Sydney Grammar School

2012 Half-Yearly Examination

FORM V
MATHEMATICS EXTENSION 1

Wednesday 29th August 2012

General Instructions
- Writing time — 3 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 117 Marks
- All questions may be attempted.

Section I – 13 Marks
- Questions 1–13 are of equal value.
- Record your solutions to the multiple choice on the sheet provided.

Section II – 104 Marks
- Questions 14–21 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection
- Write your name, class and master on each booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place your multiple choice answer sheet inside the answer booklet for Question Fourteen.
- Write your name and master on this question paper and submit it with your answers.

5A: DS	5B: TCW	5C: REP
5D: DNW	5E: LYL	5F: MLS
5G: SO	5H: BR	5I: SJE

Checklist
- SGS booklets — 8 per boy
- Multiple choice answer sheet
- Candidature — 1000 boys

Examiner
DNW

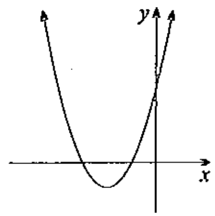
SECTION I - Multiple Choice

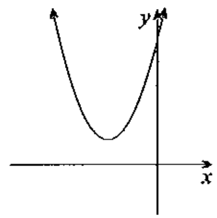
Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

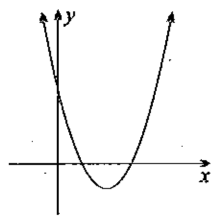
QUESTION ONE
The factors of $3x^2 - 10x - 8$ are:

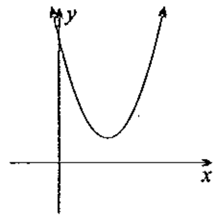
(A) $(3x+4)(x-2)$ (B) $(3x-4)(x+2)$
(C) $(3x-2)(x+4)$ (D) $(3x+2)(x-4)$

QUESTION TWO
Which of the following graphs best represents $y = (x+2)^2 - 1$?

(A)


(B)


(C)


(D)


QUESTION THREE
The derivative of $3x^4 + x^5$ is:

(A) $\frac{3}{5}x^5 + \frac{1}{6}x^6$ (B) $3x^3 + x^4$ (C) $12x^3 + 5x^4$ (D) $12x^3 + x^5$

QUESTION FOUR
The derivative of e^{3x} is:

(A) $\frac{1}{3}e^{3x}$ (B) e^{3x} (C) $3xe^{3x-1}$ (D) $3e^{3x}$

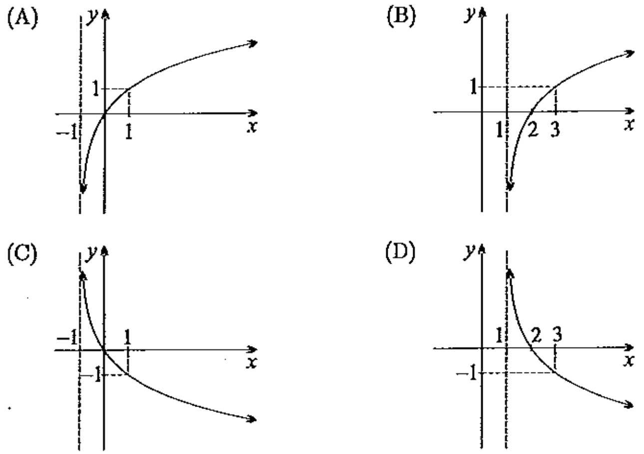
QUESTION FIVE
When $3\log p + \log(2q)$ is simplified, the result is:

(A) $\log(6pq)$ (B) $\log(2p^3q)$ (C) $\log(3pq^2)$ (D) $\log(p^3q^2)$

Exam continues next page ...

QUESTION SIX

Which of the following is the graph of $y = -\log_2(x+1)$?



