Sydney Grammar School

2013 Annual Examination

FORM V
MATHEMATICS EXTENSION 1

Wednesday 28th August 2013

General Instructions

- Writing time — 3 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 117 Marks

- All questions may be attempted.

Section I – 13 Marks

- Questions 1–13 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 104 Marks

- Questions 14–21 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your name, class and master on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Fourteen.
- Write your name and master on this question paper and submit it with your answers.

5A: DNW	5B: PKH	5C: RCF	5D: BDD
5E: KWM	5F: FMW	5G: LRP	5H: TCW

Checklist

- SGS booklets — 8 per boy
- Multiple choice answer sheet
- Candidature — 150 boys

Examiner

RCF

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

The correct factorisation of $10y^2 - 19y + 6$ is:

(A) $(5y-2)(2y-3)$ (B) $(5y-3)(2y-2)$

(C) $(5y-2)(3-2y)$ (D) $(3-5y)(2-2y)$

QUESTION TWO

x	0	5	10
$f(x)$	1	5	9

The table of values above gives three data points from an experiment modelling an unknown function $f(x)$.

Using Simpson's rule, with three function values, to approximate $\int_0^{10} f(x)\,dx$, gives the answer:

(A) 25 (B) 50 (C) 75 (D) 100

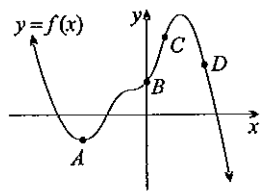
QUESTION THREE

Given that a quadratic function with integer coefficients has a positive non-square discriminant, which of the following statements about its zeroes is true?

(A) Equal zeroes (B) Distinct irrational zeroes

(C) Distinct rational zeroes (D) No real zeroes

QUESTION FOUR



For which point on the graph above is $f(x) > 0$, $f'(x) > 0$ and $f''(x) < 0$?

(A) A (B) B (C) C (D) D

Exam continues next page ...

QUESTION FIVE

A correct primitive of $2\sqrt{x}$ is:

(A) $\dfrac{x\sqrt{x}}{3}$ (B) $3x\sqrt{x}$ (C) $x\sqrt{x}$ (D) $\dfrac{4x\sqrt{x}}{3}$

QUESTION SIX

Which of the following is not equivalent to $\log_e e^2 + \log_e \left(\dfrac{1}{e}\right)$?

(A) $2\log_e e - 1$ (B) $-\log_e\left(\dfrac{1}{e}\right)$

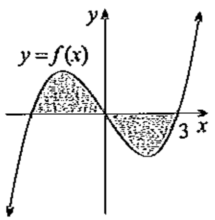
(C) $\log_e\left(\dfrac{e^3+1}{e}\right)$ (D) 1

QUESTION SEVEN

The derivative of $\dfrac{1}{(3-5x)^3}$ is:

(A) $\dfrac{-15}{(3-5x)^4}$ (B) $\dfrac{-3}{(3-5x)^2}$ (C) $\dfrac{15}{(3-5x)^4}$ (D) $\dfrac{3}{5(3-5x)^2}$

QUESTION EIGHT



Which of the following definite integrals would correctly evaluate the area shaded above, given that $f(x)$ is an odd function?

(A) $\displaystyle\int_{-3}^{3} f(x)\,dx$ (B) $\displaystyle 2\int_{0}^{3} f(x)\,dx$

(C) $\displaystyle\left|\int_{-3}^{3} f(x)\,dx\right|$ (D) $\displaystyle 2\int_{-3}^{0} f(x)\,dx$

QUESTION NINE

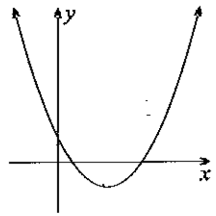
The line perpendicular to $y = 5 - 2x$ and passing through the point $(1, -3)$ has equation:

(A) $x - 2y - 7 = 0$ (B) $2x + y + 1 = 0$

(C) $x - 2y + 5 = 0$ (D) $2y + x + 5 = 0$

Exam continues overleaf ...

QUESTION TEN



Which of the following sets of statements is true for the quadratic $y = ax^2 + bx + c$ graphed above?

(A) $a > 0, c = 0, \Delta > 0$ (B) $a \neq 0, c > 0, \Delta < 0$

(C) $a < 0, c > 0, \Delta = 0$ (D) $a > 0, c > 0, \Delta > 0$

QUESTION ELEVEN

Given $a > 0$, which of the following functions is continuous but not differentiable at $x = a$?

(A) $y = \log(x + a)$ (B) $y = |x - a|$

(C) $y = ax^3$ (D) $y = \sqrt{x} + a$

QUESTION TWELVE

The quadratic equation $2x^2 - 18x + c = 0$ has one root twice the other.
What is the value of c?

(A) 3 (B) 9 (C) 18 (D) 36

QUESTION THIRTEEN

The derivative of $\ln\left(\dfrac{1}{x}\right)$ is:

(A) $-\dfrac{1}{x}$ (B) x (C) $-e$ (D) $\dfrac{1}{x^2}$

End of Section I

Exam continues next page ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION FOURTEEN (13 marks) Use a separate writing booklet. **Marks**

(a) Simplify:

(i) $\log_e \left(\frac{1}{e^4}\right)$ $\boxed{1}$

(ii) $\sqrt{18} - \sqrt{8}$ $\boxed{1}$

(b) Expand and simplify:

(i) $4 - 2(x - 3)$ $\boxed{1}$

(ii) $(2\sqrt{3} + \sqrt{5})^2$ $\boxed{1}$

(c) Find the derivative of:

(i) $x^2 + 2x + 4$ $\boxed{1}$

(ii) $\frac{1}{x}$ $\boxed{1}$

(iii) $\log_e(2x + 1)$ $\boxed{1}$

(d) Determine the exact value of $\tan 150°$. $\boxed{2}$

(e) Find a primitive of:

(i) $x^2 + 2x + 4$ $\boxed{1}$

(ii) $\frac{4}{x}$ $\boxed{1}$

(f) Find the limiting sum of the geometric series $3 + \frac{3}{2} + \frac{3}{4} + \cdots$. $\boxed{2}$

Exam continues overleaf ...

