Sydney Grammar School

2011 Annual Examination

FORM V
MATHEMATICS EXTENSION 1

Wednesday 31st August 2011

General Instructions
- Writing time — 3 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question on a new leaflet.

Structure of the paper
- Total marks — 135
- All nine questions may be attempted.
- All nine questions are of equal value.

Collection
- Write your name, class and master clearly on each leaflet.
- Hand in the nine questions in a single well-ordered pile.
- Hand in a leaflet for each question, even if it has not been attempted.
- If you use a second leaflet for a question, place it inside the first.
- Write your name on the question paper and place it inside your leaflet for Question One.

5A: BDD	5B: PKH	5C: FMW
5D: MK	5E: SJE	5F: RCF
5G: LJF	5H: SO	5I: MLS

Checklist
- Writing leaflets: 9 per boy.
- Candidature — 142 boys

Examiner
RCF

QUESTION ONE (15 marks) Start a new leaflet. **Marks**

(a) Write down the exact value of $\sin 315°$. [1]

(b) Express $\dfrac{1-\sqrt{3}}{\sqrt{2}}$ with a rational denominator. [1]

(c) Factorise $x^2 - 7x + 12$. [1]

(d) Evaluate $\log_3 \frac{1}{9}$. [1]

(e) Differentiate:

(i) $x^5 + 2x$ [1]

(ii) e^{2x+1} [1]

(f) Write down a primitive of $x^3 - 2$. [1]

(g) What is the nature of a stationary point if $\dfrac{d^2y}{dx^2} > 0$ at that point. [1]

(h) Simplify $\ln(e^3)$. [1]

(i) What is the natural domain of the function $f(x) = \ln(x+2)$? [1]

(j) Write as a single logarithm $2\log_{10} a + \log_{10} 3b$. [1]

(k) Sketch the graph of $(x-2)^2 + y^2 = 4$, clearly indicating any intercepts with the axes. [2]

(l) Solve $\tan\theta + \sqrt{3} = 0$, for $0° \le \theta \le 360°$. [2]

Exam continues next page ...

QUESTION TWO (15 marks) Start a new leaflet. **Marks**

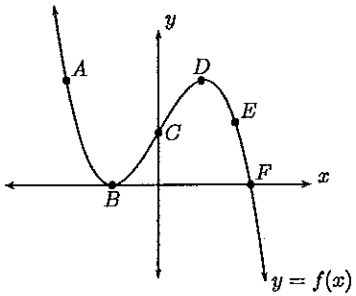
(a) (i) Sketch the parabola $y^2 = 4x$. [1]

(ii) State the co-ordinates of the focus. [1]

(b) Find the monic quadratic equation with roots 2 and -4. [2]

(c) The limiting sum of a GP with first term $\frac{1}{2}$ is 2. Find the common ratio. [2]

(d)



Which point labelled in the diagram above has:

(i) $f'(x) < 0$ and $f''(x) > 0$, [1]

(ii) $f''(x) = 0$. [1]

(e) Given points $A(-2,4)$ and $B(3,-11)$, find the co-ordinates of the point P which divides the interval AB internally in the ratio $2:3$. [2]

(f) (i) Complete the square on the expression $x^2 + 6x - 1$. [2]

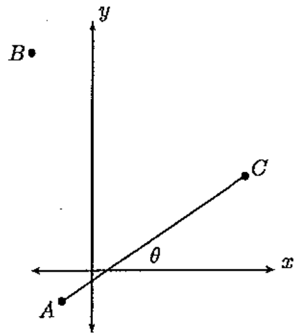
(ii) Hence, or otherwise, find the co-ordinates of the vertex of the parabola $y = x^2 + 6x - 1$. [1]

(g) (i) Graph the function $f(x) = \sqrt{4 - x^2}$. [1]

(ii) Hence evaluate the definite integral $\int_{-2}^{2} \sqrt{4 - x^2}\, dx$ without using calculus. [1]

QUESTION THREE (15 marks) Start a new leaflet. **Marks**

(a)



The points A, B and C, shown in the diagram above, have co-ordinates (-1,-1), (-2,7) and (5,3) respectively. The angle between AC and the x-axis is θ.

(i) Find the gradient of AC. [1]

(ii) Calculate the size of angle θ to the nearest degree. [1]

(iii) Find the equation of line AC. Give your answer in general form. [1]

(iv) Find the co-ordinates of D, the midpoint of AC. [1]

(v) Show that AC is perpendicular to BD. [1]

(vi) Explain why $\triangle ABC$ is isosceles. [1]

(vii) Find the area of $\triangle ABC$. [2]

(viii) Write down the co-ordinates of point E such that $ABCE$ is a rhombus. [1]

(b) (i) Differentiate $y = \log_e(2x - 1)$. [1]

(ii) Hence find the gradient of the tangent to the curve $y = \log_e(2x-1)$ at the point where $x = 3$. [1]

(c) (i) Graph $y = |2x - 2|$. Use at least one third of a page in your answer booklet for the graph. [2]