QUESTION SEVEN

Here is a table of values for $y = 2^{-x^2}$.

x	0	1	2
y	1	$\frac{1}{2}$	$\frac{1}{16}$

Applying Simpson's rule to these values, an estimate of $\int_0^2 2^{-x^2}\,dx$ is:

(A) $\frac{49}{96}$ (B) $\frac{49}{48}$ (C) $\frac{33}{32}$ (D) $\frac{49}{24}$

QUESTION EIGHT

The indefinite integral $\int (2x+1)^3\,dx$ is equal to:

(A) $(2x+1)^4 + C$

(B) $\frac{(2x+1)^4}{2} + C$

(C) $\frac{(2x+1)^4}{4} + C$

(D) $\frac{(2x+1)^4}{8} + C$

QUESTION NINE

The quadratic $Q(x) = ax^2 + bx + c$ is negative definite. Which of the following is true?

(A) $a > 0$ and $\Delta < 0$

(B) $a < 0$ and $\Delta < 0$

(C) $a > 0$ and $\Delta > 0$

(D) $a < 0$ and $\Delta > 0$

Exam continues overleaf ...

QUESTION TEN

The derivative of $\sqrt{3x^2-1}$ is:

(A) $\frac{x}{\sqrt{3x^2-1}}$ (B) $\frac{2x}{\sqrt{3x^2-1}}$ (C) $\frac{3x}{\sqrt{3x^2-1}}$ (D) $\frac{6x}{\sqrt{3x^2-1}}$

QUESTION ELEVEN

It is known that $f''(x) = (x-1)^2(x+1)$.

How many inflexion points does the graph of $y = f(x)$ have?

(A) 0 (B) 1 (C) 2 (D) 3

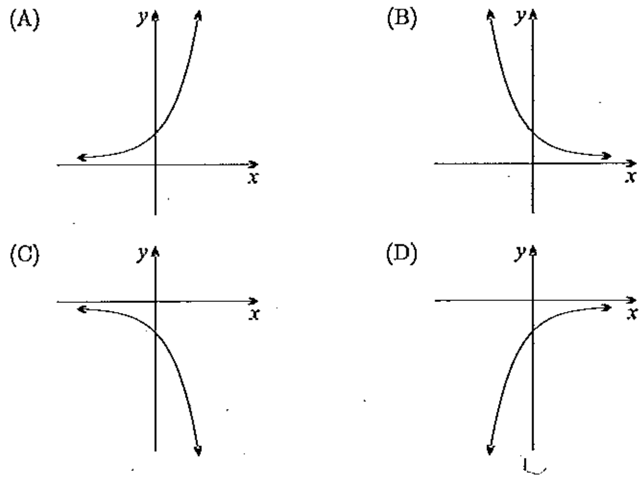
QUESTION TWELVE

What is the value of $\int_1^2 e^{x-1}\,dx$?

(A) $\frac{1}{2}e^2 - e$ (B) $e - 1$ (C) e (D) 1

QUESTION THIRTEEN

Suppose that $f'(x) > 0$ and $f''(x) < 0$ for all real values of x. Which of the following graphs best represents $y = f(x)$?



End of Section I

Exam continues next page ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION FOURTEEN (13 marks) Use a separate writing booklet. **Marks**

(a) Simplify $|-3|-|7|$. [1]

(b) Determine the exact value of $\cos 150°$. [1]

(c) Evaluate $\log_2 8$. [1]

(d) Solve $3-2x \geq 7$. [2]

(e) Find a and b if $(5-\sqrt{2})^2 = a+b\sqrt{2}$. [2]

(f) Write down the primitive of x^2+3. [2]

(g) Express $\dfrac{1}{3-\sqrt{2}}+\dfrac{1}{3+\sqrt{2}}$ in simplest form. [2]

(h) Determine the coördinates of the mid-point of AB, where $A=(3,-4)$ and $B=(7,2)$. [2]

QUESTION FIFTEEN (13 marks) Use a separate writing booklet. **Marks**

(a) Differentiate:

(i) $(3x^2+4)^5$ [2]

(ii) $x\log x$ [2]

(iii) $\dfrac{x}{3x+1}$ [2]

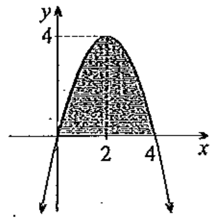
(b) Evaluate:

(i) $\displaystyle\int_1^2 \frac{1}{x^2}\,dx$ [2]

(ii) $\displaystyle\int_0^1 e^{2x+1}\,dx$ [2]

Exam continues overleaf ...