QUESTION FIFTEEN (13 marks) Use a separate writing booklet. **Marks**

(a) Solve:

(i) $(x - 2)^2 - 3 = 0$ $\boxed{1}$

(ii) $|x - 2| = 4$ $\boxed{1}$

(iii) $(x - 2)(x + 4) > 0$ $\boxed{1}$

(b) Form the monic quadratic equation with roots 3 and -4. $\boxed{1}$

(c) Let $f(x) = x^3 - 8x$.

(i) Find $f(1)$, $f'(1)$ and $f''(1)$. $\boxed{3}$

(ii) Is $f(x)$ increasing, decreasing or stationary at $x = 1$? Justify your answer. $\boxed{1}$

(iii) Is $f(x)$ concave up or down at $x = 1$? Justify your answer. $\boxed{1}$

(iv) Find the equation of the tangent to $y = f(x)$ at $x = 1$. $\boxed{1}$

(d) (i) Write down the discriminant of the quadratic expression $3x^2 - mx + 3$. $\boxed{1}$

(ii) For what values of m does the expression have no real zeroes? $\boxed{2}$

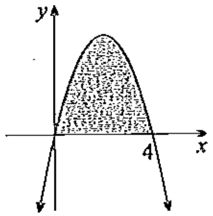
Exam continues next page ...

QUESTION SIXTEEN (13 marks) Use a separate writing booklet. **Marks**

(a) Find the equation of the curve with derivative $\dfrac{dy}{dx} = 4x - 2$ that passes through the point $(2, 5)$. **2**

(b) Evaluate $\displaystyle\int_1^3 2x^3\,dx$ **2**

(c) **2**



The graph above shows the parabola $y = 4x - x^2$. Calculate the area of the region enclosed between the curve and the x-axis.

(d) Differentiate the following functions, giving your answers in a factorised form where possible:

(i) $(3x^2 + 2)^4$ **1**

(ii) $2x(x + 7)^5$ **2**

(iii) $\dfrac{\ln 3x}{x^2}$ **2**

(e) Evaluate $\displaystyle\lim_{x \to 2} \left(\frac{x^2 - 4}{x - 2} \right)$. Show your working clearly. **2**

QUESTION SEVENTEEN (13 marks) Use a separate writing booklet. **Marks**

(a) Given the sequence $\sqrt{2}, \sqrt{18}, \sqrt{50}, \ldots$

(i) Show that the sequence is arithmetic. **1**

(ii) Find the value of the hundredth term. **1**

(iii) Find the sum of the first hundred terms. **1**

(b) Suppose that $f(x) = 2x^2 - x$.

(i) Show that $f(x + h) - f(x) = 4xh + 2h^2 - h$. **1**

(ii) Use the definition $f'(x) = \displaystyle\lim_{h \to 0} \frac{f(x + h) - f(x)}{h}$ to find $f'(x)$ from first principles. **1**

(c) Consider the curve with equation $y = 3x^4 + 8x^3 + 12$.

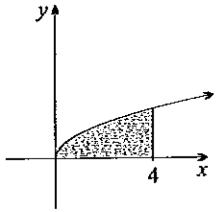
(i) Find the coordinates of any stationary points and determine their nature. **3**

(ii) Find any points of inflexion, demonstrating a change in concavity at these points. **3**

(iii) Sketch the curve showing all the points found in parts (i) and (ii). You do NOT need to find the x-intercepts. **2**

QUESTION EIGHTEEN (13 marks) Use a separate writing booklet. **Marks**

(a) [3]



The diagram above shows the region enclosed by the curve $y = 3\sqrt{x}$, the x-axis and the line $x = 4$. What is the volume of the solid of revolution generated by rotating this region about the x-axis?

(b) Find the following indefinite integrals:

(i) $\int (2x+3)^5\,dx$ [1]

(ii) $\int \dfrac{x+4}{\sqrt{x}}\,dx$ [2]

(iii) $\int \dfrac{2x}{4+x^2}\,dx$ [1]

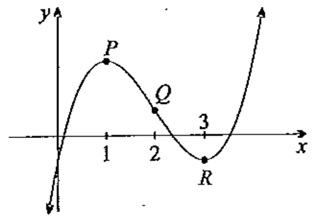
(c) (i) Write down the equation of the line with gradient m which passes through the point $P(1, -18)$. [1]

(ii) Form a quadratic equation and use the discriminant to find the values of m for which the line through P is a tangent to the parabola $y = 2x^2 + 4x - 6$. [3]

(d) Find the value of k if $\int_1^k \dfrac{1}{x^2}\,dx = \frac{1}{4}$. [2]

QUESTION NINETEEN (13 marks) Use a separate writing booklet. **Marks**

(a) [2]



The function $y = f(x)$ is sketched above. The points P and R are turning points and the point Q is a point of inflexion. Sketch a possible graph of the gradient function, $f'(x)$.

(b) (i) Sketch the curve $y = \ln(x-1)$, clearly indicating any asymptotes and any intercepts with the axes. [1]

(ii) Find the equation of the normal to $y = \ln(x-1)$ at $x = 3$. [2]

(c) The equation $x^2 - 4x + 6 = 0$ has roots m and n.

