SYDNEY GRAMMAR SCHOOL

2011 Half-Yearly Examination

FORM VI
MATHEMATICS EXTENSION 1

Thursday 3rd March 2011

General Instructions

- Writing time — 2 Hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Structure of the paper

- Total marks — 84
- All seven questions may be attempted.
- All seven questions are of equal value.

Collection

- Write your candidate number clearly on each booklet.
- Hand in the seven questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place the question paper inside your answer booklet for Question One.

Checklist

- SGS booklets — 7 per boy
- Candidature — 128 boys

Examiner
LYL

QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) Write down the exact value of:

(i) $\cos \frac{7\pi}{6}$ [1]

(ii) $\tan^{-1}(\sqrt{3})$ [1]

(b) Simplify $2\sin 2x \cos 2x$. [1]

(c) Differentiate with respect to x:

(i) $x^2 \cos x$ [2]

(ii) e^{4x} [1]

(iii) $\ln(5x+3)$ [1]

(d) Evaluate $\lim_{x\to 0}\left(\frac{\sin 7x}{x}\right)$. You must show working. [1]

(e) Find:

(i) $\int (1-x)^5\,dx$ [1]

(ii) $\int (\cos x - \sin x)\,dx$ [1]

(iii) $\int \sin 2x\,dx$ [1]

(iv) $\int e^{2-3x}\,dx$ [1]

Exam continues next page ...

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) What is the amplitude and period of the function $f(x) = 3\sin 2x$? [2]

(b) Find the acute angle between the lines $2x+y-2=0$ and $y=x+9$. Give your answer to the nearest degree. [2]

(c) Solve $\dfrac{1}{x+5} \geq 1$. [3]

(d) Consider the function $f(x) = x^2 - 9$, for $x \geq 0$.

(i) Sketch and clearly label $y = f(x)$, for $x \geq 0$. [1]

(ii) Find the equation of the inverse function. [2]

(iii) State the domain of $y = f^{-1}(x)$. [1]

(iv) Sketch the inverse function $y = f^{-1}(x)$. [1]

QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

(a) The point A is $(12, -10)$ and the point B is $(-3, -5)$. The point P divides the interval AB internally in the ratio $2:3$. Find the coordinates of P. [2]

(b) Consider the parabola with parametric equations

$$x = 4t,$$
$$y = t^2 - 1.$$

(i) Find the Cartesian equation of this parabola. [2]

(ii) State the coordinates of the vertex. [1]

(iii) State the coordinates of the focus. [2]

(c) Solve $\cos 2\theta - \sin\theta = 0$, for $0 \leq \theta \leq 2\pi$. [3]

(d) Find the exact value of $\tan\left(\cos^{-1}(-\frac{5}{7})\right)$. [2]

Exam continues overleaf ...

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

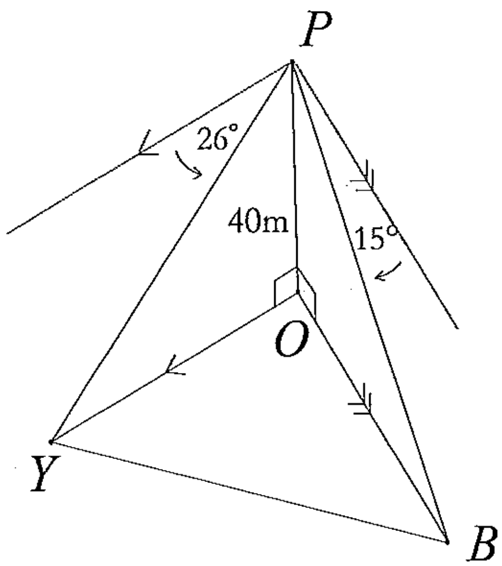
(a) Prove the identity [2]

$$\frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta} = 2\operatorname{cosec}^2\theta.$$

(b) Evaluate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan^2 x\,dx$. [3]

(c) A computer animation shows the sides of a cube increasing at a rate of 3 mm/s. Find the rate at which the volume V is increasing when the cube has a side length of 5 mm. [3]

(d)



The diagram above shows an offshore oil rig PO. A viewing platform at P is 40 metres above its base O which is at sea level. A boat B and a yacht Y are observed from P with angles of depression of 15° and 26° respectively. From O, boat B is on a bearing of 150° and yacht Y is on a bearing of 225°.