(c) [3]



The graph above shows the shaded region bounded by the x-axis and the parabola $y=4x-x^2$. Find the area of this region.

QUESTION SIXTEEN (13 marks) Use a separate writing booklet. **Marks**

(a) Determine the gradient of the tangent to $y=2x^2-x^3$ at the point where $x=2$. [2]

(b) Find the coordinates of the vertex of the parabola with equation $y=(x-4)(x+1)$. [2]

(c) Let $x=\log_a 5$ and $y=\log_a 3$. Write $\log_a 45$ in terms of x and y. [2]

(d) Consider the integral $I=\displaystyle\int_1^2 \ln x\,dx$.

(i) Find the approximate value of I using the trapezoidal rule with three function values. Give your answer correct to 2 decimal places. [3]

(ii) Give a reason why the answer to part (i) is less than the exact value of I. [1]

(e) Show that $\displaystyle\int_0^4 \frac{x}{x^2+9}\,dx = \log\tfrac{5}{3}$. [3]

QUESTION SEVENTEEN (13 marks) Use a separate writing booklet. **Marks**

(a) Solve $3\tan^2\theta - 5\sec\theta + 1 = 0$ for $0° \leq \theta \leq 360°$. Approximate your answers to the nearest degree where necessary. [4]

(b) The region between $y=\dfrac{1}{\sqrt{x}}$ and the x-axis, for $1 \leq x \leq e^2$, is rotated about the x-axis to generate a solid of revolution. Find the exact volume of this solid. [3]

(c) (i) Differentiate $y=xe^x$. [1]

(ii) Hence evaluate $\displaystyle\int_{-1}^1 xe^x\,dx$. [3]

(d) Find $\displaystyle\int (3x+1)e^{3x^2+2x+1}\,dx$. [2]

Exam continues next page ...

QUESTION EIGHTEEN (13 marks) Use a separate writing booklet. **Marks**

(a) Consider the function $y = x - \log(x+1)$, where $x > -1$.
You may assume that there is a vertical asymptote at $x = -1$ with $y \to \infty$.

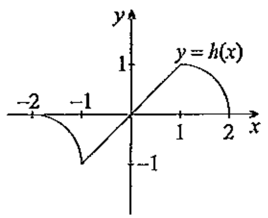
(i) Find and classify any stationary points. [2]

(ii) Explain why the curve never changes concavity. [1]

(iii) Sketch a graph of $y = x - \log(x+1)$. [1]

(iv) Hence solve $\log(1+x) \geq x$. [1]

(b) [3]



The graph of $y = h(x)$, shown above for $-2 \leq x \leq 2$, consists of a straight line and two quadrants. Use geometrical formulae to evaluate $\int_{-2}^{2} h(x)\,dx$.

(c) (i) Find the values of a, b and c if [3]

$$x^2 + 1 \equiv a(x-1)^2 + b(x-1) + c$$

for all values of x.

(ii) Hence determine $\int \frac{x^2+1}{(x-1)^2}\,dx$. [2]

QUESTION NINETEEN (13 marks) Use a separate writing booklet. **Marks**

(a) (i) What is the equation of a line through $(4, -4)$ with gradient m? [1]

(ii) Suppose that the line in part (i) is tangent to $y = \frac{2}{x}$. [3]
Use the discriminant to find the possible values of m.

(b) Solve the following by first reducing it to a quadratic equation. [4]

$$3\left(x + \frac{1}{x}\right)^2 - 16\left(x + \frac{1}{x}\right) + 20 = 0$$

(c) By using the substitution $u = x^2 + 1$, or otherwise, determine $I = \int \frac{2x}{\sqrt{x^2+1}}\,dx$. [2]

(d) In a certain geometric sequence, the sum of the first two terms is 8 and the sum of the first three terms is 26. Find the possible values of the common ratio. [3]

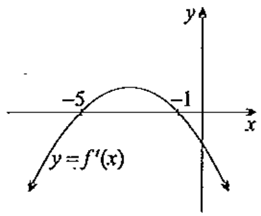
Exam continues overleaf ...