(i) Without solving the equation determine:

(α) $m + n$ [1]

(β) mn [1]

(γ) $\dfrac{1}{m} + \dfrac{1}{n}$ [1]

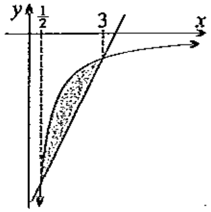
(ii) Hence, or otherwise, find a quadratic equation with integer coefficients which has roots $\dfrac{1}{m}$ and $\dfrac{1}{n}$. [2]

(d) Find the area bounded by the curve $y = x^2 - 2$, the x-axis and the lines $x = 1$ and $x = 2$. [3]

QUESTION TWENTY (13 marks) Use a separate writing booklet. **Marks**

(a) Use a suitable substitution to solve $4^x - 5 \times 2^{x+1} + 16 = 0$. $\boxed{3}$

(b) $\boxed{3}$



The graph above shows the line $y = 2x - 7$ and the hyperbola $y = -\dfrac{3}{x}$, intersecting at $x = \frac{1}{2}$ and $x = 3$.

The area of the shaded region can be written in the form $\dfrac{a}{4} + b \ln 6$, where a and b are integers. Find the values of a and b.

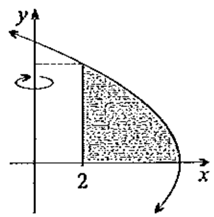
(c) Solve $4\cos(2\alpha - 45^\circ) - 2\sqrt{3} = 0$ for the domain $0^\circ \le \alpha \le 360^\circ$. $\boxed{3}$

(d) (i) Differentiate $2x^3 \ln x$. $\boxed{1}$

(ii) Hence evaluate $\displaystyle\int_1^2 x^2 \ln x \, dx$. $\boxed{3}$

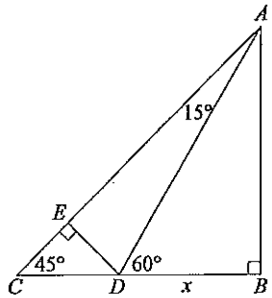
QUESTION TWENTY ONE (13 marks) Use a separate writing booklet. **Marks**

(a) $\boxed{4}$



The diagram above shows the region bounded by the curve $4x + y^2 - 24 = 0$, the x-axis and the line $x = 2$. This region is rotated around the y-axis to create a solid of revolution. Calculate the volume of this solid.

(b)



In the diagram above, ABC is a right-angled isosceles triangle. The point D is chosen on BC so that $\angle ADB = 60^\circ$, and DE is drawn perpendicular to AC. Let $BD = x$.

(i) Show that $\boxed{1}$

$$DE = \frac{(\sqrt{3} - 1)x}{\sqrt{2}}.$$

(ii) Hence, find an exact value for $\cos 15^\circ$. $\boxed{2}$

(c) (i) The function $f(t)$ satisfies $0 \leq f(t) \leq k$ for $0 \leq t \leq 1$, where k is a constant. $\boxed{1}$

Explain using a sketch why $0 \leq \int_0^1 f(t)\,dt \leq k$.

(ii) By letting $f(t) = \dfrac{1}{n-t} - \dfrac{1}{n}$, show that if $n > 1$ then $\boxed{2}$

$$0 \leq \ln\left(\frac{n}{n-1}\right) - \frac{1}{n} \leq \frac{1}{n-1} - \frac{1}{n}.$$

(iii) Hence show that $\boxed{2}$

$$0 \leq \ln 2 - \sum_{n=N+1}^{2N} \frac{1}{n} \leq \frac{1}{2N}.$$

(iv) Use the fact that $\displaystyle\sum_{n=6}^{10} \frac{1}{n} = \tfrac{1627}{2520}$ to show that $\boxed{1}$

$$6\tfrac{115}{252} \leq \ln 2^{10} \leq 7\tfrac{115}{252}.$$

End of Section II

END OF EXAMINATION

2013 FIFTH FORM EXTENSION ANNUAL

(1) $10y^2-19y+6=(5y-2)(2y-3)$ (A)

(2) $\int_0^{10} f(x)\,dx \doteq \frac{1}{6}\times 10\left(1+4\times 5+9\right)$ (B)

$= \frac{300}{6}$

$= 50$

(3) ~~Two Distinct~~ $\Delta>0$ and Irrational Δ-non-square (B)

(4) y co-ord positive, increasing and concave down (C)

(5) $f(x)=2x^{\frac{1}{2}}$ $F(x)=\frac{2x^{3/2}}{3/2}$ (D)

$=\frac{4}{3}x$

(6) $\log_e e^2+\log_e\left(\frac{1}{e}\right)=2\log_e e-1$ $\quad -\log_e\left(\frac{1}{e}\right)=-(-1)$

$=2-1$ $\quad =1$

$=1$

$\log_e\left(\frac{e^3+1}{e}\right)=\log_e(e^3+1)-\log_e e$ (C)

$=\log_e(e^3+1)-1$

(7) $y=(3-5x)^{-3}$

$\frac{dy}{dx}=-3(3-5x)^{-4}\times(-5)$ (C)

$=15$

$\left|\int_{-3}^{3} f(x)\,dx\right|=0$ $\quad \int_0^{-3} f(x)\,dx<0$

$\int_3^{10} f(x)\,dx$ (D) $\quad \int_3^{0} f(x)\,dx>0$

) $m_2=\frac{1}{2}$ $\quad y-(-3)=\frac{1}{2}(x-1)$ (A)

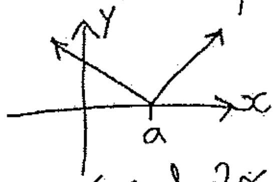
$2y+6=x-1$

$0=x-2y-7$

(10) $a>0$, $\Delta>0$, $c>0$ (D)