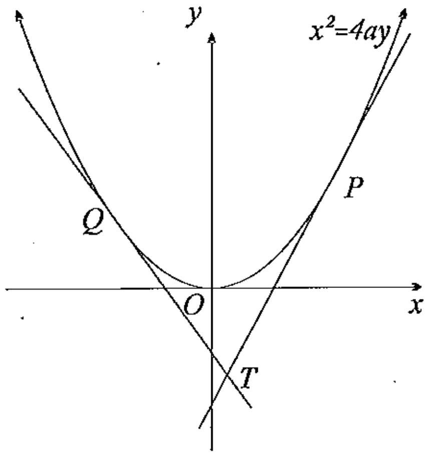
(i) Explain why $\angle YOB = 75°$. [1]

(ii) Find the distance between the vessels, YB, to the nearest metre. [3]

Exam continues next page ...

QUESTION FIVE (12 marks) Use a separate writing booklet. **Marks**

(a)



In the diagram above, two points $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$. The tangents at P and Q intersect at the point T.
The equation of the tangent at P is $y = px - ap^2$.

(i) Show that the coordinates of the point T are $(a(p+q), apq)$. **3**

(ii) Given that $pq = -2$, find the equation of the locus of T. **1**

(b) (i) Express $\sqrt{3}\cos x - \sin x$ in the form $R\cos(x + \alpha)$, where $R > 0$ and $0 \le \alpha \le \frac{\pi}{2}$. **2**

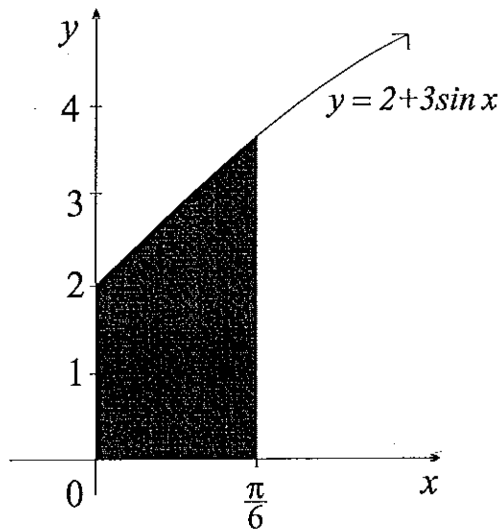
(ii) Hence, or otherwise, solve the equation $\sqrt{3}\cos x - \sin x = 1$, for $-\pi \le x \le \pi$. Give your solutions as exact values. **2**

(c) Use mathematical induction to prove that $7^{2n+1} + 11^{2n+1}$ is divisible by 3, for $n = 0, 1, 2, 3, \ldots$. **4**

Exam continues overleaf ...

QUESTION SIX (12 marks) Use a separate writing booklet. **Marks**

(a)



The diagram above shows the region bounded by the curve $y = 2 + 3\sin x$ and the x-axis from $x = 0$ to $x = \frac{\pi}{6}$. Find the exact volume of the solid generated when the region is rotated about the x-axis. **3**

(b) Consider the curve whose equation is $f(x) = \dfrac{x(x-1)}{(x+1)(x-2)}$.

(i) Find the coordinates of any intercepts with the axes. **2**

(ii) Find the equations of the vertical asymptotes of the curve. **1**

(iii) Find the equation of the horizontal asymptote of the curve. **1**

(iv) Find $f'(x)$. **2**

(v) Find the stationary point and determine its nature. **2**

(vi) Sketch the curve. **1**

Exam continues next page ...

