Sydney Grammar School

2011 Half-Yearly Examination

FORM VI
MATHEMATICS 2 UNIT

Monday 28th February 2011

General Instructions
- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Structure of the paper
- Total marks — 96
- All six questions may be attempted.
- All six questions are of equal value.

Collection
- Write your candidate number clearly on each booklet.
- Hand in the six questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place the question paper inside your answer booklet for Question One.

Checklist
- SGS booklets — 6 per boy
- Candidature — 87 boys

Examiner
FMW

QUESTION ONE (16 marks) Use a separate writing booklet. **Marks**

(a) Differentiate:

(i) $3x^4$ [1]

(ii) $(x-6)^5$ [1]

(iii) e^{3x-1} [1]

(b) Find a primitive of:

(i) $2x^3 - 3$ [1]

(ii) $2e^{2x}$ [1]

(c) Consider the parabola with equation $x^2 = 8y$.

(i) Write down the coordinates of its focus. [1]

(ii) What is the equation of its directrix? [1]

(d) Evaluate $\displaystyle\int_1^4 x\,dx$. [2]

(e) Write $\dfrac{2}{e}$ correct to 2 decimal places. [1]

(f) Consider the curve whose gradient function is $y' = (x-2)(x-3)(x+4)$. For what values of x is the curve stationary? [1]

(g) Consider the curve whose concavity function is $y'' = 3x - 2$. For what values of x is the curve concave down? [1]

(h) Write down the centre and radius of the circle with equation $(x-2)^2 + (y+3)^2 = 9$. [2]

(i) Find the gradient of the tangent to the curve $y = 2\sqrt{x}$ at the point $(9, 6)$. [2]

Exam continues next page ...

QUESTION TWO (16 marks) Use a separate writing booklet. **Marks**

(a) Sketch the graph of $y = e^x + 1$, showing any intercepts with the x or y axes and any asymptotes. 2

(b)

x	2	3	4
$f(x)$	7	5	3

2

Use Simpson's rule with the 3 function values in the table above to approximate $\int_2^4 f(x)\,dx$.

(c) Find:

(i) $\int e^{-3x+2}\,dx$ 1

(ii) $\int x(x^2-2)\,dx$ 1

(iii) $\int \frac{x^3+2x}{x}\,dx$ 1

(d) A curve has gradient function $\frac{dy}{dx} = 4x - 2$ and passes through the point $(3, 10)$. Find the equation of the curve. 2

(e) Differentiate:

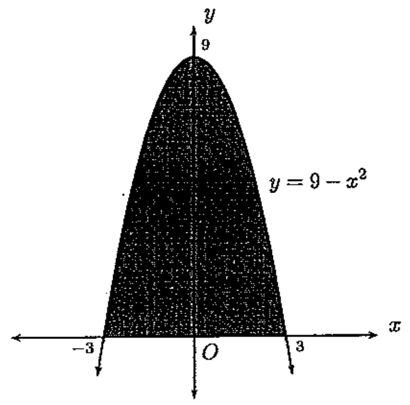
(i) $y = (5x-2)^7$ 1

(ii) $y = \frac{x}{e^x}$ 2

Question Two Continues On the Next Page

QUESTION TWO (Continued)

(f) (i) 3



Calculate the area of the shaded region in the diagram above.

(ii) Hence write down the value of $\int_0^9 \sqrt{9-y}\,dy$. 1

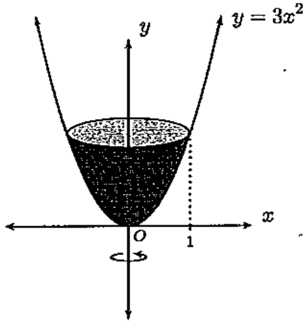
QUESTION THREE (16 marks) Use a separate writing booklet. **Marks**

(a) Find the equation of the tangent to the curve $y = \dfrac{2}{x}$ at the point where $x = 2$. $\boxed{3}$

(b) (i) Differentiate $f(x) = (4x^3 - 5)^5$. $\boxed{1}$

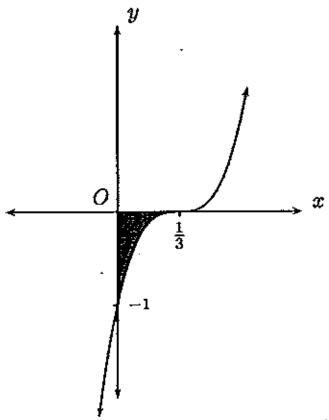
(ii) Hence find $\int x^2(4x^3-5)^4\,dx$. $\boxed{1}$

(c) $\boxed{3}$



As shown in the diagram above, the part of the curve $y = 3x^2$ between $x = 0$ and $x = 1$ is rotated about the y-axis to form a cup. Show that the volume of the cup is $\frac{3\pi}{2}$ cubic units.