(ii) Hence, or otherwise, solve the inequation $|2x - 2| \geq 4$. [2]

QUESTION FOUR (15 marks) Start a new leaflet. **Marks**

(a) Differentiate:

(i) $y = (5 - 2x)^7$ [2]

(ii) $y = \dfrac{x^4 + 2x^2}{x}$ [2]

(b) Find:

(i) $\displaystyle\int x\sqrt{x}\, dx$ [2]

(ii) $\displaystyle\int \frac{1}{2x^3}\, dx$ [2]

(iii) $\displaystyle\int \frac{1}{x-3}\, dx$ [1]

(iv) $\displaystyle\int e^{4x-2}\, dx$ [1]

(c) Evaluate $\displaystyle\int_{-1}^{3} x(x-2)\, dx$. [2]

(d) Shade the region where the inequalities $x \geq 3$, $y \leq 10$ and $2x - y > 0$ are simultaneously satisfied. Clearly indicate on your diagram the inclusion or exclusion of points on the boundary lines and corners. [3]

Exam continues overleaf ...

QUESTION FIVE (15 marks) Start a new leaflet. **Marks**

(a) Solve $\sin(\alpha + 45°) = \frac{1}{2}$, for $0° \leq \alpha \leq 360°$. [3]

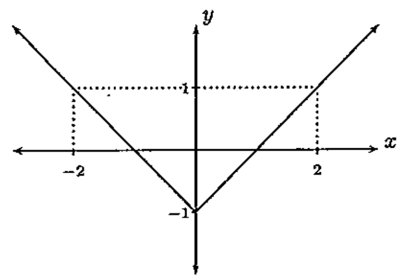
(b) The second term of a geometric series is 16 and the fifth term is 2.

(i) Find the common ratio. [1]

(ii) Find the first term. [1]

(iii) Find the sum of the first fifteen terms. Write your answer as a rational number in simplest form. [2]

(c) [1]



The diagram above shows the graph of $y = |x| - 1$. Find the value of $\displaystyle\int_0^2 |x| - 1\, dx$.

(d) Solve $\dfrac{2x-6}{x} \leq 1$ [3]

(e) Consider the parabola with equation $(y-4)^2 = -16(x+2)$.

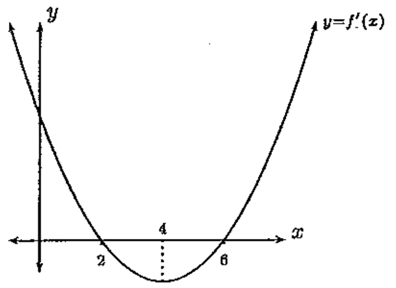
(i) What is the focal length? [1]

(ii) Sketch the curve, clearly showing the focus, the directrix and any intercepts with the axes. [3]

Exam continues next page ...

QUESTION SIX (15 marks) Start a new leaflet. **Marks**

(a) [3]



The diagram above shows the graph of the gradient function $y = f'(x)$ of the curve $y = f(x)$. Given that $f(0) = 0$, sketch the curve $y = f(x)$.

(b) Let $f(x) = \ln(x^2 + 1)$.

(i) Use the trapezoidal rule with three function values to estimate $\int_0^8 f(x)\,dx$. Write your answer correct to one decimal place. [2]

(ii) Use Simpson's rule with five function values to estimate $\int_0^8 f(x)\,dx$. Write your answer correct to three decimal places. [2]

(c) Differentiate:

(i) $y = \dfrac{\ln x}{x^2}$ [2]

(ii) $y = e^{2x}(2 - x^2)$ [2]

(d) Evaluate $\displaystyle\int_{-1}^{0} \frac{4x}{x^2+3}\,dx$. [2]

(e) Solve the equation $9^x - 10 \times 3^x + 9 = 0$. [2]

Exam continues overleaf ...

QUESTION SEVEN (15 marks) Start a new leaflet. **Marks**

(a) Suppose that the equation $2x^2 + 4x - 1 = 0$ has roots α and β. Without solving the equation, find the values of:

(i) $\alpha\beta$ [1]

(ii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta}$ [1]

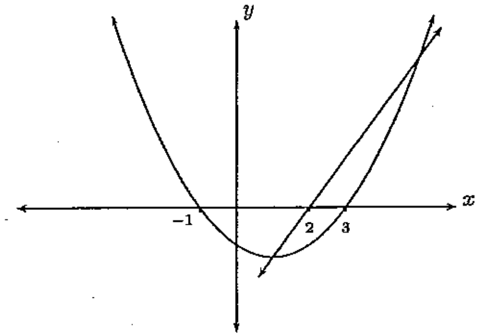
(iii) $(\alpha - \beta)^2$ [2]

(iv) $|\alpha - \beta|$ [1]

(b) (i) By forming a suitable arithmetic sequence, find how many multiples of 7 lie between 100 and 1000? [2]

(ii) What is the sum of these multiples? [1]

(c)



Consider the diagram above showing the parabola $y = 2x^2 - 4x - 6$ and the straight line $y = 8x - 16$.