QUESTION SEVEN (12 marks) Use a separate writing booklet. **Marks**

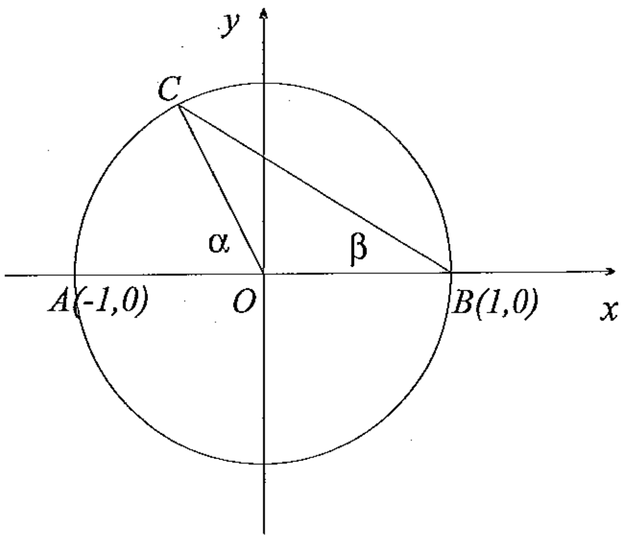
(a) Consider the functions $y = 2\sin x$ and $y = \tan x$, for $-\pi \le x \le \pi$.

(i) On the same axes, sketch the graphs of the functions $y = 2\sin x$ and $y = \tan x$, for $-\pi \le x \le \pi$. [2]

(ii) Find all the solutions of $2\sin x = \tan x$, for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. [2]

(iii) Hence solve $2\sin x \le \tan x$, for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. [1]

(b)



In the diagram above, the points $A(-1,0)$, $B(1,0)$, and C all lie on the circle with centre O and radius 1. Let $\angle COA = \alpha$ and $\angle CBO = \beta$.

(i) Given that the line BC has gradient m, find its equation. [1]

(ii) Show that the x-coordinates of B and C are solutions of the equation [2]

$$(1+m^2)x^2 - 2m^2x + m^2 - 1 = 0.$$

(iii) Using this equation, find the coordinates of C in terms of m. [2]

(iv) Hence deduce that $\tan 2\beta = \dfrac{2\tan\beta}{1-\tan^2\beta}$. [2]

END OF EXAMINATION

The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \ne -1; \quad x \ne 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \ne 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \ne 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \ne 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \ne 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \ne 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \ne 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x + \sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x + \sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Form 6

Half Yearly Ext 1 2011

Q1 a) i) $\cos \frac{7\pi}{6} = -\cos\frac{\pi}{6}$

$= -\frac{\sqrt{3}}{2}$ ✓

ii) $\tan^{-1}(\sqrt{3}) = \frac{\pi}{3}$ ✓

b) $\sin 4x$ ✓

c) i) $\frac{d}{dx}\, x^2\cos x = 2x\cos x - x^2\sin x$ ✓✓

ii) $\frac{d}{dx}\, e^{4x} = 4e^{4x}$ ✓

iii) $\frac{d}{dx}\left(\ln(5x+3)\right) = \frac{5}{5x+3}$ ✓

d) $\lim_{x\to 0}\left(\frac{\sin 7x}{x}\right) = 7\lim_{x\to 0}\frac{\sin 7x}{7x}$ ✓

$= 7$

e) i) $\int (1-x)^5\,dx = -\frac{(1-x)^6}{6} + c$ ✓

ii) $\int (\cos x - \sin x)\,dx$

$= \sin x + \cos x + c$ ✓

iii) $\int \sin 2x = -\frac{1}{2}\cos 2x + c$ ✓

(or $\sin^2 x + c$

or $-\cos^2 x + c$)

iv) $\int e^{2-3x}\,dx = -\frac{e^{2-3x}}{3} + c$ ✓

Q2 a) Amplitude $= 3$ ✓

period $T = \frac{2\pi}{2}$

$= \pi$ ✓

b) $2x + y - 2 = 0$

$y = -2x + 2$

$m_1 = -2$, $m_2 = 1$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{-2-1}{1-2\times 1}\right|$ ✓