(d) $\boxed{3}$



The diagram above shows the curve $y = (3x-1)^3$. Find the area of the shaded region.

Question Three Continues On the Next Page

Exam continues overleaf ...

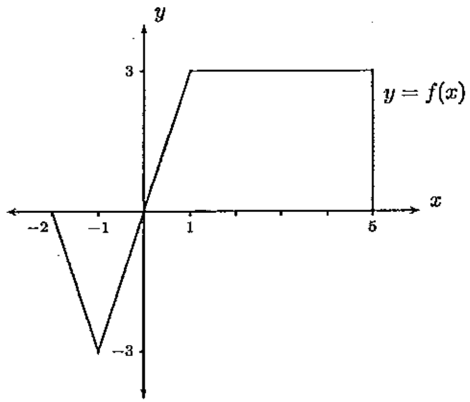
QUESTION THREE (Continued)

(e) Find the equations of the parabolas with:

(i) vertex $(0,0)$ and focus $(1,0)$, $\boxed{1}$

(ii) focus $(1,3)$ and directrix $y = -1$. $\boxed{2}$

(f) $\boxed{2}$



Given the sketch of $y = f(x)$ drawn above, use area formulae to find $\int_{-2}^{5} f(x)\,dx$.

Exam continues next page ...

QUESTION FOUR (16 marks) Use a separate writing booklet. **Marks**

(a) Consider the curve with equation $y = x^3 - 5x^2 + 7x$.

(i) Show that $y' = (3x-7)(x-1)$ and find y''. **2**

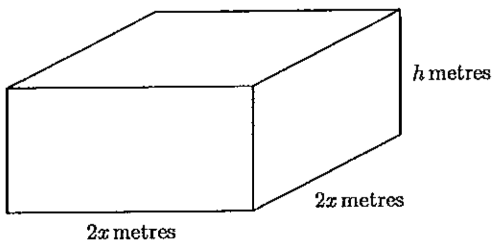
(ii) Find the two stationary points and determine their nature. **3**

(iii) Find any points of inflexion. **2**

(iv) Sketch the curve using the above information. **2**

(v) What is the maximum value of $y = x^3 - 5x^2 + 7x$ in the interval from $0 \le x \le 5$? **1**

(b)



A closed box with a square base is made from a piece of cardboard, as shown in the diagram above. The area of cardboard used is 6 square metres. Let h metres be the height of the box and let $2x$ metres be the side length of the base.

(i) Show that $h = \dfrac{3}{4x} - x$. **2**

(ii) Hence show that the volume, V cubic metres, of the box is given by the formula $V = 3x - 4x^3$. **1**

(iii) Use calculus to find the maximum volume of the box. **3**

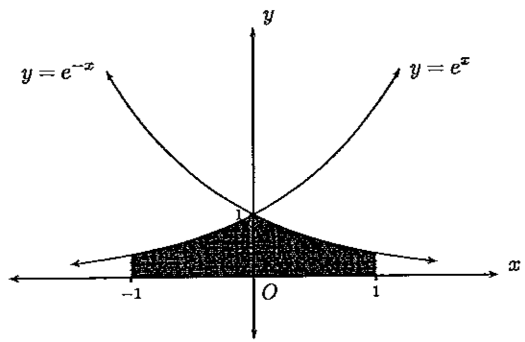
QUESTION FIVE (16 marks) Use a separate writing booklet. **Marks**

(a) (i) Express the equation $y^2 + 6y + 4x - 3 = 0$ in the form $(y-k)^2 = -4a(x-h)$. **2**

(ii) Hence find the the coordinates of the vertex and the focus of the parabola $y^2 + 6y + 4x - 3 = 0$. **2**

(b) Find $\displaystyle\int_0^3 (2 - \tfrac{1}{3}x)^{-3}\,dx$. **3**

(c) **2**



Calculate the exact area of the shaded region in the diagram above.

