SYDNEY GRAMMAR SCHOOL

2011 Annual Examination

FORM V MATHEMATICS

Wednesday 31st August 2011

General Instructions

- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- All necessary working should be shown in every question.
- Start each question on a new leaflet.

Structure of the paper

- Total marks — 108
- All nine questions may be attempted.
- All nine questions are of equal value.

Collection

- Write your name, class and master clearly on each leaflet.
- Hand in the nine questions in a single well-ordered pile.
- Hand in a leaflet for each question, even if it has not been attempted.
- If you use a second leaflet for a question, place it inside the first.
- Write your name on the question paper and place it inside your leaflet for Question One.

5P: SJG 5Q: TCW

Checklist

- Writing leaflets: 9 per boy.
- Candidature — 31 boys

Examiner
SJG

QUESTION ONE (12 marks) Start a new leaflet.

(a) Write $\tan 58°$ correct to one decimal place.

(b) Write $\log_{10} 3$ correct to three significant figures.

(c) Factorise:

(i) $4x^2 - 1$

(ii) $x^2 + 3x - 18$

(d) Simplify:

(i) $(5\sqrt{2})^2$

(ii) $8^{\frac{2}{3}}$

(e) Solve:

(i) $|x + 1| = 4$

(ii) $3x + 2 \le 11$

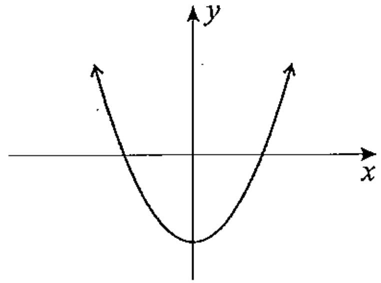
(f) Find the next term of the geometric sequence $3, -6, 12, \ldots$

(g) Differentiate:

(i) $y = 4 + 3x$

(ii) $y = 3x^2 + 7x$

(h)



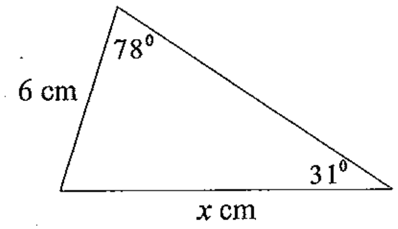
In the above diagram, state whether the given quadratic function has a positive, negative or zero discriminant.

QUESTION TWO (12 marks) Start a new leaflet.

(a) Simplify $\frac{6x}{5} - \frac{x}{6}$.

(b) Find the 20th term in the arithmetic sequence 2, 9, 16, ...

(c)



(i) Use the sine rule to find the value of x (to one decimal place) in the diagram above.

(ii) Hence, or otherwise, find the area of the triangle correct to one decimal place.

(d) Write the equation of the circle with centre $(-1, 3)$ and radius $\sqrt{5}$ units.

(e) Evaluate $\log_3 9$.

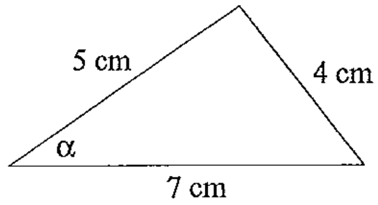
(f) Write down the natural domain of the function $g(x) = \frac{1}{\sqrt{x}}$.

(g) Evaluate $\sum_{n=1}^{4} n^2$.

Exam continues overleaf ...

QUESTION THREE (12 marks) Start a new leaflet.

(a)



In the diagram above, find α using the cosine rule, giving your answer correct to the nearest degree.

(b) Factorise $3x^2 - 8x + 4$.

(c) Express $\frac{6}{\sqrt{7} - \sqrt{5}}$ in simplest form with a rational denominator.

(d) (i) Sketch the graph of $y = -x^2 + 4$, showing all intercepts with the axes.

(ii) Hence solve $4 - x^2 > 0$.

(e) Use the quotient rule to find $\frac{dy}{dx}$, given $y = \frac{2x}{1-x}$.

(f) Find the perpendicular distance from the origin to the line $3x + 2y - 4 = 0$.

Exam continues next page ...

QUESTION FOUR (12 marks) Start a new leaflet.

(a) Solve $\log_5 x = -2$.