$= \left|\frac{-3}{-1}\right|$

$\tan\theta = 3$ ✓

$\theta \doteqdot 72^\circ$ (nearest degree)

c) $\frac{1}{x+5} \geq 1$

$(x+5) \geq (x+5)^2$

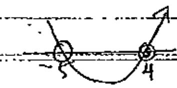
$(x+5)^2 - (x+5) \leq 0$

$(x+5)\left[(x+5) - 1\right] \leq 0$ ✓

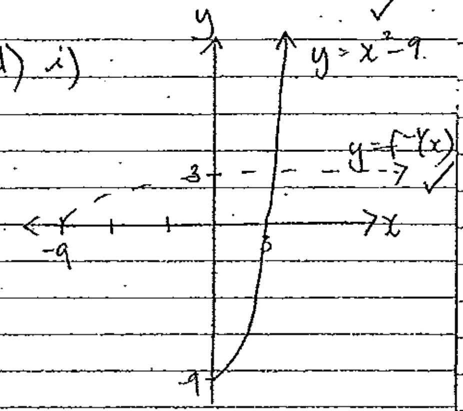
$(x+5)(x+4) \leq 0$ ✓

$-5 < x \leq -4$ ✓

$(x \neq -5)$



Q1 d) i)



✓ ✓

ii) $y = x^2 - 9$

Swap variables

$x = y^2 - 9$

$y^2 = x + 9$ ✓

$y = \sqrt{x+9}$ ✓

Range $y \geq 0$

iii) Domain $x \geq -9$ ✓

iv) on same diagram as i)

Q3 a) $A(12, -10)$ $B(-3, -5)$

$2:3$

$k:l$

$x = \frac{lx_1 + kx_2}{k+l}$

$= \frac{3\times 12 + 2\times -3}{5}$

$= \frac{36-6}{5}$

$= 6$

$y = \frac{ly_1 + ky_2}{k+l}$

$= \frac{+3\times -10 + 2\times -5}{5}$

$= \frac{-30-10}{5}$

$= -8$

$P(x, y) = (6, -8)$

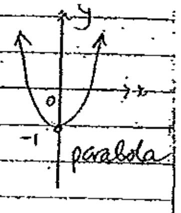
b) i) $x = 4t$ $y = t^2 - 1$

$t = \frac{x}{4}$ ✓

$y = \left(\frac{x}{4}\right)^2 - 1$

✓ $y = \frac{x^2}{16} - 1$

or $x^2 = 16(y+1)$



ii) Vertex $(0, -1)$ ✓

iii) $4a = 16$

$a = 4$ ✓ Focus $(0, 3)$ ✓

c) $\cos 2\theta - \sin\theta = 0$ $0 \leq \theta \leq 2\pi$

$1 - 2\sin^2\theta - \sin\theta = 0$ ✓

$2\sin^2\theta + \sin\theta - 1 = 0$

$(2\sin\theta - 1)(\sin\theta + 1) = 0$ ✓

$\sin\theta = \frac{1}{2}$ or $\sin\theta = -1$

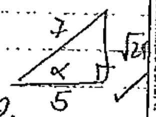
$\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{3\pi}{2}$ ✓

d) $\tan\left(\cos^{-1}\left(-\frac{5}{7}\right)\right)$

let $\alpha = \cos^{-1}\left(-\frac{5}{7}\right)$

$\cos\alpha = -\frac{5}{7}$

$0 \leq \alpha \leq \pi$ α in 2nd quad.