QUESTION TWENTY (13 marks) Use a separate writing booklet. **Marks**

(a) (i) Find the equation of the tangent to $y = \log x$ at the point $(a, \log a)$. [2]

(ii) This tangent passes through the origin. Find the value of a. [1]

(b) [2]



The graph above shows the gradient function of the curve $y = f(x)$.

What is the value of x for which the graph of $y = f(x)$ has a maximum turning point. Justify your answer.

(c) Factorise $p^3 + q^3$. [1]

(d) The quadratic equation $2x^2 - 3x - 4 = 0$ has roots p and q.

(i) Without solving the equation, determine:

(α) $p + q$ [1]

(β) pq [1]

(γ) $p^3 + q^3$ [1]

(ii) Hence or otherwise find a quadratic equation with integer coefficients which has roots p^3 and q^3. [1]

(e) The function $f(t)$ is even and hence [3]

$$\int_{-x}^{x} f(t)\,dt = 2\int_{0}^{x} f(t)\,dt.$$

Let $F(x) = \int_{0}^{x} f(t)\,dt$.

By considering $F(x) - F(-x)$, and using the properties of definite integrals, show that $F(x)$ is odd.

Exam continues next page ...

QUESTION TWENTY ONE (13 marks) Use a separate writing booklet. **Marks**

(a) The function $f(x)$ is defined as follows:

$$f(x) = \begin{cases} e^{2x} & \text{for } x < 0 \\ ax^2 + bx + c & \text{for } x \geq 0 \end{cases}$$

It is known that $f(x)$ is continuous and differentiable for all real values of x.
It is also known that $f(1) = 0$.

(i) Find $f'(x)$. [1]

(ii) Show that $a = -3$, $b = 2$ and $c = 1$. [3]

(iii) Sketch a graph of $y = f(x)$. [1]

(iv) Hence determine the global maximum of $f(x)$. [1]

(b) (i) Write a^x as a power of e. [1]

(ii) Hence show that $\dfrac{d}{dx}\left(a^x\right) = a^x \log a$. [1]

(c) Let $g(x) = a^x - x^a$ where $a > e$ and is constant, and $x \geq 0$.
You may assume that $g(x)$ is continuous for all $x \geq 0$.
Note that $g(a) = 0$.

(i) Evaluate $g(0)$. [1]

(ii) Show that $g'(a) > 0$. [1]

(iii) Explain why $y = g(x)$ has at least two x-intercepts. [1]

(iv) In this part assume that $y = g(x)$ has exactly two x-intercepts. One is at $x = a$. Let the other be at $x = b$. By considering the sign of $g'(b)$, show that [2]

$$b < \frac{a}{\log a}.$$

End of Section II

END OF EXAMINATION

The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\ x > 0$

Multiple Choice (with explanations of errors)

Q 1 (D) (A) $3x^2 - 2x - 8$, (B) $3x^2 + 2x - 8$, (C) $3x^2 + 10x - 8$

Q 2 (A) (B) $y = (x+2)^2 + 1$, (C) $y = (x-2)^2 - 1$, (D) $y = (x-2)^2 + 1$

Q 3 (C) (A) primitive, (B) failure to multiply by index, (D) only first term

Q 4 (D) (A) primitive, (B) f' missing from $f'e^f$, (C) confused with $\frac{d}{dx}\left(x^3\right)$

Q 5 (B) (A) $3\log p \neq \log(3p)$, (C) $3\log p \neq \log(3p)$ and $\log(2q) \neq \log(q^2)$,
(D) $\log(2q) \neq \log(q^2)$

Q 6 (C) (A) $y = \log_2(x+1)$, (B) $y = \log_2(x-1)$, (D) $y = -\log_2(x-1)$

Q 7 (B) (A) $\frac{1}{6}h(f_0 + 4f_1 + f_2)$, (C) $\frac{1}{2}h(f_0 + 2f_1 + f_2)$ (Trapezoidal rule)
(D) $\frac{2}{3}h(f_0 + 4f_1 + f_2)$