(i) Find the x co-ordinates of the points of intersection of the parabola and the line. [1]

(ii) Find the area enclosed between the parabola and the line. [3]

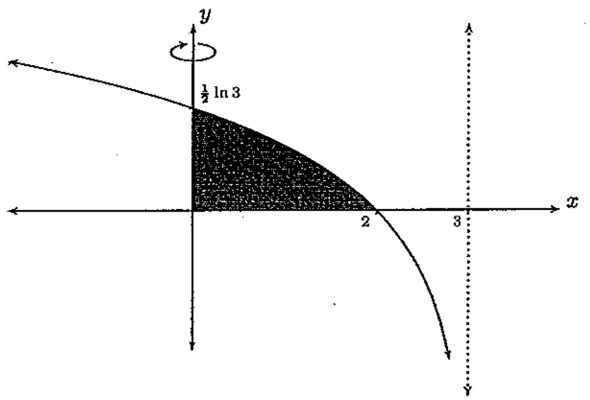
(d) (i) Show that the x co-ordinates of the points of intersection of the line $y = mx$ and the circle $(x+3)^2 + y^2 = 1$ satisfy the equation $(1 + m^2)x^2 + 6x + 8 = 0$. [1]

(ii) Hence find the equations of the two tangents to the circle $(x+3)^2 + y^2 = 1$ that pass through the origin. [2]

Exam continues next page ...

QUESTION EIGHT (15 marks) Start a new leaflet. **Marks**

(a)



In the diagram above the shaded region is bounded by the curve $y = \frac{1}{2}\ln(3 - x)$ and the co-ordinate axes.

(i) Find x as a function of y. **1**

(ii) Hence find the volume of the solid of revolution formed when the shaded region is rotated about the y-axis. **3**

(b) Consider the curve $y = (x - 2)e^{-x}$.

(i) Find any x or y intercepts. **2**

(ii) Find any stationary points and determine their nature. **3**

(iii) Find any points of inflection. **3**

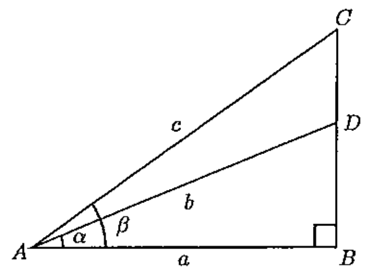
(iv) By considering the behaviour of the function for large values of x, determine the equation of the horizontal asymptote. **1**

(v) Sketch the curve $y = (x - 2)e^{-x}$, clearly showing all the above information. **2**

Exam continues overleaf ...

QUESTION NINE (15 marks) Start a new leaflet. **Marks**

(a)



In the diagram above, $\angle BAD = \alpha$, $\angle BAC = \beta$ and sides AB, AD and AC have lengths a, b and c respectively.

(i) Write down an expression for the area of $\triangle ACD$ in terms of b, c, α and β. **1**

(ii) Hence show that $\sin(\beta - \alpha) = \sin\beta\cos\alpha - \sin\alpha\cos\beta$. **2**

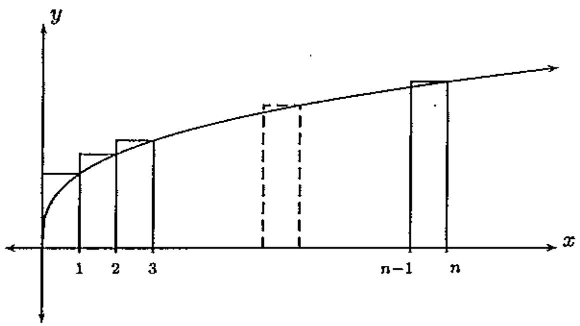
(b) (i) Differentiate $\sqrt{x}e^{\sqrt{x}}$. **1**

(ii) Hence find $\int e^{\sqrt{x}}\,dx$. **2**

(iii) Evaluate $\int_0^4 e^{\sqrt{x}}\,dx$. **1**

Exam continues next page ...

(c)



The diagram above shows the curve $y = \sqrt[3]{x}$, together with a set of n rectangles of unit width.

(i) By considering the areas of these rectangles, explain why **[1]**

$$\sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \ldots + \sqrt[3]{n} > \int_0^n \sqrt[3]{x}\,dx.$$

(ii) By drawing another set of rectangles and considering their areas, show that **[2]**

$$\sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \ldots + \sqrt[3]{n} < \int_1^{n+1} \sqrt[3]{x}\,dx.$$

(iii) Using parts (i) and (ii) find the value of $\sum_{n=1}^{100} \sqrt[3]{n}$, giving your answer accurate to as many significant figures as can be justified. **[2]**

(d) Given that e^x is the limiting sum of the infinite series **[3]**

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots$$

where $n! = n \times (n-1) \times (n-2) \times \cdots \times 2 \times 1$,

find an expression for the limiting sum of the infinite series

$$1 + 2x + \frac{3x^2}{2!} + \frac{4x^3}{3!} + \cdots + \frac{(n+1)x^n}{n!} + \cdots$$