$\tan\alpha = -\frac{\sqrt{24}}{5}$ ✓

Q4a) LHS $= \frac{1}{1-\cos\theta} + \frac{1}{1+\cos\theta}$

$= \frac{(1+\cos\theta) + (1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}$ ✓

$= \frac{2}{1-\cos^2\theta}$

$= \frac{2}{\sin^2\theta}$ ✓

$= 2\csc^2\theta$

= RHS

b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \tan^2 x \, dx$

$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (\sec^2 x - 1)\, dx$

$= \left[\tan x - x\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$ ✓

$= \left(\tan\frac{\pi}{3} - \frac{\pi}{3}\right) - \left(\tan\frac{\pi}{4} - \frac{\pi}{4}\right)$

$= \sqrt{3} - \frac{4\pi}{12} - 1 + \frac{3\pi}{12}$ ✓

$= \sqrt{3} - 1 - \frac{\pi}{12}$

c) let l be the side length

$\frac{dl}{dt} = 3 \qquad V = l^3$

$\frac{dV}{dt} = \frac{dV}{dl} \times \frac{dl}{dt}$ ✓

$= 3l^2 \times 3$ ✓

$= 9l^2$

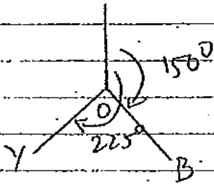
$\frac{dV}{dt}(\text{at } l=5) = 9 \times 25$ ✓

$= 225 \text{ mm}^3/\text{s}$

d) i) $\angle PYO = 26^\circ$ (alternate ∠ on ∥ lines)

$\angle PBO = 15^\circ$

Plan view

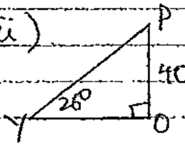


$\angle YOB$ = Bearing of Y − Bearing of B

$= 225^\circ - 150^\circ$ ✓

$= 75^\circ$ (adjacent ∠s)

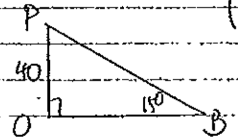
ii)



$\tan 26^\circ = \frac{40}{YO}$

$YO = \frac{40}{\tan 26^\circ}$

$(= 40\cot 26^\circ$ or $40\tan 64^\circ)$



$\tan 15^\circ = \frac{40}{BO}$

$BO = \frac{40}{\tan 15^\circ}$ ✓

$(= 40\cot 15^\circ$ or $40\tan 75^\circ)$

In △YOB cosine rule

$YB^2 = BO^2 + YO^2 - 2 \times BO \times YO \cos 75^\circ$

$= 40^2(\tan^2 75 + \tan^2 64 - 2 \times \tan 75 \tan 64 \cos 75^\circ)$ ✓

$\doteqdot 22674$ ✓

$YB = 151$ m

Q5 i) Equation of P

$y = px - ap^2$ ①

Equation of Q

$y = qx + aq^2$ ② ✓

① = ②

$px - ap^2 = qx - aq^2$

$px - qx = ap^2 - aq^2$ ✓

$x(p-q) = a(p-q)(p+q)$

$x = a(p+q)$ ③

Sub ③ into ①

$y = pa(p+q) - ap^2$ ✓

$= ap^2 + apq - ap^2$

$= apq$

ii) $pq = -2$ ✓

$y = -2a$ is the locus of T

as a is constant.