(b) Suppose that α and β are the roots of the equation $-x^2+8x-9=0$. Without solving the equation, find:

(i) $\alpha+\beta$

(ii) $\alpha\beta$

(iii) $\dfrac{1}{\alpha}+\dfrac{1}{\beta}$

(c) (i) Write x^2-4x-1 in the form $(x-h)^2+k$ by completing the square.

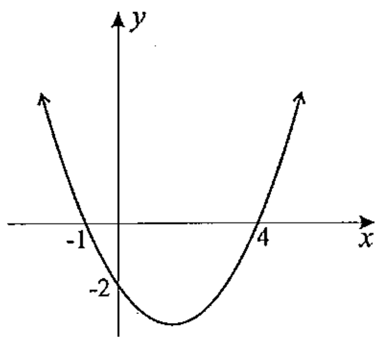
(ii) Hence, or otherwise, find the coordinates of the vertex of the parabola $y=x^2-4x-1$.

(d) A footballer is running a series of laps at training. The first lap takes him 42·5 seconds, and each successive lap takes 5 seconds more than the previous one.

(i) Which lap will be the first to take more than 1 min 30 secs?

(ii) How many minutes will it take him to run 20 laps?

(e)



Find the equation of the parabola in the diagram above.

QUESTION FIVE (12 marks) Start a new leaflet.

(a) Consider the curve $y=4x^3-3x^2+2$. Find the x-coordinates of the points where the tangent to the curve is horizontal.

(b) Consider the curve $y=2x^2-5x+3$.

(i) Show that the point $P(2,1)$ lies on the curve.

(ii) Find the gradient of the tangent to the curve at P.

(iii) Find the equation of the tangent to the curve at P.

(iv) Show that the equation of the normal to the curve at P is $x+3y-5=0$.

(v) Let Q and R be the x-intercept and y-intercept of the normal respectively. Find the coordinates of Q and R.

(vi) Hence calculate the area of the triangle ΔOQR, where O is the origin.

QUESTION SIX (12 marks) Start a new leaflet.

(a) Differentiate:

(i) $y = \dfrac{1}{x^2+2}$

(ii) $y = x^4(x+1)^3$ [In part (ii), write the derivative in fully factorised form.]

(b) Simplify fully $\log_4 5 + 2\log_4 6 - \log_4 45$.

(c) (i) Express $0{\cdot}\dot{3}\dot{6}$ as an infinite geometric series.

(ii) Using the formula for the limiting sum of a geometric series, express $0{\cdot}\dot{3}\dot{6}$ as a fraction in lowest terms.

(d) (i) Write $\cot\alpha$ in terms of $\sin\alpha$ and $\cos\alpha$.

(ii) Prove the identity $\sin\alpha + \cot\alpha\cos\alpha = \operatorname{cosec}\alpha$.

QUESTION SEVEN (12 marks) Start a new leaflet.

(a) Bill has released a new iPhone app. In its first month of release, Bill sells 10 000 copies of the app. In each successive month, he sells 20% fewer copies than in the month before.

(i) Show that the numbers of copies sold in successive months form a geometric progression.

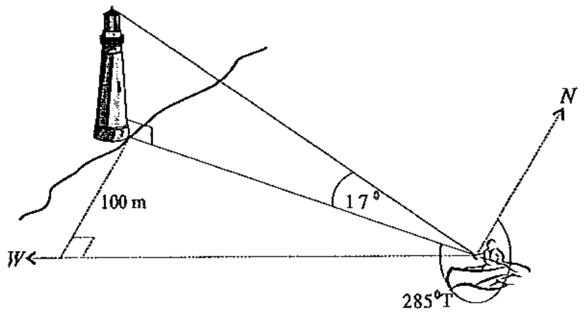
(ii) How many copies of the app does Bill sell in the first 10 months?

(b) An arrow is fired from a stationary position. Its height h at time t is given by $h = 17 + 6t - t^2$, where h is measured in metres and t in seconds.

(i) Find the time at which the arrow reaches its greatest height.

(ii) Hence find the maximum height reached by the arrow.