Q 8 (D) (A) $\int 8(2x+1)^3\,dx$ (B) $\int 4(2x+1)^3\,dx$ (C) $\int 2(2x+1)^3\,dx$

Q 9 (B) (A) positive definite (C) indefinite (D) indefinite

Q 10 (C) (A) $\frac{d}{dx}\left(\frac{1}{3}\sqrt{3x^2-1}\right)$ (B) $\frac{d}{dx}\left(\frac{2}{3}\sqrt{3x^2-1}\right)$ (D) $\frac{d}{dx}\left(2\sqrt{3x^2-1}\right)$

Q 11 (B) (A) f'' changes sign at $x = -1$ (C) f'' does not change sign at $x = 1$
(D) f'' only has two zeros

Q 12 (B) (A) $\int e^{x-1}\,dx \neq \frac{e^x}{x}$ (C) $e^0 \neq 0$ (D) $\int e^{x-1}\,dx \neq (x-1)e^{x-2}$

Q 13 (D) (A) $f'' > 0$ (B) $f'' > 0,\ f' < 0$ (C) $f' < 0$

QUESTION FOURTEEN (13 marks)

(a) $|-3| - |7| = -4$ ✓

(b) $\cos 150° = -\frac{\sqrt{3}}{2}$ ✓

(c) $\log_2 8 = 3$ ✓

(d)
$$3 - 2x \geq 7$$
$$-2x \geq 4 \quad ✓$$
$$x \leq -2 \quad ✓$$

(e)
$$(5 - \sqrt{2})^2 = 25 - 10\sqrt{2} + 2 \quad ✓$$
$$= 27 - 10\sqrt{2} \quad ✓$$
so $a = 27$ and $b = -10$.

(f) $\int x^2 + 3\,dx = \frac{1}{3}x^3 + 3x + C$ ✓✓

[2nd mark for constant.]

(g)
$$\frac{1}{3-\sqrt{2}} - \frac{1}{3+\sqrt{2}} = \frac{3+\sqrt{2}-(3-\sqrt{2})}{3^2-2} \quad ✓$$
$$= \frac{2\sqrt{2}}{7} \quad ✓$$

(h)
$$\text{mid-point} = \left(\frac{3+7}{2}, \frac{-4+2}{2}\right) \quad ✓$$
$$= (5, -1) \quad ✓$$

Total for Question 14: 13 Marks

QUESTION FIFTEEN (13 marks)

(a) (i)
$$\frac{d}{dx}(3x^2+4)^5 = 5 \times (3x^2+4)^4 \times 3x \times 2 \quad \text{(chain rule)} \quad ✓$$
$$= 30x(3x^2+4)^4 \quad ✓$$

(ii)
$$\frac{d}{dx}(x\log x) = 1 \times \log x + x \times \frac{1}{x} \quad \text{(product rule)} \quad ✓$$
$$= \log x + 1 \quad ✓$$

(iii)
$$\frac{d}{dx}\left(\frac{x}{3x+1}\right) = \frac{(3x+1) \times 1 - x \times 3}{3x+1)^2} \quad \text{(quotient rule)} \quad ✓$$
$$= \frac{1}{(3x+1)^2} \quad ✓$$

(b) (i) $\displaystyle\int_1^2 \frac{1}{x^2}\,dx = \left[-\frac{1}{x}\right]_1^2$ ✓

$= -\frac{1}{2} + 1$

$= \frac{1}{2}$ ✓

(ii) $\displaystyle\int_0^1 e^{2x+1}\,dx = \left[\tfrac{1}{2}e^{2x+1}\right]_0^e$ ✓

$= \frac{1}{2}e^3 - \frac{1}{2}e$ ✓

$= \frac{1}{2}e(e^2 - 1)$.