(Concave Up) (Two Distinct Zeroes) Y-intercept positive

(11) (B)



(12) Let roots be α and 2α (D)

$3\alpha=-\frac{b}{a}=\frac{18}{2}$ $\quad 2\alpha^2=\frac{c}{a}=\frac{c}{2}$

$\therefore \alpha=3$ $\quad \therefore 18=\frac{c}{2}$

$c=36.$

(13) $y=\ln\left(\frac{1}{x}\right)$ (A)

$\frac{dy}{dx}=\frac{1}{\left(\frac{1}{x}\right)}\times(-x^{-2})$

$=x\times\left(\frac{-1}{x^2}\right)$

$=\left(-\frac{1}{x}\right)$

13

(14) a) (i) $\log_e\left(\frac{1}{e^4}\right) = (-4)$ ✓

(ii) $\sqrt{18} - \sqrt{8} = 3\sqrt{2} - 2\sqrt{2}$
$= \sqrt{2}$ ✓

c) (i) $y = x^2 + 2x + 4$
$\frac{dy}{dx} = 2x + 2$ ✓

(ii) $y = \frac{1}{x}$
$= x^{-1}$
$\frac{dy}{dx} = -x^{-2}$
$= -\frac{1}{x^2}$ ✓

(iii) $y = \log_e(2x+1)$
$\frac{dy}{dx} = \frac{2}{2x+1}$ ✓

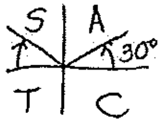
e) (i) $f(x) = x^2 + 2x + 4$
$F(x) = \frac{x^3}{3} + x^2 + 4x + C$ ✓

(ii) $f(x) = \frac{4}{x}$
$= 4x^{-1}$
$F(x) = 4\ln|x| + C$ ✓

b) (i) $4 - 2(x-3) = 4 - 2x + 6$
$= 10 - 2x$ ✓

(ii) $(2\sqrt{3}+\sqrt{5})(2\sqrt{3}+\sqrt{5})$
$= 12 + 2\sqrt{15} + 2\sqrt{15} + 5$
$= 17 + 4\sqrt{15}$ ✓

d) $\tan 150° = -\tan 30°$ ✓
$= \left(-\frac{1}{\sqrt{3}}\right)$ ✓

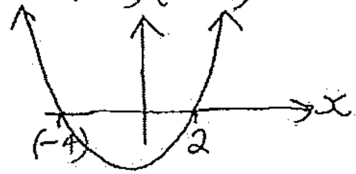


f) GP $a = 3$ $r = \frac{1}{2}$ ✓
$S_\infty = \frac{a}{1-r} = \frac{3}{1-\frac{1}{2}}$
$= 6$ ✓

(15) a) (i) $(x-2)^2 - 3 = 0$
$x - 2 = \pm\sqrt{3}$
$x = 2 \pm \sqrt{3}$ ✓

(ii) $|x-2| = 4$
$x - 2 = 4$ or $x - 2 = (-4)$
$x = 6$ or (-2) ✓

(iii) $(x-2)(x+4) > 0$



$x < (-4)$ or $x > 2$ ✓

b) $a(x-\alpha)(x-\beta) = 0$
$\therefore (x-3)(x+4) = 0$
or $x^2 + x - 12 = 0$ } ✓ (either : must include equals zero)

c) (i) $f(x) = x^3 - 8x$ $f(1) = 1 - 8 = (-7)$ ✓
$f'(x) = 3x^2 - 8$ $f'(1) = 3 - 8 = (-5)$ ✓
$f''(x) = 6x$ $f''(1) = 6$ ✓

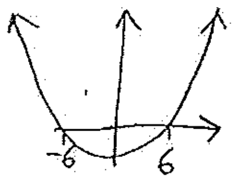
(ii) $f'(1) < 0$ $\therefore$ Curve is decreasing at $x = 1$ ✓

(iii) $f''(1) > 0$ $\therefore$ Curve is concave up at $x = 1$ ✓

(iv) $m = (-5)$ $(1, -7)$
$\therefore$ Tangent $y + 7 = -5(x-1)$ ✓
$5x + y + 2 = 0$ (or $y = -5x - 2$)

d) (i) $\Delta = b^2 - 4ac$
$= (-m)^2 - 4 \times 3 \times 3$
$= m^2 - 36.$ ✓

(ii) No real zeroes $\Delta < 0$ ✓
$\therefore (m-6)(m+6) < 0$
$(-6) < m < 6$ ✓



16) a) $\frac{dy}{dx} = 4x - 2$

$y = 2x^2 - 2x + C$ ✓

Given $(2,5)$

$5 = 2\times2^2 - 2\times2 + C$

$5 = 8 - 4 + C$ ✓

$1 = C$

$\therefore y = 2x^2 - 2x + 1$

b) $\int_1^3 2x^3\,dx$

$= \left[\frac{2x^4}{4}\right]_1^3$ ✓

$= \frac{1}{2}(3^4 - 1^4)$ ✓

$= 40$

c) $\int_0^4 4x - x^2\,dx = \left[2x^2 - \frac{x^3}{3}\right]_0^4$ ✓

$= \left(32 - \frac{64}{3}\right) - 0$

$= \frac{96}{3} - \frac{64}{3}$

$= \frac{32}{3}u^2$ (or $10\frac{2}{3}u^2$) ✓

d) (i) $y = (3x^2+2)^4$

Chain Rule

$\frac{dy}{dx} = 4(3x^2+2)^3 \times 6x$

$= 24x(3x^2+2)^3$ ✓

(ii) $y = 2x(x+7)^5$

Product Rule

$\frac{dy}{dx} = 2(x+7)^5 + 2x\times5(x+7)^4$ ✓

$= 2(x+7)^4[x+7+5x]$ ✓

$= 2(x+7)^4(6x+7)$

(iii) $y = \frac{\ln 3x}{x^2}$

Quotient Rule

$\frac{dy}{dx} = \frac{x^2\times\frac{3}{3x} - \ln 3x \times 2x}{x^4}$ ✓

$= \frac{x - 2x\ln 3x}{x^4}$

$= \frac{1 - 2\ln 3x}{x^3}$ ✓

e) $\lim_{x\to2}\left(\frac{x^2-4}{x-2}\right)$

$= \lim_{x\to2}\left(\frac{(x+2)\cancel{(x-2)}}{\cancel{(x-2)}}\right)$ ✓ (Working Reqd)