(c)



The diagram above shows how a canoeist can see a lighthouse on a bearing of 285°T from his canoe. The base of the lighthouse is at sea level. Looking up at the top of the lighthouse, he can see the light at an angle of elevation of 17°. Find, correct to the nearest metre:

(i) the distance from the canoe to the base of the lighthouse,

(ii) the height of the lighthouse.

(d) On the same set of axes, sketch the graphs of $y = 2^x$ and $y = \log_2 x$.

(e) (i) On the same set of axes, sketch the graphs of $y = |2x-5|$ and $y = x-1$. Give the coordinates of any points of intersection.

(ii) On your graph, shade the region where $y \le x-1$ and $y \ge |2x-5|$.

QUESTION EIGHT (12 marks) Start a new leaflet.

(a) Find $\lim_{x \to 3} \left(\dfrac{x^2 - x - 6}{x^2 + 2x - 15} \right)$.

(b) Find the equation of the line that passes through the point $(2, 1)$ and also through the point of intersection of the lines $x - 2y + 3 = 0$ and $2x + y - 1 = 0$.

(c) (i) Find the discriminant of the quadratic expression $mx^2 - mx + (m - 3)$.

(ii) Hence find the value of m for which the equation $mx^2 - mx + (m - 3) = 0$ has two equal roots.

(d) Find all solutions to the equation $2\cos 2\theta = 1$, for $0° \le \theta \le 360°$.

Exam continues overleaf ...

QUESTION NINE (12 marks) Start a new leaflet.

(a) Using the substitution $u = 3^x$, solve $9^x - 12 \times 3^x + 27 = 0$.

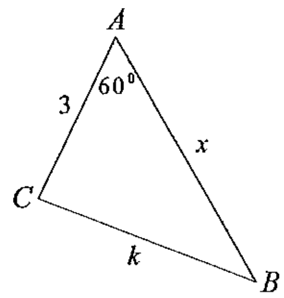
(b) Find the equation of the normal to the curve $y = \sqrt{3 - x}$ at the point $A(-1, 2)$.

(c) Consider the geometric series $1 + \cos^2\theta + \cos^4\theta + \cdots$

(i) Find an expression for the limiting sum in simplest form.

(ii) If the limiting sum is equal to 4, find all possible values of θ given $0° \le \theta \le 360°$.

(d)



The diagram above shows $\triangle ABC$ with $AC = 3$, $AB = x$, $BC = k$ and $\angle BAC = 60°$.

(i) Use the cosine rule to show that $k^2 = x^2 - 3x + 9$.

(ii) Explain why k must be at least $\frac{3\sqrt{3}}{2}$.

(iii) Find $\angle ACB$ when $k = \frac{3\sqrt{3}}{2}$.