(c) Area $= \displaystyle\int_0^4 4x - x^2\,dx$ ✓

$= \left[2x^2 - \frac{1}{3}x^3\right]_0^4$ ✓

$= \frac{32}{3}$. ✓

Total for Question 15: 13 Marks

QUESTION SIXTEEN (13 marks)

(a) $y = 2x^2 - x^3$

so $\dfrac{dy}{dx} = 4x - 3x^2$ ✓

at $x = 2$: $\dfrac{dy}{dx} = 4 \times 2 - 3 \times 2^2$

$= -4$. ✓

(b) x-intercepts $= -1, 4$ ✓

so vertex is at $x = \frac{-1+4}{2} = \frac{3}{2}$

where $y = -6\frac{1}{4}$ ✓

[or any other valid method.]

(c) $\log_a 45 = \log_a 3^2 + \log_a 5$ ✓

$= 2\log_a 3 + \log_a 5$

$= 2y + x$ ✓

(d) (i) Let $y = \ln x$ then:

x	1	$\frac{3}{2}$	2
y	0	$\ln\frac{3}{2}$	$\ln 2$

✓

so $I \doteqdot \dfrac{(\frac{1}{2})}{2}\left(0 + 2 \times \ln\frac{3}{2} + \ln 2\right)$ ✓

$\doteqdot 0{\cdot}38$ (to two decimal places) ✓

(ii) $y' = \dfrac{1}{x}$ so $y'' = -\dfrac{1}{x^2}$. Thus $y'' < 0$ for all x in the domain and the curve is concave down. ✓

(e) $\displaystyle\int_0^4 \frac{x}{x^2+9}\,dx = \frac{1}{2}\left[\log(x^2+9)\right]_0^4$ ✓

$= \frac{1}{2}(\log 25 - \log 9)$ ✓

$= \log 5 - \log 3$ ✓

$= \log\frac{5}{3}$.

Total for Question 16: 13 Marks

QUESTION SEVENTEEN (13 marks)

(a) $3\tan^2\theta - 5\sec\theta + 1 = 0$

so $3(\sec^2\theta - 1) - 5\sec\theta + 1 = 0$ ✓

$3\sec^2\theta - 5\sec\theta - 2 = 0$

$(3\sec\theta + 1)(\sec\theta - 2) = 0$ ✓

thus $\sec\theta = -\frac{1}{3}$ or 2

$\sec\theta = -\frac{1}{3}$ has no real solutions. ✓

$\sec\theta = 2$ has solutions $\theta = 60^\circ$ or 300°. ✓

(b) Volume $= \pi\displaystyle\int_1^{e^2} y^2\,dx$

$= \pi\displaystyle\int_1^{e^2} \frac{1}{x}\,dx$ ✓

$= \pi\Big[\log x\Big]_1^{e^2}$ ✓

$= \pi(\log e^2 - \log 1)$

$= 2\pi$. ✓

(c) (i) $y = xe^x$

$\dfrac{dy}{dx} = 1 \times e^x + x \times e^x$ (product rule) ✓

$= e^x + xe^x$

(ii) Rearrange part (i) to get

$$xe^x = \frac{dy}{dx} - e^x$$

so $\displaystyle\int_{-1}^{1} xe^x\,dx = \int_{-1}^{1} \frac{dy}{dx} - e^x\,dx$ ✓

$= \Big[y - e^x\Big]_{-1}^{1}$ ✓

$= (e - e) - (-e^{-1} - e^{-1})$

$= 2e^{-1}$ ✓

(d) $\displaystyle\int (3x+1)e^{3x^2+2x+1}\,dx = \tfrac{1}{2}.\int (6x+2)e^{3x^2+2x+1}\,dx$ ✓

$= \tfrac{1}{2}e^{3x^2+2x+1} + C$ ✓

[Do not penalise lack of a constant.]

Total for Question 17: 13 Marks

QUESTION EIGHTEEN (13 marks)

(a) (i) $y = x - \log(x+1)$

so $y' = 1 - \dfrac{1}{x+1}$

$= \dfrac{x}{x+1}$

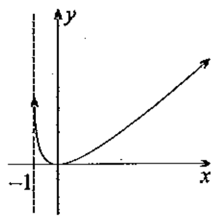
thus there is a stationary point at $(0,0)$. ✓

$y'' = \dfrac{1}{(x+1)^2}$

so at $x = 0$, $y'' = 1$ and it is a minimum stationary point. ✓

(ii) $y'' > 0$ for all x in the domain,
thus y'' never changes sign,
hence the concavity never changes. ✓