$= 4$ ✓

17) a) $\sqrt{2}, \sqrt{18}, \sqrt{50}$

$\sqrt{2}, 3\sqrt{2}, 5\sqrt{2}$

(i) $t_3 - t_2 = 5\sqrt{2} - 3\sqrt{2}$
$= 2\sqrt{2}$
$t_2 - t_1 = 3\sqrt{2} - \sqrt{2}$
$= 2\sqrt{2}$ ✓

$\therefore$ AP $a = \sqrt{2}$ $d = 2\sqrt{2}$.

(ii) $t_{100} = a + 99d$

$= \sqrt{2} + 198\sqrt{2}$

$= 199\sqrt{2}$ ✓

(iii) $S_{100} = \frac{100}{2}(\sqrt{2} + 199\sqrt{2})$ $S_n = \frac{n}{2}(a+l)$

$= 10000\sqrt{2}$ ✓

b)(i) $f(x) = 2x^2 - x$

$f(x+h) = 2(x+h)^2 - (x+h)$
$= 2(x^2+2xh+h^2) - x - h$
$= 2x^2 + 4xh + 2h^2 - x - h$ ✓

$\therefore f(x+h) - f(x) = 4xh + 2h^2 - h$

(ii) $f'(x) = \lim_{h\to0}\left(\frac{4xh+2h^2-h}{h}\right)$

$= \lim_{h\to0}(4x+2h-1)$ ✓

$= 4x - 1$

c) $g(x) = 3x^4 + 8x^3 + 12$

(i) $g'(x) = 12x^3 + 24x^2$
$= 12x^2(x+2)$ ✓

Stat pts $g'(x) = 0$

$\therefore x = 0$ or $x = (-2)$

$g(0) = 12$ $g(-2) = 3\times16 - 64 + 12$
$= (-4)$

Stat pts $(0,12)$ and $(-2,-4)$

x	-3	-2	-1	0	1
$g'(x)$	-108	0	12	0	36

✓

$\therefore (-2,-4)$ Minimum turning point
$(0,12)$ Stationary point of inflection ✓

(ii) $g''(x) = 36x^2 + 48x$
$= 12x(3x+4)$

Possible pts of inflection

$g''(x) = 0$

$x = 0$ or $x = -\frac{4}{3}$ ✓

$g(0) = 12$ $g(-\frac{4}{3}) =$

$3\times(-\frac{4}{3})^4 + 8(-\frac{4}{3})^3 + 12$

$= 9\frac{13}{27} - 18\frac{26}{27} + 12$

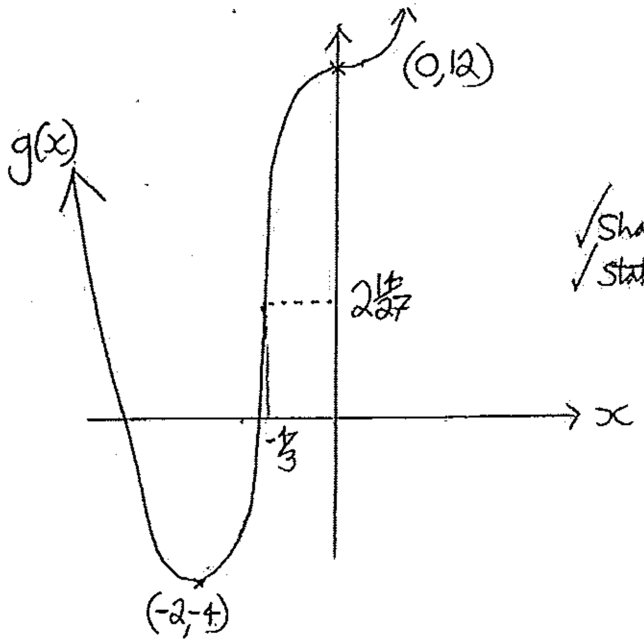
$= 2\frac{14}{27}$ ✓

$(0,12)$ is stationary point of inflection from (i)

x	-2	$-\frac{4}{3}$	-1	0	1
$g''(x)$	48	0	-12	0	84

✓

hence $(-\frac{4}{3}, 2\frac{14}{27})$ is a point of inflection too.



✓ Shape
✓ Stat pts & P.O.I.