END OF EXAMINATION

2011 SGS : Form V 2 Unit Yearly Examination

SOLUTIONS

Q1. a) $\tan 58° = 1.6$ ✓

b) $\log_{10} 3 = 0.477$ ✓

c) i) $4x^2 - 1 = (2x+1)(2x-1)$ ✓

ii) $x^2 + 3x - 18 = (x+6)(x-3)$ ✓

d) i) $(5\sqrt{2})^2 = 50$ ✓

ii) $8^{2/3} = 4$ ✓

e) i) $|x+1| = 4 \quad \therefore \ x+1 = 4$ or $-x-1 = 4$ ✓ both

$x = 3$ or $x = -5$

ii) $3x + 2 \leq 11$

$3x \leq 9$

$x \leq 3$ ✓

f) 3, −6, 12, −24 ✓

g) i) $y = 4 + 3x \quad \therefore \ y' = 3$ ✓

ii) $y = 3x^2 + 7x \quad \therefore \ y' = 6x + 7$ ✓

h) $\Delta > 0$ or "the discriminant is positive" ✓ either

Q2. a) $\dfrac{36x - 5x}{30}$ ✓ $= \dfrac{31x}{30}$ ✓

b) $T_n = a + (n-1)d$

$T_{20} = 2 + (19)7$ ✓

$= 135$ ✓

c) i) $\dfrac{x}{\sin 78°} = \dfrac{6}{\sin 31°}$ ✓

$\therefore \ x = \dfrac{6 \sin 78°}{\sin 31°}$

$= 11.395\ldots$

$\doteqdot 11.4$ ✓

ii) $\angle A = 180 - 78 - 31$

$\angle A = 71°$

Area $= \frac{1}{2} \times 6 \times x \times \sin 71°$ ✓

$= 32.3226\ldots$

$\doteqdot 32.3 \text{ cm}^2$ ✓

d) $(x+1)^2 + (y-3)^2 = 5$ ✓

e) $\log_3 9 = 2$ ✓

f) $x > 0$ ✓

g) $\sum_{n=1}^{4} n^2 = 1^2 + 2^2 + 3^2 + 4^2$

$= 30$ ✓

Q3. a) $\cos\alpha = \dfrac{a^2+b^2-c^2}{2ab}$ $\quad \therefore \cos\alpha = \dfrac{7^2+5^2-4^2}{2\times7\times5}$ ✓

$= \dfrac{58}{70}$

$\therefore \alpha = \cos^{-1}\left(\dfrac{58}{70}\right)$

$\alpha = 34°$ (nearest degree) ✓

b) $3x^2 - 8x + 4 = 3x^2 - 6x - 2x + 4$

$= 3x(x-2) - 2(x-2)$

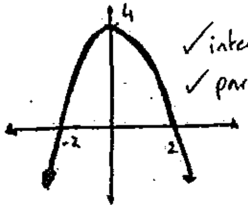
$= (x-2)(3x-2)$ ✓

c) $\dfrac{6}{\sqrt7 - \sqrt5} = \dfrac{6(\sqrt7+\sqrt5)}{7-5}$

$= \dfrac{6\sqrt7 + 6\sqrt5}{2}$

$= 3\sqrt7 + 3\sqrt5$ ✓

d) i)



✓intercepts
✓parabola

ii) $4 - x^2 > 0 \iff$ when is the parabola above the x-axis?

$\therefore -2 < x < 2$ ✓

e) $y = \dfrac{2x}{1-x}$, $\quad u = 2x \iff u' = 2$ ✓

$v = 1-x \iff v' = -1$

$\dfrac{dy}{dx} = \dfrac{u'v - v'u}{v^2}$

$= \dfrac{2(1-x)+1(2x)}{(1-x)^2}$ ✓

$= \dfrac{2-2x+2x}{(1-x)^2}$

$= \dfrac{2}{(1-x)^2}$ ✓

f) $d = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$

$d = \dfrac{|0+0-4|}{\sqrt{3^2+2^2}}$ ✓

$= \dfrac{4}{\sqrt{13}}$ or $\dfrac{4\sqrt{13}}{13}$ ✓either

Q4. a) $\log_5 x = -2 \iff 5^{-2} = x$

$\therefore x = \dfrac{1}{25}$ ✓

b) i) $\alpha + \beta = -\dfrac{b}{a}$ $\quad \therefore \alpha+\beta = 8$ ✓

ii) $\alpha\beta = \dfrac{c}{a}$ $\quad \therefore \alpha\beta = 9$ ✓

iii) $\dfrac{1}{\alpha} + \dfrac{1}{\beta} = \dfrac{\beta+\alpha}{\alpha\beta} = \dfrac{8}{9}$ ✓

c) i) $x^2 - 4x - 1 = x^2 - 4x + 4 - 4 - 1$

$= (x-2)^2 - 5$ ✓

ii) vertex $(h,k) = (2,-5)$ ✓

d) i) 42.5, 47.5, 52.5, ...

