^9}{9}\right] = +\frac{\pi 8^9}{27}$ ✓

5. a)



b) graph $y = x + \frac{1}{2}$

3 solutions ✓

6.(a) $T = S + Be^{kt}$

$\frac{dT}{dt} = kBe^{kt}$ ✓

But $Be^{kt} = T - S$

and $kBe^{kt} = k(T-S)$

$\therefore \frac{dT}{dt} = ke^{kt} = k(T-S)$

b) $\frac{dT}{dt}$ proportional to difference

$\therefore 40 = 20 + 60e^{2k}$ ✓

$2k = \ln\frac{1}{3} \to \therefore k = -0.54$ ✓

Extra hour

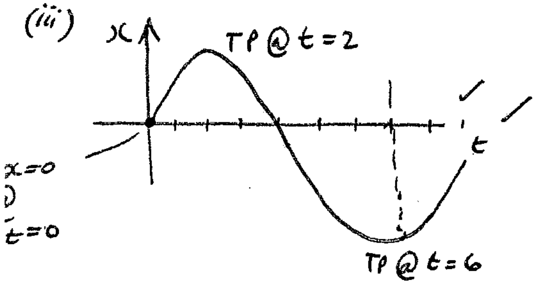
$T = 20 + 60e^{-3(0.54)} = 32°$ ✓

Q7 (i) Accel greatest when $\frac{dv}{dt}$ is max

from graph it appears ✓ to be between $t=6$ ✓ and $t=7$

(ii) Since area under v/t graph is distance travelled from $x=0$ ✓ to $x=5$ graph appears to be linear, ✓ Area above = Area below at $t=4$

(iii)



Q8 a)

$$\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$

$$=\frac{k+1}{2k+3} \quad ✓$$

Q 8 cont.

Show true $n=1$

$$\frac{1}{1\times3}=\frac{1}{2(1)+1}=\frac{1}{3} \quad \text{True}$$

Assume True $n=k$, i.e.

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\frac{1}{5\times7}+\dots\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$$

show true for $n=k+1$, i.e.

$$\frac{1}{1\times3}+\frac{1}{3\times5}+\dots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2(k+1)-1)(2(k+1)+1)}=\frac{k+1}{2(k+1)+1}$$

But

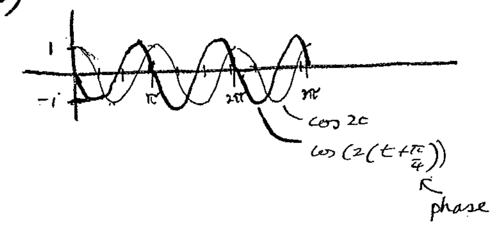
$$\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}=\frac{k+1}{2(k+1)+1} \quad \text{QED}$$

Q9

a) $x=\cos(2t+\frac{\pi}{2})$

$x(0)=\cos\frac{\pi}{2}=0$

b)



Each trough or peak represents a 'travel' <u>to</u> the amplitude and <u>back</u>

3 troughs, 3 peaks = <u>12 units</u>

Q10

$\frac{dv}{dt}=a=x-1$

but $\frac{d}{dx}(\frac{1}{2}v^2)=a=x-1$

$\therefore \frac{1}{2}v^2=\int x-1\,dx$

$\frac{1}{2}v^2=\frac{x^2}{2}-x+c$ ✓

But @ $x=0$ $v=1$

$\therefore \frac{1}{2}(1)=c$

$\therefore c=\frac{1}{2}$ ✓

i.e. $\frac{1}{2}v^2=\frac{x^2}{2}-x+\frac{1}{2}$

or $v^2=x^2-2x+1$

i.e. $v^2=(x-1)^2$ ✓

b) $v=\pm(x-1)$

But BC's state $x=0$, $v=1$

$\therefore v=-(x-1)$

$\boxed{v=1-x}$ ✓

i.e. $\frac{dx}{dt}=1-x$

$\int\frac{dx}{1-x}=\int dt$

$+\ln(1-x)=-t+c$ ✓

i.e. $e^{-t+c}=1-x$

$x=1-e^{-t+c}$

10 b) cont.

But $x=0$ $t=0$

$\therefore 0=1-e^c$

$\therefore e^c=1$

Hence $\boxed{x=1-e^{-t}}$ ✓

11 a) $\frac{dV}{dt}=20\ cm^3/s$ ✓

b) $\frac{dr}{dt}=\frac{dr}{dV}\cdot\frac{dV}{dt}$

$\frac{dV}{dr}=4\pi r^2$ ✓

$\therefore \frac{dr}{dt}=\frac{1}{4\pi r^2}\cdot\frac{dV}{dt}$

$=\frac{1}{4\pi r^2}\cdot 20$ ✓

$=\frac{5}{\pi r^2}$ cm/s

c) $\frac{dA}{dt}=\frac{dA}{dr}\cdot\frac{dr}{dt}$

$=8\pi r\cdot\frac{5}{\pi r^2}$ ✓

$=\frac{40}{r}$ ✓

$\frac{dA}{dt}_{(r=5)}=\frac{40}{5}=8\ cm^2/s$