b.) i) $\sqrt{3}\cos x - \sin x = R\cos(x+\alpha)$

$\sqrt{3}\cos x - \sin x = R\cos x\cos\alpha - R\sin x\sin\alpha$

$= R\cos\alpha\cos x - R\sin\alpha\sin x$

$R\cos\alpha = \sqrt{3} \qquad -R\sin\alpha = -1$

$R^2 = 4$

$R = 2 \quad R > 0$ ✓

$2\cos\alpha = \sqrt{3} \qquad -2\sin\alpha = -1$

$\cos\alpha = \frac{\sqrt{3}}{2}, \qquad \sin\alpha = \frac{1}{2}$

$\alpha = \frac{\pi}{6}$

So $\sqrt{3}\cos x - \sin x = 2\cos\left(x + \frac{\pi}{6}\right)$

ii). $2\cos\left(x + \frac{\pi}{6}\right) = 1, \quad -\pi \le x \le \pi$

let $u = x + \frac{\pi}{6} \qquad -\frac{5\pi}{6} \le u \le \frac{7\pi}{6}$

$2\cos u = 1$

$\cos u = \frac{1}{2}$

$u = \frac{\pi}{3}$ or $u = -\frac{\pi}{3}$ ✓

$x + \frac{\pi}{6} = \frac{\pi}{3} \qquad x + \frac{\pi}{6} = -\frac{\pi}{3}$

$x = \frac{\pi}{6} \qquad x = -\frac{\pi}{2}$ ✓

c) A. For $n = 0$

$7^{2n+1} + 11^{2n+1} = 7 + 11$ ✓

$= 18$ div by 3

B. Suppose k is such that $7^{2k+1} + 11^{2k+1}$ is div by 3 for some integer value of $k > 0$

ie. $7^{2k+1} + 11^{2k+1} = 3m$ for some integer m. *

C. The statement must be proved for $n = k+1$ ✓

$7^{2k+3} + 11^{2k+3}$

$= 7^{2k+1} \times 7^2 + 11^{2k+3}$

$= (3m - 11^{2k+1})7^2 + 11^{2k+3}$ ✓ using the inductive hypothesis *

$= 3m \times 7^2 + 11^{2k+1}(11^2 - 7^2)$

$= 3m \times 7^2 + 11^{2k+1} \times 72$

$= 3(7^2 m + 24 \times 11^{2k+1})$

which is divisible by 3 ✓

So by steps A, B and mathematical induction, the statement is true for $n = 0, 1, 2, 3, \ldots$

Q6 a) $y = 2 + 3\sin x$

$y^2 = (2 + 3\sin x)^2$

$= 4 + 12\sin x + 9\sin^2 x$

$V = \pi \int_0^{\frac{\pi}{6}} (4 + 12\sin x + 9\sin^2 x)\,dx$

$= \pi\Big[4x - 12\cos x\Big]_0^{\frac{\pi}{6}} + 9\pi\int_0^{\frac{\pi}{6}}\left(\frac{1}{2} - \frac{1}{2}\cos 2x\right)dx$

$= \pi\left(\frac{2\pi}{3} - 12 \times \frac{\sqrt{3}}{2} + 12\right) + 9\pi\Big[\frac{x}{2} - \frac{1}{4}\sin 2x\Big]_0^{\frac{\pi}{6}}$

$= \pi\left(\frac{2\pi}{3} - 6\sqrt{3} + 12\right) + 9\pi\left(\frac{\pi}{12} - \frac{1}{4} \times \frac{\sqrt{3}}{2}\right)$

$= \pi\left(\frac{2\pi}{3} - 6\sqrt{3} + 12 + \frac{3\pi}{4} - \frac{9\sqrt{3}}{8}\right)$