(18) a) $V = \pi\int_a^b y^2\,dx$

$= \pi\int_0^4 9x\,dx$ ✓

$= \pi\left[\frac{9x^2}{2}\right]_0^4$ ✓

$= 72\pi\ u^3$ ✓

b) (i) $\int(2x+3)^5\,dx = \frac{(2x+3)^6}{6\times 2} + C$

$= \frac{1}{12}(2x+3)^6 + C$ ✓

(ii) $\int \frac{x+4}{\sqrt{x}}\,dx = \int x^{\frac{1}{2}} + 4x^{-\frac{1}{2}}\,dx$ ✓

$= \frac{2x^{3/2}}{3} + 8x^{\frac{1}{2}} + C$ ✓

$= \frac{2}{3}(\sqrt{x})^3 + 8\sqrt{x} + C$

(iii) $\int \frac{2x}{4+x^2}\,dx = \ln(4+x^2) + C$ ✓

c) (i) $y+18 = m(x-1)$ ✓

$y = mx - m - 18$

(ii) Line meets parabola $\Rightarrow$ solve simultaneously ✓

$2x^2+4x-6 = mx-m-18$

$2x^2 - mx + 4x + 12 + m = 0$

$2x^2 + (4-m)x + (12+m) = 0$ ✓

$\Delta = (4-m)^2 - 4\times 2\times(12+m)$

$= 16 - 8m + m^2 - 96 - 8m$ ✓

$= m^2 - 16m - 80$

Tangency $\Rightarrow \Delta = 0$

$\therefore m^2 - 16m - 80 = 0$

$(m-20)(m+4) = 0$

$m = 20$ or (-4) ✓

d) $\int_1^k \frac{1}{x^2}\,dx = \int_1^k x^{-2}\,dx$

$= \left[\frac{x^{-1}}{-1}\right]_1^k$

$= \left[-\frac{1}{x}\right]_1^k$

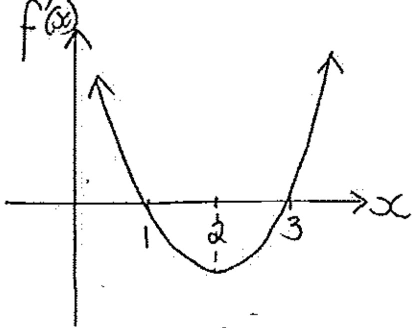
$= -\frac{1}{k} + 1$ ✓

$\therefore 1 - \frac{1}{k} = \frac{1}{4}$

$\frac{1}{k} = \frac{3}{4}$

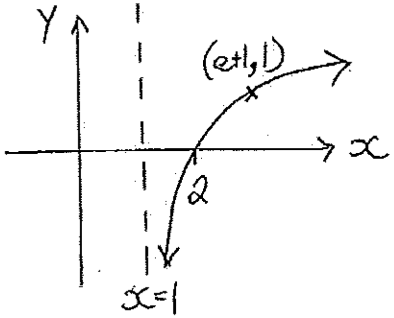
$k = \frac{4}{3}$ ✓

(19) a) $f'(x)$



$\checkmark$ Shape
$\checkmark$ x values corresponding

b) (i)



$\checkmark$ Shape, x-int + asymptote

(ii) $y = \ln(x-1)$

$\frac{dy}{dx} = \frac{1}{x-1}$

$\left(\frac{dy}{dx}\right)_{x=3} = \frac{1}{3-1} = \frac{1}{2}$ $\quad (3, \ln 2)$ $\checkmark$

$m_{tang} = \frac{1}{2}$ $\quad m_{norm} = (-2)$

Eqn of normal $\quad y - \ln 2 = -2(x-3)$ $\checkmark$

$y + 2x - (\ln 2 + 6) = 0$

(or equivalent)

c) (i) $x^2 - 4x + 6 = 0$.

α) $m+n = -b/a = 4$ $\checkmark$

β) $mn = c/a = 6$ $\checkmark$

γ) $\frac{1}{m} + \frac{1}{n} = \frac{n+m}{mn} = \frac{4}{6}$

$= 2/3$ $\checkmark$

(ii) $\frac{1}{m} + \frac{1}{n} = 2/3$

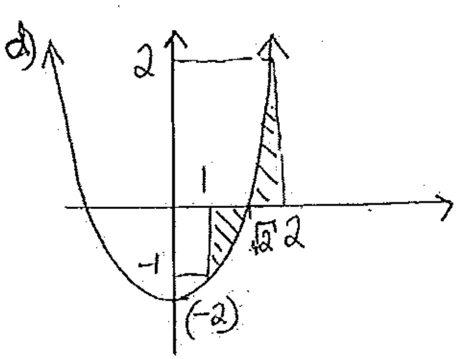
$\frac{1}{m} \times \frac{1}{n} = \frac{1}{mn}$

$= 1/6.$ $\checkmark$

$\therefore x^2 - \frac{2}{3}x + \frac{1}{6} = 0$

$6x^2 - 4x + 1 = 0$ $\checkmark$ (Needs RHS $=0$)

d)



Area $= \left|\int_1^{\sqrt{2}} x^2 - 2\,dx\right| + \int_{\sqrt{2}}^{2} x^2 - 2\,dx$ $\checkmark$

$= \left|\left[\frac{x^3}{3} - 2x\right]_1^{\sqrt{2}}\right| + \left[x^3/3 - 2x\right]_{\sqrt{2}}^{2}$

$= \left|\left(\frac{2\sqrt{2}}{3} - 2\sqrt{2}\right) - \left(\frac{1}{3} - 2\right)\right| + \left(\frac{8}{3} - 4\right) - \left(\frac{2\sqrt{2}}{3} - 2\sqrt{2}\right)$

$= \left|\frac{-4\sqrt{2}}{3} + \frac{5}{3}\right| + \frac{4\sqrt{2}}{3} - \frac{4}{3}$

$= \frac{4\sqrt{2} - 5}{3} + \frac{4\sqrt{2} - 4}{3}$

$= \frac{8\sqrt{2} - 9}{3}\ u^2$ $\checkmark$

(20) a) $4^x - 5\times 2^{x+1} + 16 = 0$

Let $u = 2^x$ $\quad (2^x)^2 - 10\times 2^x + 16 = 0$ ✓

$u^2 - 10u + 16 = 0$

$(u-8)(u-2) = 0$ ✓

$u = 8$ or 2

$2^x = 8$ or 2 ✓

$x = 3$ or 1

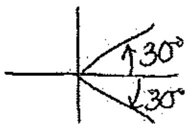
c) $4\cos(2\alpha - 45^\circ) - 2\sqrt{3} = 0$