(d) (i) Sketch the region bounded by the curves $y = e^{2x}$, $y = e^{-x}$ and the line $x = 1$. **1**

(ii) If this region is rotated about the x-axis, find the exact volume of the solid formed. **3**

(e) Find the value of k given that $\displaystyle\int_k^0 \frac{1}{e^x}\,dx = e^2 - 1$. **3**

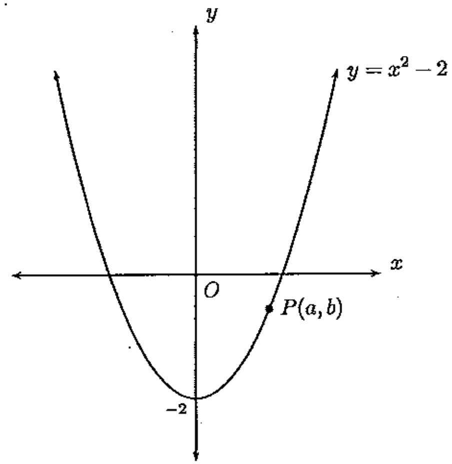
QUESTION SIX (16 marks) Use a separate writing booklet. **Marks**

(a) (i) Copy and complete the following table using exact values: [1]

x	-1	0	1
$f(x)=\frac{1}{1+e^{-x}}$			

(ii) Use the trapezoidal rule with the 3 function values from your table to find the value of $\int_{-1}^{1} \frac{1}{1+e^{-x}}\,dx$. [3]

(b)



The diagram above shows the curve $y=x^2-2$ and the point $P(a,b)$ on the curve.

(i) Find the equation of the normal at P. [2]

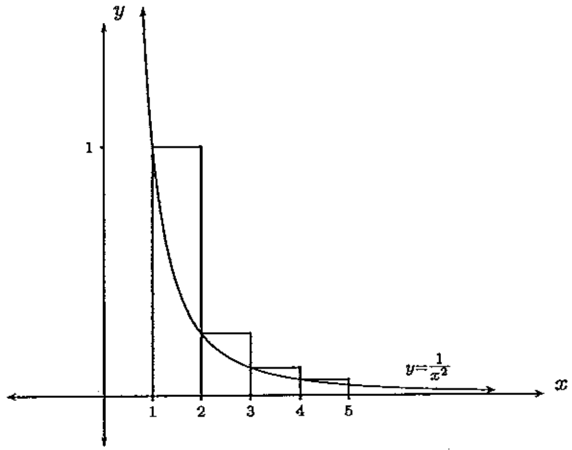
(ii) Find all possible points P on the curve such that the normal at P passes through $(0,0)$. [3]

Question Six Continues On the Next Page

QUESTION SIX (Continued)

(c) (i) Evaluate $\int_1^5 \frac{1}{x^2}\,dx$. [2]

(ii) [2]



The diagram above shows part of the curve $y=\frac{1}{x^2}$. Rectangles of width 1 unit are constructed as shown. Use the rectangles in the diagram and part (i) to explain why

$$1\times\frac{1}{1^2}+1\times\frac{1}{2^2}+1\times\frac{1}{3^2}+1\times\frac{1}{4^2}>\frac{4}{5}.$$

(iii) Show that [3]

$$1+\frac{1}{4}+\frac{1}{9}+\cdots+\frac{1}{10\,000}>\frac{100}{101}.$$

END OF EXAMINATION

The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\ x > 0$

Q.1

(a) (i) $\frac{d}{dx}(3x^4) = 12x^3$ ✓

(ii) $\frac{d}{dx}\left((x-6)^5\right) = 5(x-6)^4$ ✓

(iii) $\frac{d}{dx}\left(e^{3x-1}\right) = 3e^{3x-1}$ ✓

do not penalise omission of constants

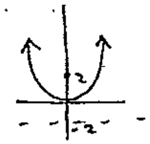
(b) (i) $\frac{2x^4}{4} - 3x = \frac{1}{2}x^4 - 3x + C$ ✓

(ii) $e^{2x} + C$ ✓

(c) $x^2 = 8y$
$= 4(2)y$

(i) focus $(0,2)$ ✓

(ii) directrix $y = -2$ ✓



(d) $\int_1^4 x\,dx = \left[\frac{x^2}{2}\right]_1^4$ ✓

$= \frac{16}{2} - \frac{1}{2}$

$= \frac{15}{2}$ ✓

(e) $\frac{2}{e} \doteqdot 0.74$ (2 d.p) ✓

(f) $y' = 0$ at $x = 2, 3, -4$ ✓

(g) $3x - 2 < 0$
$3x < 2$
$x < \frac{2}{3}$ ✓

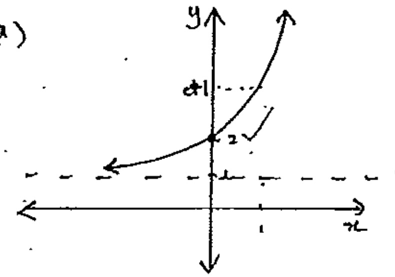
(h) centre $(2,-3)$ ✓ radius 3 ✓

/16

(i) $y = 2x^{\frac{1}{2}}$
$y' = 2 \times \frac{1}{2} x^{-\frac{1}{2}}$ ✓ at $x = 9$
$y' = 9^{-\frac{1}{2}} = \frac{1}{3}$ ✓