$= \pi\left(\frac{17\pi}{12} - \frac{57\sqrt{3}}{8} + 12\right)\ u^3$

b) i) $(0,0)$, $(1,0)$

ii) vertical asymptotes at $x = -1$ and $x = 2$

iii) $y = 1$

iv) $f(x) = \frac{x(x-1)}{(x+1)(x-2)}$

$f'(x) = \frac{(x+1)(x-2)(2x-1) - x(x-1)(2x-1)}{(x+1)^2(x-2)^2}$

$= \frac{(2x-1)\left[x^2 - x - 2 - (x^2 - x)\right]}{(x+1)^2(x-2)^2}$

$= \frac{-2(2x-1)}{(x+1)^2(x-2)^2}$

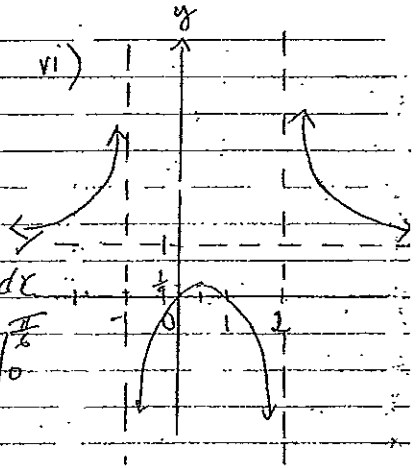
v) $f'(x) = 0$ when $x = \frac{1}{2}$

$y = \frac{1}{9}$

x	0	$\frac{1}{2}$	1
$f'(x)$	$+$	0	$-$

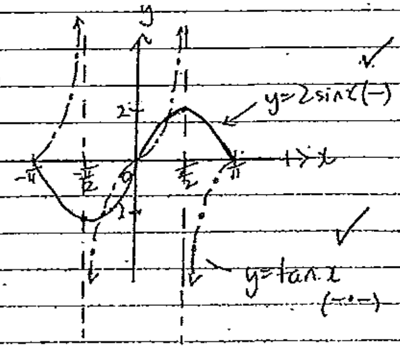
$\left(\frac{1}{2}, \frac{1}{9}\right)$ is a maximum turning point.

vi)



Q7

a) i)



ii) $2\sin x = \tan x \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

$2\sin x = \frac{\sin x}{\cos x}$

$2\sin x\cos x - \sin x = 0$

$\sin x(2\cos x - 1) = 0$

$\sin x = 0 \quad \cos x = \frac{1}{2}$

$x = 0$ or $x = \pm\frac{\pi}{3}$

iii) $0 < x < \frac{\pi}{3}$ or $-\frac{\pi}{2} < x < -\frac{\pi}{3}$

b) i) $B(1,0)$ $\quad y - 0 = m(x-1)$

$y = mx - m$

ii) B & C are the solutions of the simultaneous equations

$y = mx - m$ ①

$x^2 + y^2 = 1$ ②

Sub ① into ②

$x^2 + (mx - m)^2 = 1$

$x^2 + m^2x^2 - 2m^2x + m^2 = 1$

$x^2(1 + m^2) - 2m^2x + (m^2 - 1) = 0$

as required.

iii) Given the x-coordinate of B is 1, let x-coordinate of C be β.

$\left(\alpha + \beta = \frac{-b}{a}\right)$

$1 + \beta = \frac{2m^2}{1 + m^2}$

$\beta = \frac{2m^2}{1 + m^2} - 1$

$= \frac{2m^2 - 1(1 + m^2)}{1 + m^2}$

$= \frac{m^2 - 1}{m^2 + 1}$

y-coordinate of C is given by

$y = m\left(\frac{m^2 - 1}{m^2 + 1} - 1\right)$

$= m\left(\frac{m^2 - 1 - m^2 - 1}{m^2 + 1}\right)$

$= \frac{-2m}{m^2 + 1}$

$C\left(\frac{m^2 - 1}{m^2 + 1}, \frac{-2m}{m^2 + 1}\right)$

iv) ΔOBC is isosceles (radii)

$\alpha = 2\beta$ (exterior angle of Δ)

$\tan\alpha = \tan 2\beta$ (α is acute)

But $\tan(\pi - \beta) = m$

so $-\tan\beta = m$

$\tan(\pi - \alpha) =$ gradient of OC

$-\tan\alpha = \frac{-2m}{1 + m^2} \div \frac{m^2 - 1}{m^2 + 1}$

$= \frac{-2m}{m^2 - 1}$

So $\tan 2\beta = \frac{2m}{m^2 - 1}$

$= \frac{-2\tan\beta}{\tan^2\beta - 1}$

$= \frac{2\tan\beta}{1 - \tan^2\beta}$

as required