END OF EXAMINATION

The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}}\,dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}}\,dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \ x > 0$

FIFTH FORM ANNUAL 2011

Question 1

a) $\sin 315^\circ = (-\sin 45^\circ) = \left(-\frac{1}{\sqrt{2}}\right)$ ✓

b) $\frac{1-\sqrt{3}}{\sqrt{2}} = \frac{\sqrt{2}-\sqrt{6}}{2}$ ✓

c) $x^2-7x+12 = (x-3)(x-4)$ ✓

d) $\log_3 \frac{1}{9} = (-2)$ ✓

e)(i) $y = x^5+2x$

$\frac{dy}{dx} = 5x^4+2$ ✓

(ii) $y = e^{2x+1}$

$\frac{dy}{dx} = 2e^{2x+1}$ ✓

f) $f(x) = x^3-2$

$F(x) = \frac{x^4}{4} - 2x + C$ ✓

g) Minimum turning point ✓

h) $\ln e^3 = 3$ ✓

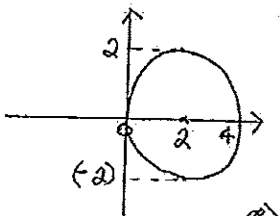
i) $x+2>0$

$x > (-2)$ ✓

j) $2\log_{10} a + \log_{10} 3b$

$= \log_{10}(a^2 \times 3b) = \log_{10} 3a^2b$ ✓

k)



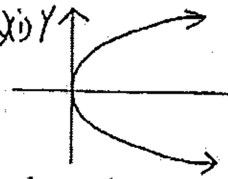
✓ SHAPE
✓ INTS

l) $\tan\theta = (-\sqrt{3})$

$\theta = 120^\circ, 300^\circ$ ✓✓

Question 2

a)(i)



✓ ORIENTATION & VERTEX

(ii) $S(1,0)$ ✓

b) $(x-2)(x+4) = 0$ ✓ LHS

OR $x^2+2x-8=0$ ✓ RHS

c) $S_\infty = \frac{a}{1-r}$ $\quad 2 = \frac{1/2}{1-r}$ ✓

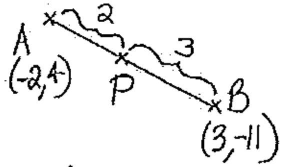
$1-r = \frac{1}{4}$

$r = \frac{3}{4}$ ✓

d)(i) Point A ✓

(ii) Point C ✓

e) $A(-2,4)$ $\quad B(3,-11)$ $\quad 2:3$



$x = \frac{2\times 3 + 3\times(-2)}{2+3}$ ✓

$= 0$

$y = \frac{2\times(-11)+3\times 4}{2+3}$

$= (-2)$

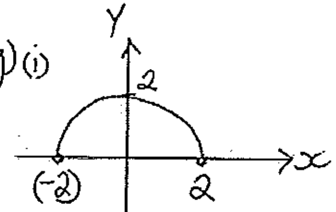
P is $(0,-2)$ ✓

f)(i) x^2+6x-1

$= (x+3)^2 - 10$ ✓✓

(ii) $V(-3,-10)$ ✓

2g)(i)



✓

(ii) $\int^2 \sqrt{4-x^2}\,dx = \frac{1}{2}\pi r^2$

$= \frac{1}{2}\times\pi\times 2^2$

$= 2\pi$ ✓

Question 3

a)(i) $A(-1,-1)$ $\quad C(5,3)$

$m_{AC} = \frac{3-(-1)}{5-(-1)} = \frac{4}{6} = \frac{2}{3}$ ✓

(ii) $\tan\theta = \frac{2}{3}$ $\quad \theta = \tan^{-1}\frac{2}{3}$

$\doteqdot 34^\circ$ ✓

(iii) $y+1 = \frac{2}{3}(x+1)$

$3y+3 = 2x+2$

$0 = 2x-3y-1$ ✓

(iv) $D\left(\frac{-1+5}{2}, \frac{-1+3}{2}\right) = (2,1)$ ✓

(v) $m_{BD} = \frac{7-1}{-2-2} = \frac{6}{-4} = -\frac{3}{2}$ SHOW

$\therefore m_{AC}\times m_{BD} = \frac{2}{3}\times\left(-\frac{3}{2}\right) = (-1)$ ✓

$\therefore AC \perp BD$

(vi) $\triangle ABD \equiv \triangle CBD$ (SAS) ✓

hence $AB = CB$ (Corresponding sides in congruent triangles)

OR

$\triangle ABC$ is Isosceles since median BD is also altitude

OR

SHOW $AB = BC = \sqrt{65}$

(vii) Area $= \frac{1}{2}\times AC\times BD$

$= \frac{1}{2}\times\sqrt{4^2+6^2}\times\sqrt{4^2+6^2}$ ✓

$= \frac{1}{2}\times 52$

$= 26\,u^2$ ✓

(viii) $B \to A$ $\quad$ +1, −8

hence $C \to E$ $\quad$ +1, −8

$E(6,-5)$ ✓

OR

Use D as midpoint of BE since diagonals of a rhombus bisect.