$\cos(2\alpha - 45^\circ) = \dfrac{\sqrt{3}}{2}$

$0^\circ \leqslant \alpha \leqslant 360^\circ$

$0^\circ \leqslant 2\alpha \leqslant 720^\circ$

$-45^\circ \leqslant (2\alpha - 45^\circ) \leqslant 675^\circ$ ✓



$2\alpha - 45^\circ = -30^\circ, 30^\circ, 330^\circ, 390^\circ$ ✓

$2\alpha = 15^\circ, 75^\circ, 375^\circ, 435^\circ$

$\alpha = 7\tfrac{1}{2}^\circ, 37\tfrac{1}{2}^\circ, 187\tfrac{1}{2}^\circ, 217\tfrac{1}{2}^\circ$ ✓

d) $y = 2x^3 \log x$

$\dfrac{dy}{dx} = 6x^2 \log x + 2x^3 \times \left(\dfrac{1}{x}\right)$

$= 2x^2(1 + 3\log x)$

$\dfrac{d}{dx}(2x^3\log x) = 2x^2 + 6x^2 \log x$ ✓

b) Area between curves

$A = \displaystyle\int_{\frac{1}{2}}^{3} \left(-\frac{3}{x}\right) - (2x-7)\,dx$

$= \displaystyle\int_{\frac{1}{2}}^{3} 7 - 2x - \frac{3}{x}\,dx$ ✓

$= \left[7x - x^2 - 3\ln x\right]_{\frac{1}{2}}^{3}$

$= (21 - 9 - 3\ln 3) - \left(\frac{7}{2} - \frac{1}{4} - 3\ln\frac{1}{2}\right)$ ✓

$= 12 - 3\ln 3 - \frac{13}{4} + 3\ln\frac{1}{2}$

$= \frac{35}{4} + 3\left(\ln\frac{1}{2} - \ln 3\right)$

$= \frac{35}{4} + 3\ln\frac{1}{6}$

$= \frac{35}{4} - 3\ln 6$

$\therefore\ a = 35 \quad b = (-3)$ ✓

Integrating both sides wrt x

$\left[2x^3 \log x\right]_1^2 = \displaystyle\int_1^2 2x^2\,dx + \int_1^2 6x^2 \log x\,dx$ ✓

$\therefore \displaystyle\int_1^2 x^2 \log x\,dx = \frac{1}{6}\left[16\log 2 - 2\log 1 - \int_1^2 2x^2\,dx\right]$

$= \frac{1}{6}\left[16\log 2 - \left[\frac{2x^3}{3}\right]_1^2\right]$ ✓

$= \frac{1}{6}\left[16\log 2 - \frac{16}{3} + \frac{2}{3}\right]$

$= \frac{8}{3}\log 2 - \frac{7}{9}$ ✓

(21) a) $V = \displaystyle\int_0^4 \pi x^2\,dy - \int_0^4 \pi 2^2\,dy$ ✓ or $\displaystyle\int_0^4 \pi x^2\,dy - \pi \times 2^2 \times 4$ (Vol of Cylindrical Hole)

$x = 6 - \frac{y^2}{4}$

$x^2 = 36 - 3y^2 + \frac{y^4}{16}$ ✓

$V = \pi \displaystyle\int_0^4 36 - 3y^2 + \frac{y^4}{16}\,dy - 16\pi$

$= \pi\left[36y - y^3 + \frac{y^5}{80}\right]_0^4 - 16\pi$ ✓

$= \left(144 - 64 + \frac{4^5}{80}\right)\pi - 16\pi$

$= 64\pi + \frac{4^3}{5}\pi$

$= \frac{320\pi + 64\pi}{5}$

$= \frac{384\pi}{5}\ u^3$ ✓

(21) at $x=2$ $\quad y^2 = 24-8=16 \quad y=4 \ (y>0)$

a) $V = \int_0^4 \pi x^2\,dy - \int_0^4 \pi 2^2\,dy$ ✓ Limits $\quad$ OR $\int_0^4 \pi x^2\,dy - \pi\times 2^2\times 4$ (Vol of Cylindrical Hole)

$$x = 6 - \frac{y^2}{4}$$

$$x^2 = 36 - 3y^2 + \frac{y^4}{16}$$

$$V = \pi\int_0^4 36 - 3y^2 + \frac{y^4}{16}\,dy - 16\pi$$ ✓ Correct Integrand

$$= \pi\left[36y - y^3 + \frac{y^5}{80}\right]_0^4 - 16\pi$$ ✓ Correct Primitive

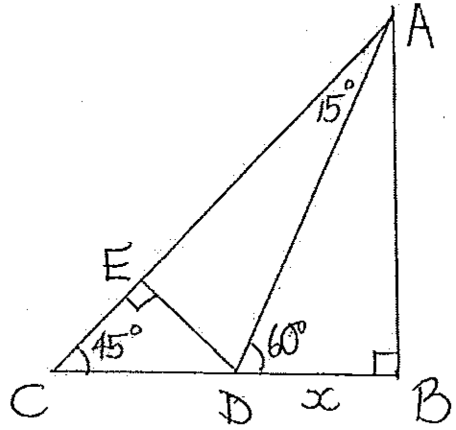
$$= \left(144 - 64 + \frac{4^5}{80}\right)\pi - 16\pi$$

$$= 64\pi + \frac{4^3}{5}\pi$$

$$= \frac{320\pi + 64\pi}{5}$$

$$= \frac{384\pi}{5}\ u^3 \quad \left(\doteq 76\tfrac{4}{5}\pi\right)$$ ✓ Answer

b)



in $\triangle BAD$ $\quad \tan 60° = \frac{AB}{x}$ $\quad \therefore AB = \sqrt{3}x$