Q2

(a)



(b) $\int_2^4 f(x)\,dx \doteq 1 \times \frac{1}{3}\left[7 + 4(5) + 3\right]$

$= 10$

(c) (i) $\int e^{-3x+2}\,dx = -\frac{1}{3}e^{-3x+2} + C$

(ii) $\int x(x^2-2)\,dx = \int x^3 - 2x\,dx$

$= \frac{x^4}{4} - x^2 + C$

(iii) $\int \frac{x^3+2x}{x}\,dx = \int x^2 + 2\,dx$

$= \frac{x^3}{3} + 2x + C$

(do not penalise omission of constants!)

(d) $\frac{dy}{dx} = 4x - 2$

$y = 2x^2 - 2x + C$

$10 = 2(3)^2 - 2(3) + C$

$10 = 12 + C$

$C = -2$

$y = 2x^2 - 2x - 2$

(e) (i) $y = (5x-2)^7$

$y' = 7(5x-2)^6 \times 5$

$= 35(5x-2)^6$

(ii) $y = \frac{x}{e^x}$

$y' = \frac{e^x \times 1 - x \times e^x}{(e^x)^2}$

$= \frac{e^x(1-x)}{e^{2x}}$

$= \frac{1-x}{e^x}$

(f) (i) $A = 2\int_0^3 (9-x^2)\,dx$

$= 2\left[9x - \frac{x^3}{3}\right]_0^3$

$= 2\left(27 - \frac{27}{3} - 0\right)$

$= 36$ square units

(ii) $\int_0^9 \sqrt{9-y}\,dx = \frac{1}{2} \times 36$

$= 18$

(note: $x^2 = 9 - y$
$x = \sqrt{9-y}$ – RHS of curve given)

/16

Q3

(a) $y = \frac{2}{x}$

$= 2x^{-1}$

$y' = -2x^{-2}$

$= \frac{-2}{x^2}$

at $x = 2$, $y' = -\frac{2}{2^2}$
$y = 1$

$= -\frac{1}{2}$

tangent:

$y - 1 = -\frac{1}{2}(x-2)$

$2y - 2 = -x + 2$

$x + 2y - 4 = 0$

(b) (i) $f(x) = (4x^3-5)^5$

$f'(x) = 5(4x^3-5)^4 \times 12x^2$

$= 60x^2(4x^3-5)^4$

(ii) $\int x^2(4x^3-5)^4\,dx$

$= \frac{1}{60}\int 60x^2(4x^3-5)^4\,dx$

$= \frac{1}{60}(4x^3-5)^5 + C$

(c) $y = 3x^2$

$\frac{y}{3} = x^2$

$V = \pi\int_0^3 \frac{y}{3}\,dy$

$= \pi\left[\frac{y^2}{6}\right]_0^3$

$= \pi\left(\frac{9}{6} - 0\right)$

$= \frac{3\pi}{2}$ cubic units as required

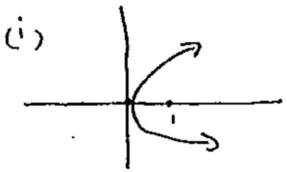
(d) $A = -\int_0^{\frac{1}{3}} (3x-1)^3\,dx$

$= -\left[\frac{(3x-1)^4}{4 \times 3}\right]_0^{\frac{1}{3}}$

$= -\left(0 - \frac{1}{12}\right)$

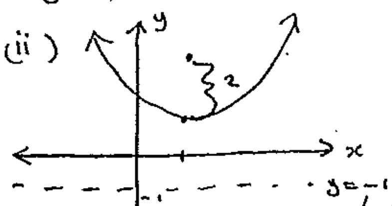
$= \frac{1}{12}$ square unit

(e) (i)



$y^2 = 4x$

(ii)