b)(i) $y = \log_e(2x-1)$

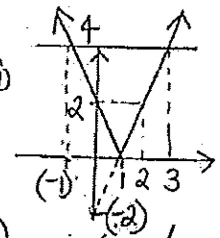
$\frac{dy}{dx} = \frac{1}{2x-1}\times 2$

$= \frac{2}{2x-1}$ ✓

(ii) $\left(\frac{dy}{dx}\right)_{x=3} = \frac{2}{6-1} = \frac{2}{5}$ ✓

Gradient of ~~tangent~~ at $x=3$ is $\frac{2}{5}$.

c)(i)



✓ SHAPE
✓ INTS

(ii) $x \le (-1)$ ✓ OR $x \ge 3$ ✓

Question 4

a)(i) $y = (5-2x)^7$

$\frac{dy}{dx} = 7(5-2x)^6 \times (-2)$ ✓✓

$= -14(5-2x)^6$

(ii) $y = \frac{x^4+2x^2}{x}$ ✓

$= x^3+2x$ ✓

$\frac{dy}{dx} = 3x^2+2$ ✓

b)(i) $\int x^{3/2}\,dx$ ✓ $= \frac{x^{5/2}}{5/2}+c$

$= \frac{2x^{5/2}}{5}+c$ ✓

(ii) $\int \frac{1}{2}x^{-3}\,dx$ ✓ $= \frac{1}{2}\frac{x^{-2}}{-2}+c$

$= \frac{-1}{4x^2}+c$ ✓

(iii) $\int \frac{1}{x-3}\,dx = \ln|x-3|+c$ ✓

(iv) $\int e^{4x-2}\,dx = \frac{e^{4x-2}}{4}+c$ ✓

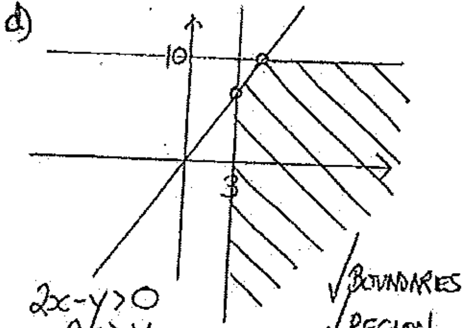
c) $\int_{-1}^{3} x(x-2)\,dx = \int_{-1}^{3} x^2-2x\,dx$

$= \left[\frac{x^3}{3}-x^2\right]_{-1}^{3}$ ✓

$= (9-9)-(-\frac{1}{3}-1)$

$= 4/3$ ✓

d)



$2x-y>0$

$2x>y$

✓ BOUNDARIES

✓ REGION

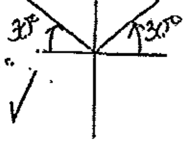
✓ CORNERS

Question 5

a) $\sin(\alpha+45°)$ $\quad 0 \le \alpha \le 360°$

$45° \le \alpha+45° \le 405°$ ✓

$\alpha+45° = 150°, 390°$ ✓



$\alpha = 105°, 345°$ ✓

b)(i) GP $\quad t_2 = ar = 16$ ①

$t_5 = ar^4 = 2$ ②

② ÷ ① $\quad r^3 = \frac{2}{16} = \frac{1}{8}$

$r = \frac{1}{2}$ ✓

(ii) $ar = 16$

$\frac{a}{2} = 16$

$a = 32 \qquad t_1 = 32$ ✓

(iii) $S_{15} = \frac{a(1-r^{15})}{1-r}$

$= \frac{32(1-(\frac{1}{2})^{15})}{\frac{1}{2}}$ ✓

$= 64(1-(\frac{1}{2})^{15}) = \frac{32767}{512}$ ✓

$(\doteqdot 63.998)$

Approx decimal earns one mark only.

c) $\int_0^2 |x-1|\,dx = 0$ ✓

(since triangles of equal area but one above & one below x axis)

d) $\frac{2x-6}{x} \le 1 \qquad$ Domain $x \neq 0$.

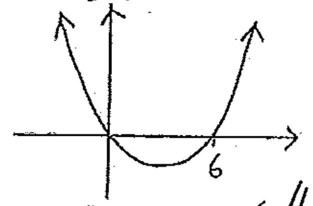
$(\times x^2)$ ✓

$(2x-6)x \le x^2$ ✓

$2x^2-6x-x^2 \le 0$

$x^2-6x \le 0$

$x(x-6) \le 0$



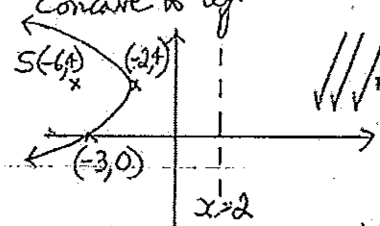
hence $0 < x \le 6$ ✓✓

e)(i) $4a = 16$

$a = 4.$ Focal Length is 4 ✓

(ii) Vertex $(-2,4)$ $\quad$ ∴ Focus $(-6,4)$

Concave to left



✓✓✓ SHAPE, FOCUS & DIRECTRIX, INT.