$\cos 60° = \frac{x}{AD}$ $\quad \therefore AD = \frac{x}{\cos 60°}$ $\quad AD = 2x$

in $\triangle ABC$ $\quad \angle CAB = 45°$ (Angle Sum $\triangle CAB$)

in $\triangle DAB$ $\quad \angle DAB = 30°$ (Angle Sum $\triangle DAB$)

$\therefore \angle CAD = 15°$ (Adjacent Angles)

in $\triangle ABC$ $\quad CB = AB$ (Equal Sides of Isosceles $\triangle$)

$\therefore CB = \sqrt{3}x$ $\quad \therefore CD = CB - BD = (\sqrt{3}-1)x$

By Pythag $AC^2 = (\sqrt{3}x)^2 + (\sqrt{3}x)^2 = 6x^2$

$AC = \sqrt{6}x$

in $\triangle CED$ $\quad EC = ED$ (Isosceles $\triangle$)

By Pythag. $\therefore 2DE^2 = (\sqrt{3}-1)^2 x^2$

$DE^2 = \left[\left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)x\right]^2$ ✓ Show

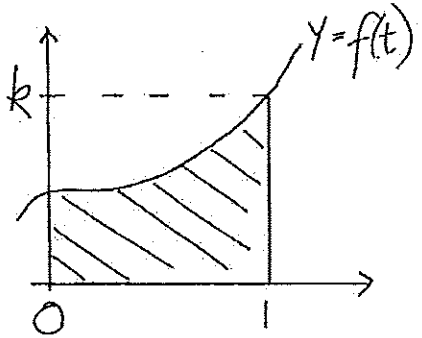
$DE = \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)x$

in $\triangle DEA$ $\quad \cos 15° = \frac{EA}{AD}$

$EA = AC - EC = \sqrt{6}x - \left(\frac{\sqrt{3}-1}{\sqrt{2}}\right)x = \left(\frac{\sqrt{12}-\sqrt{3}+1}{\sqrt{2}}\right)x = \left(\frac{\sqrt{3}+1}{\sqrt{2}}\right)x$ ✓

$\therefore \cos 15° = \frac{\sqrt{3}+1}{2\sqrt{2}}$ ✓ $\quad \left(\text{OR } \frac{\sqrt{2+\sqrt{3}}}{2}\right)$

(i)



$\int_0^1 f(t)\,dt$ represents shaded area

Rectangular area $1\times k = k$

$\therefore\ 0 \leqslant \int_0^1 f(t)\,dt \leqslant k$ ✓ (diagram + explanation)

(ii) $f(t) = \frac{1}{n-t} - \frac{1}{n}$ if $n>1$ $f(t)$ has its maximum value in the interval $0<t<1$ since first denominator is smallest at this point, hence $k = \frac{1}{n-1} - \frac{1}{n}$.

thus $0 \leqslant \int_0^1 \frac{1}{n-t} - \frac{1}{n}\,dt \leqslant \frac{1}{n-1} - \frac{1}{n}$ from (i) ✓

$0 \leqslant \left[-\ln(n-t) - \frac{t}{n}\right]_0^1 \leqslant \frac{1}{n-1} - \frac{1}{n}$

$0 \leqslant \left(-\ln(n-1) - \frac{1}{n}\right) - \left(-\ln n - 0\right) \leqslant \frac{1}{n-1} - \frac{1}{n}$

$0 \leqslant \ln n - \ln(n-1) - \frac{1}{n} \leqslant \frac{1}{n-1} - \frac{1}{n}$ ✓

$0 \leqslant \ln\left(\frac{n}{n-1}\right) - \frac{1}{n} \leqslant \frac{1}{n-1} - \frac{1}{n}$ ∎

(iii) Sum result (ii) from $n=N+1$ to $n=2N$

$$0 \leqslant \left\{\ln\left(\frac{N+1}{N}\right) + \ln\left(\frac{N+2}{N+1}\right) + \ln\left(\frac{N+3}{N+2}\right) + \ldots + \ln\left(\frac{2N}{2N-1}\right) - \frac{1}{N+1} - \frac{1}{N+2} - \frac{1}{N+3} - \ldots - \frac{1}{2N}\right\} \leqslant \left(\frac{1}{N} - \frac{1}{N+1}\right) + \left(\frac{1}{N+1} - \frac{1}{N+2}\right) + \ldots + \left(\frac{1}{2N-1} - \frac{1}{2N}\right)$$ ✓

Telescoping the series and applying log laws.

$$0 \leqslant \ln\left(\frac{N+1}{N} \times \frac{N+2}{N+1} \times \ldots \times \frac{2N-1}{2N-2} \times \frac{2N}{2N-1}\right) - \sum_{n=N+1}^{2N} \frac{1}{n} \leqslant \frac{1}{N} - \frac{1}{2N}$$ ✓

$$0 \leqslant \ln 2 - \sum_{n=N+1}^{2N} \frac{1}{n} \leqslant \frac{1}{2N}$$ ∎

(iv) putting $N=5$,

$$0 \leqslant \ln 2 - \sum_{n=6}^{10} \frac{1}{n} \leqslant \frac{1}{10}$$

$$0 \leqslant \ln 2 - \frac{1627}{2520} \leqslant \frac{1}{10}$$

$$\frac{1627}{2520} \leqslant \ln 2 \leqslant \frac{1627}{2520} + \frac{252}{2520}$$

$(\times 10)$ $$\frac{1627}{252} \leqslant 10\ln 2 \leqslant \frac{1879}{252}$$ ✓

$$6\tfrac{115}{252} \leqslant \ln(2^{10}) \leqslant 7\tfrac{115}{252}$$ ∎