$a = 42.5$

$d = 5$

$T_n > 90$ secs

$\therefore 42.5 + (n-1)\times 5 > 90$ ✓

$5(n-1) > 47.5$

$5n - 5 > 47.5$

$5n > 52.5$

$n > 10.5$

$\therefore$ the 11^{th} lap will be the first to take more than 90 secs. ✓

ii) $S_n = \dfrac{n}{2}\left(2a + (n-1)d\right)$

$\therefore 1800 = \dfrac{n}{2}\left(2\times42.5 + 5(n-1)\right)$ ✓

$3600 = n(85 + 5n - 5)$

$3600 = 80n + 5n^2$

$720 = 16n + n^2$

$n^2 + 16n - 720 = 0$ $\quad \therefore n = 20$ laps ✓

$(n-20)(n+36) = 0$

e) $y = a(x-\alpha)(x-\beta)$

$\therefore y = a(x+1)(x-4)$

$y = a(x^2 - 3x - 4)$ ✓

$y = ax^2 - 3ax - 4a$

y-intercept is $(0,-2)$

$\therefore -4a = -2$

$a = \frac{1}{2}$

$\therefore y = \frac12 x^2 - \frac32 x - 2$

or $y = \frac12(x^2 - 3x - 4)$ ✓any version

or $y = \frac12(x+1)(x-4)$

Q5. a) $y = 4x^3 - 3x^2 + 2 \Leftrightarrow \frac{dy}{dx} = 12x^2 - 6x$ ✓

tangent horizontal $\Leftrightarrow \frac{dy}{dx} = 0$

$12x^2 - 6x = 0$ ✓

$6x(2x-1) = 0$

$x = 0$ or $x = \frac{1}{2}$ ✓ both

b) $y = 2x^2 - 5x + 3$

i) at P: LHS $= 1$

RHS $= 2 \times 2^2 - 5 \times 2 + 3$

$= 1$ ✓

LHS = RHS $\therefore P(2,1)$ is on the curve

ii) $\frac{dy}{dx} = 4x - 5$ ✓

at P: $\frac{dy}{dx} = 4 \times 2 - 5$

$= 3$ ✓

iii) $y - y_0 = m(x - x_0) \quad \therefore y - 1 = 3(x-2)$

$y = 3x - 6 + 1$

$y = 3x - 5$ ✓

iv) gradient of normal : $m_\perp = -\frac{1}{3}$

$\therefore y - 1 = -\frac{1}{3}(x-2)$ ✓

$y - 1 = -\frac{1}{3}x + \frac{2}{3}$

$y = -\frac{1}{3}x + \frac{5}{3}$

$3y = -x + 5$

$\therefore x + 3y - 5 = 0$ ✓

v) at R: $x = 0$ $\quad$ at Q: $y = 0$

$\therefore 3y = 5$ $\quad$ $\therefore x = 5$

R is $(0, \frac{5}{3})$ ✓ $\quad$ Q is $(5, 0)$ ✓

vi) Area $= \frac{1}{2} \times OQ \times OR$

$= \frac{1}{2} \times 5 \times \frac{5}{3}$ $\quad \therefore$ Area $= \frac{25}{6}$ units^2 ✓

Q6. a) i) $y = (x^2+2)^{-1}$ $\quad \therefore u = x^2 + 2 \quad y = u^{-1}$

$\frac{du}{dx} = 2x \quad \frac{dy}{du} = -u^{-2}$ ✓

$\therefore \frac{dy}{dx} = \frac{du}{dx} \times \frac{dy}{du}$

$= 2x \times \frac{-1}{u^2}$

$= \frac{-2x}{u^2}$

$\frac{dy}{dx} = \frac{-2x}{(x^2+2)^2}$ ✓

ii) $y = x^4(x+1)^3$ $\quad \therefore u = x^4 \quad v = (x+1)^3$ ✓

$u' = 4x^3 \quad v' = 3(x+1)^2$

$\therefore y' = u'v + v'u$

$= 4x^3(x+1)^3 + 3(x+1)^2x^4$ ✓

$= x^3(x+1)^2[4(x+1) + 3x]$

$= x^3(x+1)^2[7x+4]$ ✓

b) $\log_4 5 + 2\log_4 6 - \log_4 45 = \log_4[5 \times 6^2 \div 45]$ ✓

$= \log_4 4$

$= 1$ ✓

c) i) $0.\dot{3}\dot{6} = 0.36 + 0.0036 + 0.000036 + \ldots$

$a = 0.36 \quad r = 0.01$ ✓

ii) $S_\infty = \frac{a}{1-r}$

$= \frac{0.36}{0.99}$ ✓

$= \frac{4}{11}$ ✓

d) i) $\cot\alpha = \frac{\cos\alpha}{\sin\alpha}$ ✓

ii) LHS : $\sin\alpha + \cot\alpha\cos\alpha$

$= \sin\alpha + \frac{\cos^2\alpha}{\sin\alpha}$

$= \frac{\sin^2\alpha + \cos^2\alpha}{\sin\alpha}$ ✓

$= \frac{1}{\sin\alpha}$

$=$ RHS as required

Q7. a) i) Month 1: 10000

Month 2: $10000 \times 0.8 = 8000$

Month 3: $8000 \times 0.8 = 6400$

$$\frac{T_3}{T_2} = \frac{6400}{8000} = 0.8 \quad ; \quad \frac{T_2}{T_1} = \frac{8000}{10000} = 0.8$$