Intercepts $y=0 \qquad 16 = -16(x+2)$

$(x+2) = -1$

$x = -3$

Question 6

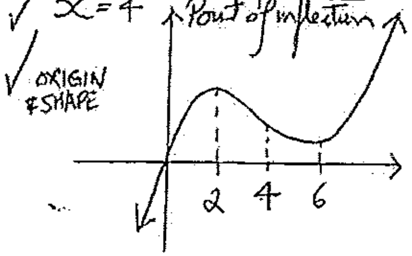
a) Stat pts @ $x=2$ and $x=6$

✓ $x=2$ Max t.p.

✓ $x=6$ Min t.p.

✓ $x=4$ Point of inflection

✓ ORIGIN & SHAPE



b)(i)

x	0	2	4	6	8
$f(x)$	ln 1 = 0	ln 5	ln 17	ln 37	ln 65

$\int_0^8 f(x)\,dx \doteqdot \frac{1}{2}\times 4[\ln 1+\ln 17]$ ✓

$+\frac{1}{2}\times 4[\ln 17+\ln 65]$

$\doteqdot 2[2\ln 17+\ln 65]$

$\doteqdot 19.7$ (1dp) ✓

(ii) $\frac{1}{6}\times 4\times(0+4\ln 5+\ln 17)$ ✓

$+\frac{1}{6}\times 4(\ln 17+4\ln 37+\ln 65)$

$\doteqdot 20.481$ (3dp) ✓

c)(i) $y = \frac{\ln x}{x^2}$

$\frac{dy}{dx} = \frac{x^2(\frac{1}{x}) - \ln x(2x)}{x^4}$ ✓

$= \frac{x-2x\ln x}{x^4}$

$= \frac{1-2\ln x}{x^3}$ ✓

(ii) $y = e^{2x}(2-x^2)$

$\frac{dy}{dx} = 2e^{2x}(2-x^2) + e^{2x}(-2x)$ ✓

$= 2e^{2x}[2-x-x^2]$ ✓

$= -2e^{2x}(x^2+x-2)$

$= -2e^{2x}(x+2)(x-1)$

d) $\int_{-1}^{0} \frac{4x}{x^2+3}dx = 2\int_{-1}^{0} \frac{2x}{x^2+3}dx$

$= [2\ln(x^2+3)]_{-1}^{0}$ ✓

$= 2\ln 3 - 2\ln 4$

$= 2\ln \frac{3}{4}$ or $-2\ln\frac{4}{3}$ ✓

e) $9^x - 10\times 3^x + 9 = 0$

$(3^x)^2 - 10\times 3^x + 9 = 0$

$(3^x - 9)(3^x - 1) = 0$ ✓

$3^x = 9$ or $3^x = 1$

$\therefore x = 2$ or $x = 0$ ✓

Question 7

a) $2x^2 + 4x - 1 = 0$

$a=2 \quad b=4 \quad c=(-1)$

(i) $\alpha\beta = c/a = (-\frac{1}{2})$ ✓

(ii) $\frac{1}{\alpha}+\frac{1}{\beta} = \frac{\beta+\alpha}{\alpha\beta}$

$= \frac{-b/a}{+c/a}$

$= -b/c$

$= 4$ ✓

(iii) $(\alpha-\beta)^2 = \alpha^2 - 2\alpha\beta + \beta^2$

$= \alpha^2 + 2\alpha\beta + \beta^2 - 4\alpha\beta$

$= (\alpha+\beta)^2 - 4\alpha\beta$ ✓

$= (-2)^2 - 4\times(-\frac{1}{2})$

$= 4 + 2 = 6$ ✓

(iv) $|\alpha-\beta| = \sqrt{(\alpha-\beta)^2}$

$= \sqrt{6}$ ✓

b)(i) AP $a = 105 \quad d = 7$

$t_n = a + (n-1)d$

$= 105 + 7(n-1)$ ✓

$t_n < 1000$

$105 + 7(n-1) < 1000$

$98 + 7n < 1000$

$7n < 902$

$n < \frac{902}{7}$

$n < 128\frac{6}{7}$

$t_{128} = 105 + 7\times 127$

$= 105 + 889$

$= 994$

$\therefore$ 128 multiples of 7 between 100 and 1000. ✓

(ii) $S_{128} = \frac{n}{2}(a + l)$

$= \frac{128}{2}(105 + 994)$

$= 70336$ ✓

c) (i) $2x^2 - 4x - 6 = 8x - 16$

$2x^2 - 12x + 10 = 0$

$x^2 - 6x + 5 = 0$

$(x-1)(x-5) = 0$

$x = 1$ or $x = 5$ ✓

(ii) Area $= \int_1^5 (8x-16) - (2x^2-4x-6)\,dx$ ✓

$= \int_1^5 12x - 10 - 2x^2\,dx$