$(x-1)^2 = 8(y-1)$

(f) $\int_{-2}^{5} f(x)\,dx = -\frac{1}{2} \times 2 \times 3 + \frac{1}{2} \times 3(5+4)$

$= -3 + \frac{27}{2}$

$= 10\frac{1}{2}$

/16

Q4

(a) $y = x^3 - 5x^2 + 7x$

(i) $y' = 3x^2 - 10x + 7$
$= 3x^2 - 3x - 7x + 7$
$= 3x(x-1) - 7(x-1)$
$= (3x-7)(x-1)$ ✓

$y'' = 6x - 10$ ✓

(ii) $y' = 0$ at $x = \frac{7}{3}$ or $x = 1$
$y = \frac{49}{27}$ $\quad y = 3$
$y'' = 4 > 0$ $\quad y'' = -4 < 0$ ✓✓

$\left(\frac{7}{3}, \frac{49}{27}\right)$ is a minimum turning point
$(1, 3)$ is a maximum turning point ✓

(iii) $y'' = 0$
$6x - 10 = 0$
$6x = 10$
$x = \frac{5}{3}$ ✓
$y = \frac{65}{27}$

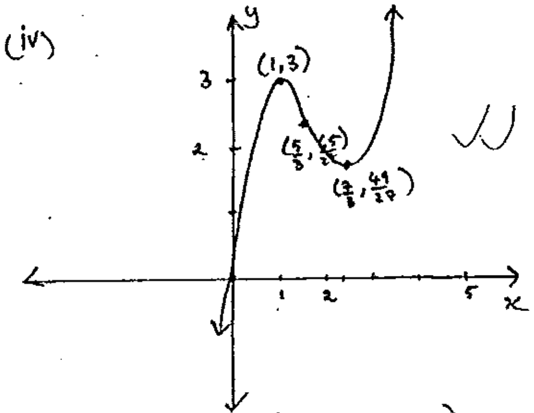
check

x	1	$\frac{5}{3}$	$\frac{7}{3}$
y''	-4	0	4
	∩	·	∪

✓

$\left(\frac{5}{3}, \frac{65}{27}\right)$ is a point of inflexion

(iv)



✓✓

(v) check $x = 5$ (end points)
$y = 5^3 - 5 \times 5^2 + 7 \times 5$
$= 35$

maximum value is 35 ✓

(b) (i) $2 \times 2x \times 2x + 4 \times 2x \times h = 6$
$8x^2 + 8xh = 6$
$8xh = 6 - 8x^2$
$h = \frac{6}{8x} - \frac{8x^2}{8x}$ ✓ either
$= \frac{3}{4x} - x$ as required ✓

(ii) $V = 2x \times 2x \times h$
$= 4x^2\left(\frac{3}{4x} - x\right)$
$= 3x - 4x^3$ ✓

(iii) $V' = 3 - 12x^2$ $\quad V'' = -24x$
$= 3(1 - 4x^2)$
$V' = 0$ at $4x^2 = 1$
$x^2 = \frac{1}{4}$
$x = \frac{1}{2}, \ x > 0$ ✓
$V = 3(\frac{1}{2}) - 4(\frac{1}{2})^3$ ✓
$= 1$
$V'' = -24 < 0$ ✓

the maximum volume is $1\,m^3$

need to show this

16

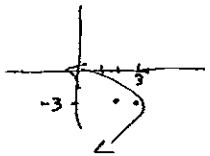
Q5

(a) (i) $y^2 + 6y + 4x - 3 = 0$

$y^2 + 6y + 9 = -4x + 3 + 9$

$(y+3)^2 = -4x + 12$

$(y+3)^2 = -4(x-3)$



(ii) vertex $(3, -3)$

focus $(2, -3)$

(b) $\int_0^3 (2 - \frac{1}{3}x)^{-3}\,dx = \left[\frac{(2-\frac{1}{3}x)^{-2}}{-2 \times -\frac{1}{3}}\right]_0^3$

$= \frac{3}{2}\left((2-1)^{-2} - 2^{-2}\right)$

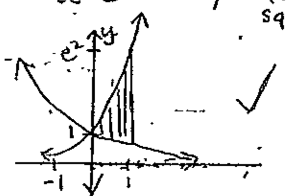
$= \frac{9}{8}$

(c) $\therefore\ A = 2\int_0^1 e^{-x}\,dx$

$= 2\left[-e^{-x}\right]_0^1$

$= 2\left(-\frac{1}{e} - (-1)\right) = 2\left(1 - \frac{1}{e}\right)$ square units

(d) (i)