(iii)



✓

(iv) If $\log(x+1) \geq x$

then $0 \geq x - \log(x+1)$

which, from the graph, is only true when $x = 0$. ✓

(b) In this case, areas below the x-axis are negative,

area quadrant $= \frac{\pi}{4}$

area triangle $= \frac{1}{2}$, ✓

hence $\displaystyle\int_2^{2} h(x)\,dx = -\Big((1 - \tfrac{\pi}{4}) + \tfrac{1}{2}\Big) + (\tfrac{1}{2} + \tfrac{\pi}{4})$ ✓

$= \frac{\pi}{2} - 1$ ✓

(c) (i) Since $x^2 + 1 \equiv a(x-1)^2 + b(x-1) + c$

at $x = 1$: $2 = c$ ✓

at $x = 2$: $5 = a + b + c$

at $x = 0$: $1 = a - b + c$ ✓

solving simultaneously,

$a = 1$

and $b = 2$ ✓

(ii) From part (i):

$\displaystyle\int \frac{x^2+1}{(x-1)^2}\,dx = \int \frac{(x-1)^2 + 2(x-1) + 2}{(x-1)^2}\,dx$

$= \displaystyle\int 1 + \frac{2}{x-1} + \frac{2}{(x-1)^2}\,dx$ ✓

$= x + 2\log(x-1) - \dfrac{2}{x-1} + C$ ✓

[Do not penalise lack of a constant.]

Total for Question 18: 13 Marks

QUESTION NINETEEN (13 marks)

(a) (i) $y = mx - 4(m+1)$ ✓

[or equivalent.]

(ii) Substitute $y = \dfrac{2}{x}$ into part (i) to get:

$\dfrac{2}{x}mx - 4(m+1)$

or $mx^2 - 4(m+1)x - 2 = 0$ ✓

$\Delta = 0$ since the two are tangent, so:

$16(m+1)^2 + 8m = 0$ ✓

or $2m^2 + 5m + 2 = 0$

so $(2m+1)(m+2) = 0$

thus $m = -\frac{1}{2}$ or -2. ✓

(b) $$3\left(x+\frac{1}{x}\right)^2-16\left(x+\frac{1}{x}\right)+20=0.$$

Put $\lambda=(x+\frac{1}{x})$ to get:

$$3\lambda^2-16\lambda+20=0$$

or $(3\lambda-10)(\lambda-2)=0$ (or equivalent.) ✓

thus $\lambda=2$ or $\frac{10}{3}$. ✓

When $\lambda=2$ $$x+\frac{1}{x}=2$$
$$x^2-2x+1=0$$
so $x=1$. ✓

When $\lambda=\frac{10}{3}$ $$x+\frac{1}{x}=\frac{10}{3}$$
$$3x^2-10x+3=0$$
$$(3x-1)(x-3)=0$$
so $x=3$ or $\frac{1}{3}$. ✓

(c) $$I=\int\frac{2x}{\sqrt{x^2+1}}\,dx$$
$$=2\int\frac{x}{\sqrt{x^2+1}}\,dx \quad ✓$$
$$=2\sqrt{x^2+1}+C \quad ✓$$

(d) $$a(r^2+r+1)=26$$
$$a(r+1)=8 \quad ✓$$
Dividing the first by the second:
$$\frac{r^2+r+1}{r+1}=\frac{13}{4} \quad ✓$$
so $4r^2+4r+4=13r+13$

or $4r^2-9r-9=0$

thus $(4r+3)(r-3)=0$

hence $r=-\frac{3}{4}$ or 3 ✓

Total for Question 19: 13 Marks

QUESTION TWENTY (13 marks)

(a) (i) $$y=\log x$$
so $y'=\frac{1}{x}$

and $y'(a)=\frac{1}{a}$ ✓

Thus the tangent has equation:
$$y=\tfrac{1}{a}x+\log a-1 \quad \text{(or equivalent)} \quad ✓$$

(ii) At the origin:
$$0=\log a-1$$
so $\log a=1$

or $a=e$. ✓

(b) There is a stationary point at $x=-1$ where $f'(x)=0$. ✓
Since the sign of f' changes from positive to negative, ✓
it is a maximum stationary point (local maximum).