$= \left[6x^2 - 10x - \frac{2x^3}{3}\right]_1^5$ ✓

$= \left(150 - 50 - \frac{250}{3}\right) - \left(6 - 10 - \frac{2}{3}\right)$

$= 104 - \frac{248}{3}$

$= 104 - 82\frac{2}{3}$

$= 21\frac{1}{3}u^2$ ✓

d)(i) $y = mx$ ①

$(x+3)^2 + y^2 = 1$ ②

sub ① into ②

$x^2 + 6x + 9 + m^2x^2 = 1$ show

$(1+m^2)x^2 + 6x + 8 = 0$ ✓

$\Delta = 36 - 4\times(1+m^2)\times 8$

$= 36 - 32(1+m^2)$

$= 4 - 32m^2$ ✓

For tangency $\Delta = 0$

$m^2 = 1/8$

$m = \pm\frac{1}{2\sqrt{2}}$

$\therefore$ Eqns of tangents

$y = \frac{\sqrt{2}x}{4}$ or $y = -\frac{\sqrt{2}x}{4}$ ✓

Question 8

a)(i) $y = \frac{1}{2}\ln(3-x)$ ①

$V_y = \pi\int x^2\,dy$

from ① $2y = \ln(3-x)$

$e^{2y} = (3-x)$

$x = 3 - e^{2y}$ ✓

(ii) $x^2 = 9 - 6e^{2y} + e^{4y}$

$V = \pi\int_0^{\frac{1}{2}\ln 3} 9 - 6e^{2y} + e^{4y}\,dy$ ✓

$= \pi\left[9y - \frac{6e^{2y}}{2} + \frac{e^{4y}}{4}\right]_0^{\frac{1}{2}\ln 3}$ ✓

$= \pi\left[\left(\frac{9}{2}\ln 3 - 9 + \frac{9}{4}\right) - \left(0 - 3 + \frac{1}{4}\right)\right]$

$= \pi\left(\frac{9}{2}\ln 3 - 4\right)\ u^3$ ✓

b) $y = (x-2)e^{-x}$

(i) y ints. $x = 0$

$y = -2\times e^{-0} = -2 \quad (0,-2)$ ✓

x ints $y = 0$

$e^{-x} \neq 0 \therefore x - 2 = 0$

$x = 2$

$(2, 0)$ ✓

(ii) $\frac{dy}{dx} = -e^{-x}(x-2) + 1\times e^{-x}$

$= e^{-x}[3 - x]$ ✓

Stat pts $\frac{dy}{dx} = 0$.

$e^{-x} \neq 0 \quad \therefore x = 3 \quad (3, \frac{1}{e^3})$ ✓

$y = 1/e^3$

$\frac{d^2y}{dx^2} = -e^{-x}(3-x) + e^{-x}(-1)$

$= e^{-x}[x - 4]$

$\therefore \left(\frac{d^2y}{dx^2}\right)_{x=3} = -\frac{1}{e^3} < 0 \therefore$ Max turning pt @ $(3, \frac{1}{e^3})$ ✓

(iii) Possible POI $\frac{d^2y}{dx^2} = e^{-x}(x-4)$ ✓

$\frac{d^2y}{dx^2} = 0 \quad e^{-x} \neq 0 \quad \therefore x = 4$ ✓

$y = \frac{2}{e^4}$

x	3	4	5
$\frac{d^2y}{dx^2}$	$-\frac{1}{e^3}$ ∩	0 •	$\frac{1}{e^5}$ ∪

✓

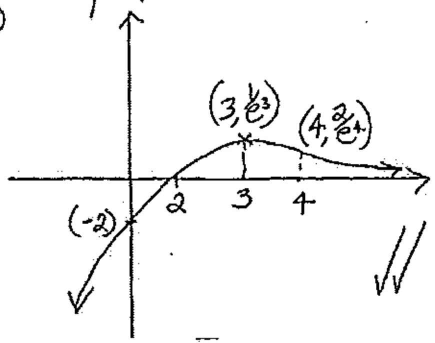
There is a change in concavity hence $(4, \frac{2}{e^4})$ is a point of inflection

(iv) $y = (x-2)e^{-x}$

as $x \to \infty \quad xe^{-x} \to 0^+$

$\therefore$ (x axis) $y = 0$ is an asymptote ✓

(v)



✓✓

Question 9

a) i) Area $\triangle ACD = \frac{1}{2}bc\sin(\angle ACD)$

$= \frac{1}{2}bc\sin(\beta - \alpha)$ ✓

OR

Area $\triangle ABD = \frac{1}{2}ab\sin\alpha$

Area $\triangle ABC = \frac{1}{2}ac\sin\beta$.