$\therefore$ GP with $a = 10000$, $r = 0.8$

ii) $S_{10} = \dfrac{a(1-r^{10})}{1-r} = \dfrac{10000(1-0.8^{10})}{0.2}$

$= 44631$ copies

b) i) h_{max} is at vertex $\Leftrightarrow t = \dfrac{-b}{2a}$

$t = \dfrac{-6}{-2}$

$t = 3$ secs

ii) height at vertex: $h = 17 + 6 \times 3 - 3^2$

$h = 26$ metres

c) i) Let x be the distance to the base of the lighthouse.

$\sin 15° = \dfrac{100}{x} \quad \therefore \quad x = \dfrac{100}{\sin 15°}$

$x = 386$ metres (to nearest metre)

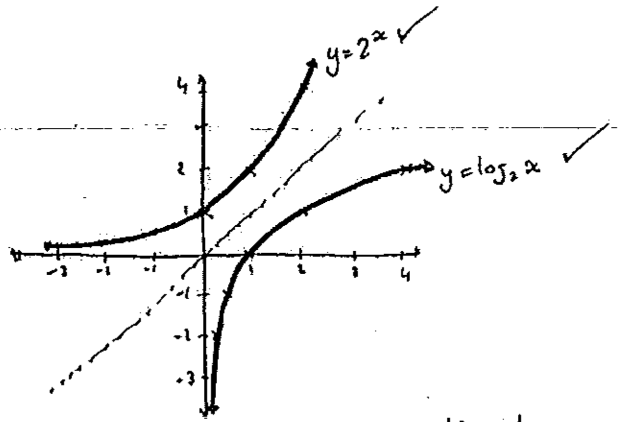
ii) Let height of lighthouse be h.

$\tan 17° = \dfrac{h}{x} \quad \therefore \quad h = 386 \times \tan 17°$

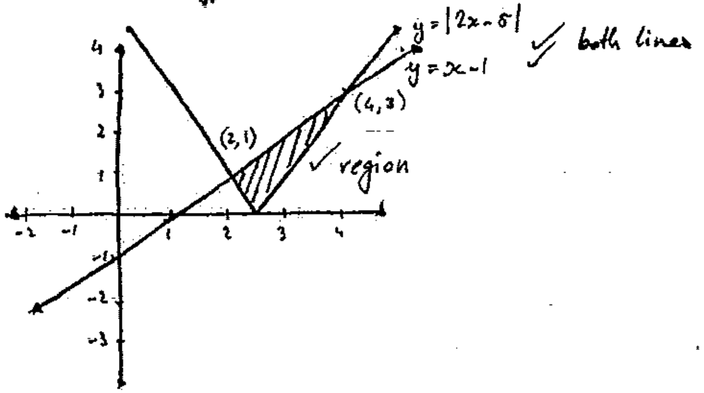
$h = 118$ metres (to nearest metre)

Q7. (cont)

d)



e)



Q8. a) $\lim_{x\to 3}\left(\frac{x^2-x-6}{x^2+2x-15}\right) = \lim_{x\to 3}\left(\frac{(x-3)(x+2)}{(x-3)(x+5)}\right)$