(c) $$p^3+q^3=(p+q)(p^2-pq+q^2) \quad ✓$$

(d) (i) (α) $$p+q=\tfrac{-b}{a}=\tfrac{3}{2} \quad ✓$$

(β) $$pq=\tfrac{c}{a}=-2 \quad ✓$$

(γ) $$p^3+q^3=(p+q)\left((p+q)^2-3pq\right)=\tfrac{99}{8} \quad ✓$$

(ii) $(pq)^3=-8$, so an equation with those roots is
$$x^2-\tfrac{99}{8}x-8=0$$
hence $8x^2-99x-64=0$. ✓

(e) $$F(x)-F(-x)=\int_0^x f(t)\,dt-\int_0^{-x}f(t)\,dt$$
$$=\int_0^x f(t)\,dt+\int_{-x}^0 f(t)\,dt \quad \text{(reversing the direction)} \quad ✓$$
$$=\int_{-x}^x f(t)\,dt \quad \text{(combining regions)} \quad ✓$$
$$=2\int_0^x f(t)\,dt$$
$$=2F(x).$$
Hence $-F(-x)=F(x)$, ✓

that is, $F(x)$ is odd.

Total for Question 20: 13 Marks

QUESTION TWENTY ONE (13 marks)

(a) (i) $f'(x) = \begin{cases} 2e^{2x} & \text{for } x < 0 \\ 2ax + b & \text{for } x > 0 \end{cases}$ ☑

(ii) $f(x)$ is continuous so $f(0) = \lim_{x \to 0^-} f(x)$

i.e. $c = 1$ ☑

$f(x)$ is differentiable so $\lim_{x \to 0^-} f'(x) = \lim_{x \to 0^+} f'(x)$

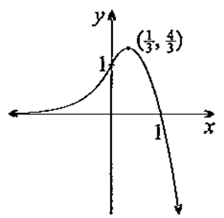
i.e. $2 = b$ ☑

$f(1) = 0$ so

$a + b + c = 0$

thus $a = -3$ ☑

(iii)



[Vertex coordinates may be omitted.] ☑

(iv) The global maximum is at the vertex of the parabola, where $x = \frac{1}{3}$

thus $f_{\max} = \frac{4}{3}$ ☑

(b) (i) $a^x = e^{x \log a}$ ☑

(ii) $y = a^x$

$= e^{x \log a}$

so $y' = e^{x \log a} \times \log a$

$= a^x \log a$ ☑

(c) For $g(x) = a^x - x^a$

(i) $g(0) = a^0 - 0^a \quad (a > 0)$

$= 1$ ☑

(ii) $g'(x) = a^x \log a - ax^{a-1}$ (by part (i))

so $g'(a) = a^a \log a - aa^{a-1}$

$= a^a(\log a - 1)$.

Now $a > e$

so $\log a > 1$ ☑

hence $g'(a) > 0$.

(iii) $g(a) = 0$ so $x = a$ is an x-intercept

$g'(a) > 0$ so there is at least one value $x = c$,

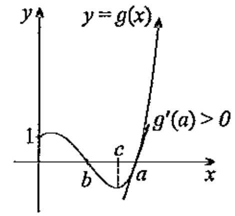
$c < a$, for which $g(c) < 0$.

But $g(0) = 1$ and $g(x)$ is continuous.

Thus $g(x)$ changes sign between c and 0.

Hence $g(x)$ must have another x-intercept.

[Or any other valid argument.] ☑



(iv) It follows that $g'(b) < 0$.

now $g'(b) = a^b \log a - ab^{a-1}$

$= a^b \log a - \frac{a}{b} \times b^a$

$= a^b \log a - \frac{a}{b} \times a^b$ (since $g(b) = 0$) ☑

$= a^b \left(\log a - \frac{a}{b} \right)$

thus $\log a - \frac{a}{b} < 0$ ☑

hence $b < \frac{a}{\log a}$

Total for Question 21: 13 Marks

DNW