$\therefore$ Area $\triangle ACD = \frac{1}{2}ac\sin\beta - \frac{1}{2}ab\sin\alpha$ ①

Equating area expressions

$\frac{1}{2}bc\sin(\beta-\alpha) = \frac{1}{2}ac\sin\beta - \frac{1}{2}ab\sin\alpha$

in $\triangle ABD$

$\cos\alpha = \frac{a}{b}$

$\therefore a = b\cos\alpha$ ② ✓

in $\triangle ABC$

$\cos\beta = \frac{a}{c}$

$\therefore a = c\cos\beta$ ③

sub ② and ③ into ①

$\frac{1}{2}bc\sin(\beta-\alpha) = \frac{1}{2}(b\cos\alpha)c\sin\beta - \frac{1}{2}(c\cos\beta)b\sin\alpha$ ✓

$(\div \frac{1}{2}bc)$

$\sin(\beta-\alpha) = \sin\beta\cos\alpha - \sin\alpha\cos\beta$

b) $y = \sqrt{x}e^{\sqrt{x}}$

$= x^{\frac{1}{2}}e^{x^{\frac{1}{2}}}$

(i) $\frac{dy}{dx} = \frac{1}{2}x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}} + x^{\frac{1}{2}} \times \frac{1}{2}x^{-\frac{1}{2}}e^{x^{\frac{1}{2}}}$

$= \frac{e^{\sqrt{x}}}{2\sqrt{x}} + \frac{e^{\sqrt{x}}}{2}$ (*) ✓

(ii) $\int e^{\sqrt{x}}\,dx$ Hence integrate result (*)

$\sqrt{x}e^{\sqrt{x}} + C = \int \frac{e^{\sqrt{x}}}{2\sqrt{x}}\,dx + \int \frac{e^{\sqrt{x}}}{2}\,dx$ ✓

$\sqrt{x}e^{\sqrt{x}} + C = e^{\sqrt{x}} + \frac{1}{2}\int e^{\sqrt{x}}\,dx$

$\therefore \frac{1}{2}\int e^{\sqrt{x}}\,dx = e^{\sqrt{x}}(\sqrt{x}-1) + C$ ✓

$\int e^{\sqrt{x}}\,dx = 2e^{\sqrt{x}}(\sqrt{x}-1) + d$ ✓

(iii) $\int_0^4 e^{\sqrt{x}}\,dx = \left[2e^{\sqrt{x}}(\sqrt{x}-1)\right]_0^4$

$= 2e^2 + 2 = 2(e^2+1)$ ✓

c) Area under cube root curve is OVERestimated by upper rectangles since $\sqrt[3]{x}$ is concave down.

Height of each rectangle is x coord at right hand edge cube rooted, width of each rectangle is one

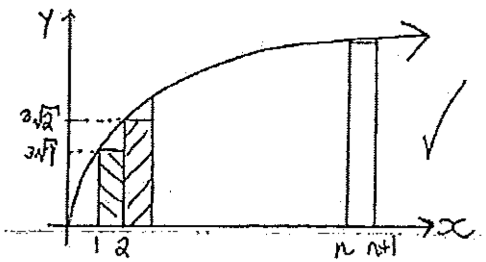
i.e first rectangle $A_1 = 1 \times \sqrt[3]{1}$

last rectangle $A_n = 1 \times \sqrt[3]{n}$ ✓

hence

$\sqrt[3]{1} + \sqrt[3]{2} + \ldots + \sqrt[3]{n} > \int_0^n \sqrt[3]{x}\,dx$

(ii) This set of rectangles must underestimate area under curve hence take height as cube root of x co-ordinate at left hand end of rectangle to get lower rectangles as illustrated below ✓



✓

hence

$\int_1^{n+1} \sqrt[3]{x}\,dx > \sqrt[3]{1} + \sqrt[3]{2} + \ldots + \sqrt[3]{n}$

note change to limits of integral $1 \to n+1$

(iii) $\int_0^{100} \sqrt[3]{x}\,dx < \sqrt[3]{1} + \sqrt[3]{2} + \sqrt[3]{3} + \ldots + \sqrt[3]{100} < \int_1^{101} \sqrt[3]{x}\,dx$

$\left[\frac{3x^{\frac{4}{3}}}{4}\right]_0^{100} < \sum_{n=1}^{100} \sqrt[3]{n} < \left[\frac{3x^{\frac{4}{3}}}{4}\right]_1^{101}$ ✓

$\frac{3}{4}(100^{\frac{4}{3}}) < \sum_{n=1}^{100} \sqrt[3]{n} < \frac{3}{4}(101^{\frac{4}{3}} - 1)$

$348.1\ldots < \sum_{n=1}^{100} \sqrt[3]{n} < 352.0\ldots$

$\sum_{n=1}^{100} \sqrt[3]{n} \doteqdot 350$ (2sf) ✓

(Must consider both bounds for two marks)

d) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots + \frac{x^n}{n!} + \ldots$

$xe^x = x + \frac{x^2}{1!} + \frac{x^3}{2!} + \frac{x^4}{3!} + \ldots + \frac{x^{n+1}}{n!} + \ldots$ ✓

$\frac{d}{dx}(xe^x) = 1 + 2x + \frac{3x^2}{2!} + \frac{4x^3}{3!} + \ldots + \frac{(n+1)x^n}{n!} + \ldots$ ✓

LHS $= xe^x + 1 \times e^x$

$= e^x(x+1)$

Hence required result is $e^x(x+1)$ ✓