$V = \pi\int_0^1 (e^{2x})^2 - (e^{-x})^2\,dx$

$= \pi\int_0^1 e^{4x} - e^{-2x}\,dx$

$= \pi\left[\frac{e^{4x}}{4} + \frac{e^{-2x}}{2}\right]_0^1$

$= \pi\left(\frac{e^4}{4} + \frac{1}{2e^2} - \left(\frac{1}{4} + \frac{1}{2}\right)\right)$

$= \frac{\pi}{4}\left(e^4 + \frac{2}{e^2} - 3\right)$ cubic units

(e) $\int_k^0 \frac{1}{e^x}\,dx = \int_k^0 e^{-x}\,dx$

$= \left[-e^{-x}\right]_k^0$

$= -1 + e^{-k}$

$-1 + e^{-k} = e^2 - 1$

$e^{-k} = e^2$

$-k = 2$

$k = -2$

16

Q6

(a)

(i)

x	-1	0	1
$f(x)$	$\frac{1}{1+e}$	$\frac{1}{1+1} = \frac{1}{2}$	$\frac{1}{1+e^{-1}}$

(ii) $I \doteq \frac{1}{2}\left(\frac{1}{1+e} + 2 \times \frac{1}{2} + \frac{1}{1+e^{-1}}\right)$

$= \frac{1}{2}\left(\frac{1}{1+e} + 1 + \frac{1}{1+\frac{1}{e}}\right)$

$= \frac{1}{2}\left(\frac{1}{1+e} + \frac{1+e}{1+e} + \frac{1}{\frac{e+1}{e}}\right)$

$= \frac{1}{2}\left(\frac{1}{1+e} + \frac{1+e}{1+e} + \frac{e}{1+e}\right)$

$= \frac{1}{2}\left(\frac{2+2e}{1+e}\right)$

$= \frac{1}{2} \times 2\left(\frac{1+e}{1+e}\right)$

$= 1$

(b) $y = x^2 - 2$

(i) $y' = 2x$

at $x = a$

$y' = 2a$

normal has equation

$y - b = -\frac{1}{2a}(x - a)$

(ii) if $(0,0)$ is on the normal

$0 - b = -\frac{1}{2a}(0 - a)$

$-b = \frac{1}{2}$

$b = -\frac{1}{2}$

$-\frac{1}{2} = x^2 - 2$

$x^2 = \frac{3}{2}$

$x = \sqrt{\frac{3}{2}}$ or $-\sqrt{\frac{3}{2}}$

P is $\left(\sqrt{\frac{3}{2}}, -\frac{1}{2}\right)$ or $\left(-\sqrt{\frac{3}{2}}, -\frac{1}{2}\right)$

(c) (i) $\int_1^5 \frac{1}{x^2}\,dx = \left[\frac{x^{-1}}{-1}\right]_1^5$

$= -\frac{1}{5} - (-1)$

$= \frac{4}{5}$

(ii) the area of the four rectangles is given by

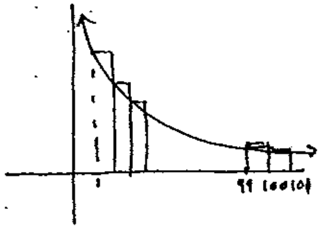
$1 \times \frac{1}{1^2} + 1 \times \frac{1}{2^2} + 1 \times \frac{1}{3^2} + 1 \times \frac{1}{4^2}$

the area under the curve is less than the area of the rectangles as parts are above the curve

the area under the curve is $\frac{4}{5}$ from (i)

so $1 \times \frac{1}{1^2} + 1 \times \frac{1}{2^2} + 1 \times \frac{1}{3^2} + 1 \times \frac{1}{4^2} > \frac{4}{5}$

(iii) extend the diagram to $x = 101$



the area under the curve is given by

$\int_1^{101} \frac{1}{x^2}\,dx = \left[-x^{-1}\right]_1^{101}$

$= -\frac{1}{101} + 1$

$= \frac{100}{101}$

the area of the rectangles is

$1 \times \frac{1}{1^2} + 1 \times \frac{1}{2^2} + \dots \; 1 \times \frac{1}{100^2}$

using the same reasoning in (ii)

$\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{100^2} > \frac{100}{101}$

$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \frac{1}{99^2} + \frac{1}{100^2} > \frac{99}{100} + \frac{1}{100^2}$

$1 + \frac{1}{4} + \frac{1}{9} + \dots + \frac{1}{10000} > \frac{9901}{10000}$

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