$= \lim_{x\to 3}\left(\frac{x+2}{x+5}\right)$

$= \frac{5}{8}$

b) line through intersection: $x-2y+3+k(2x+y-1)=0$

at $P(2,1)$: $2-2+3+k(4+1-1)=0$

$3+4k=0$

$k=-\frac{3}{4}$

$\therefore x-2y+3-\frac{3}{4}(2x+y-1)=0$

$x-2y+3-\frac{3}{2}x+\frac{3}{4}y-\frac{1}{4}=0$

$-\frac{1}{2}x-\frac{11}{4}y+\frac{15}{4}=0 \Rightarrow 2x+11y-15=0$

c) i) $\Delta = b^2-4ac$

$\Delta = (-m)^2-4\times m(m-3)$

$= m^2-4m^2+12m$

$= -3m^2+12m$

$= 3m(4-m)$

ii) two equal roots $\Leftrightarrow \Delta=0$

$\therefore 3m(4-m)=0$

$m=0$ or $m=4$

BUT $m=0$ does not fit the original equation

$\therefore m=4$

d) $2\cos(2\theta)=1$

$\cos(2\theta)=\frac{1}{2}$ $\quad u=2\theta$, $0^\circ \le u \le 720^\circ$

$\therefore \cos u = \frac{1}{2}$

$u=60^\circ$ $\therefore \theta=30^\circ$

$u=300^\circ$ $\therefore \theta=150^\circ$ correct solutions

4 values $u=420^\circ$ $\therefore \theta=210^\circ$

$u=660^\circ$ $\therefore \theta=330^\circ$

Q9. a) $u=3^x$ $\therefore 3^{2x}-12\times 3^x+27=0$

$u^2-12u+27=0$

$(u-9)(u-3)=0$

$u=9$ or $u=3$

$3^x=9$ or $3^x=3$

$\therefore x=2$ or $x=1$ both

b) $y=\sqrt{3-x}$; $u=3-x$ $\quad y=\sqrt{u}$

$\frac{du}{dx}=-1$ $\quad \frac{dy}{du}=\frac{1}{2\sqrt{u}}$

$\frac{dy}{dx}=\frac{dy}{du}\times\frac{du}{dx} \Leftrightarrow \frac{dy}{dx}=\frac{1}{2\sqrt{u}}\times -1$

$= \frac{-1}{2\sqrt{3-x}}$

at $A(-1,2)$: $\frac{dy}{dx}=\frac{-1}{2\sqrt{3+1}}$

$=-\frac{1}{4}$ (gradient of tangent)

$\therefore$ gradient of normal: $m_\perp=4$

$\therefore y-y_0=m(x-x_0)$

$y-2=4(x+1)$

$y=4x+6$

P.T.O.

Q9. (cont.)

c) i) $a = 1$, $r = \cos^2\theta$

$S_\infty = \frac{a}{1-r}$

$= \frac{1}{1-\cos^2\theta}$

$S_\infty = \frac{1}{\sin^2\theta}$ ✓

ii) $S_\infty = 4 \Leftrightarrow \frac{1}{\sin^2\theta} = 4$

$\therefore \sin^2\theta = \frac{1}{4}$

$\therefore \sin\theta = \frac{1}{2}$ or $\sin\theta = -\frac{1}{2}$ ✓

$\theta = 30°$ or $\theta = 210°$

or $\theta = 150°$ or $\theta = 330°$ ✓ 4 values

d) i) cosine rule: $k^2 = x^2 + 3^2 - 2 \times 3 \times x \times \cos 60°$

$k^2 = x^2 + 9 - 6x \times \frac{1}{2}$ ✓

$k^2 = x^2 - 3x + 9$ as required

ii) k is minimised when k^2 is minimised

k^2 is minimised at vertex of $x^2 - 6x + 9$

vertex $= \frac{-b}{2a}$ $\therefore$ $x = \frac{3}{2}$ ✓

$\therefore$ when $x = \frac{3}{2}$, $k^2 = \left(\frac{3}{2}\right)^2 - 3 \times \frac{3}{2} + 9$

$k^2 = \frac{27}{4}$ (minimum value)

$\therefore$ minimum $k = \sqrt{\frac{27}{4}}$

$= \frac{3\sqrt{3}}{2}$ as required ✓

iii) when $k = \frac{3\sqrt{3}}{2}$, $x^2 + k^2 = 3^2$

$\therefore$ ΔABC is right-angled at B

$\therefore \sin\angle ACB = \frac{x}{3}$

$= \frac{\frac{3}{2}}{3}$

$\sin\angle ACB = \frac{1}{2}$

$\therefore \angle ACB = 30°$ ✓