SYDNEY GRAMMAR SCHOOL

2012 Half-Yearly Examination

FORM VI
MATHEMATICS 2 UNIT

Monday 20th February 2012

General Instructions
- Writing time — 2 hours
- Write using black or blue pen.
- Board-approved calculators and templates may be used.
- A list of standard integrals is provided at the end of the examination paper.

Total — 90 Marks
- All questions may be attempted.

Section I – 10 Marks
- Questions 1–10 are of equal value.

Section II – 80 Marks
- Questions 11–15 are of equal value.
- All necessary working should be shown in every question.
- Start each question in a new booklet.

Collection

Section I Questions 1–10
- Place your multiple choice answer sheet inside the answer booklet for Question Eleven.

Section II Questions 11–15
- Start each of these questions in a new booklet.
- Write your candidate number clearly on each booklet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Place the question paper inside your answer booklet for Question Eleven.

Checklist
- SGS booklets — 5 per boy
- Candidature — 80 boys

Examiner
JMR

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

Question One

Which of the following is equal to $e^x(e^x - \frac{1}{e})$?

(A) $e^{x^2} - e$

(B) $e^{2x} - e$

(C) $e^{x^2} - e^{x+1}$

(D) $e^{2x} - e^{x-1}$

Question Two

For the function $y = x^3 + 1$, which one of the following statements is true?

(A) The function is odd.

(B) The function is even.

(C) The function is increasing for all values of $x > 0$.

(D) There is a triple root at $x = -1$.

Question Three

The definite integral $\displaystyle\int_0^4 (x^2 + 2)\,dx$ is equal to

(A) $\dfrac{88}{3}$

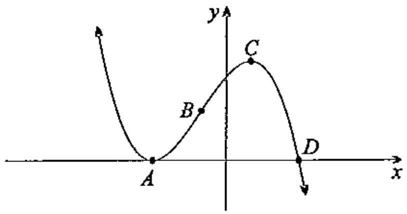
(B) 18

(C) $\dfrac{64}{3}$

(D) $23\frac{1}{3}$

Exam continues next page ...

Question Four



Given the function $y = f(x)$ above, which of the following statements is false?

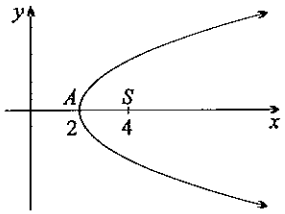
(A) There is a local minimum at A.

(B) The concavity changes at B.

(C) There is a global maximum at C.

(D) The zeroes occur at points A and D.

Question Five



For the parabola sketched above, point A is the vertex and point S is the focus. The equation of the parabola could be

(A) $(y-2)^2 = 8x$

(B) $y^2 = 8(x-2)$

(C) $(y-2)^2 = 8(x-2)$

(D) $(y+2)^2 = 8(x-2)$

Exam continues overleaf ...

Question Six

Which of the following is a primitive of $\dfrac{10}{x^2}$?

(A) $-5x^2 + C$

(B) $-10x^3 + C$

(C) $-\dfrac{10}{x} + C$

(D) $-\dfrac{10}{3x^2} + C$

Question Seven

The graph of the locus of the point $P(x, y)$ that moves so that its distance from a point $A(1,1)$ is twice the distance from another point $B(4,1)$ would be a

(A) vertical line

(B) parabola with a vertical axis of symmetry

(C) parabola with a horizontal axis of symmetry

(D) circle

Question Eight

The number of solutions to the equation $e^{x+1} + x^2 + 2 = 0$ may be found by sketching graphs. Which of the following statements is true?

(A) We should sketch $y = e^{x+1}+2$ and $y = -x^2$ to show there are no solutions.

(B) We should sketch $y = x^2+2$ and $y = e^{x+1}$ to show there are two solutions.

(C) We should sketch $y = e^{x+1}$ and $y = -x^2-2$ to show there are two solutions.

(D) We should sketch $y = e^{x+1}+2$ and $y = x^2$ to show only one solution.

Question Nine

The gradient of a line that is perpendicular to $3x + 5y - 5 = 0$ is

(A) $-\frac{1}{3}$

(B) $\frac{5}{3}$

(C) $-\frac{5}{3}$

(D) $-\frac{3}{5}$

Exam continues next page ...

Question Ten

The equation of a line with gradient $\frac{3}{2}$ and y-intercept $\frac{1}{2}$ is

(A) $y = 3(2x - 1)$

(B) $y = \dfrac{2x+1}{3}$

(C) $3y = 1 - 2x$

(D) $y = \dfrac{1}{2}(3x+1)$

——— End of Section I ———

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

Question Eleven (16 marks) Use a separate writing booklet. **Marks**

(a) Use your calculator to find $\dfrac{e^3}{2}$ correct to two decimal places. **1**

(b) Simplify $\dfrac{(e^x)^4}{e^x}$. **1**

(c) A parabola has equation $x^2 = 8y$. Find: **3**

(i) the coordinates of the vertex,

(ii) the coordinates of the focus,

(iii) the equation of the directrix.

(d) Differentiate: **3**

(i) $\dfrac{x^4}{2}$

(ii) $3e^{2x}$

(iii) $(2x-1)^5$

(e) Find a primitive of: **3**

(i) $x + 16$

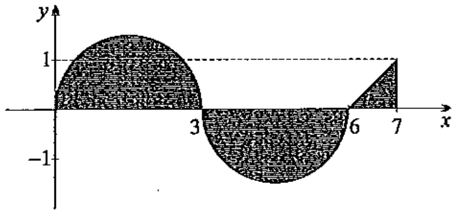
(ii) e^{4x+1}

(iii) $\sqrt{x}$

(f) Sketch on a number plane the locus of a point P which moves so that it is always 3 units from the origin. Write down the equation of the locus of P. **2**

Question ELEVEN (Continued)

(g)



The function $y = f(x)$, for $0 \le x \le 7$, is shown above. The curves are semicircular arcs.

(i) Find $\int_0^7 f(x)\,dx$. **[1]**

(ii) Find the exact total area of the shaded parts. **[2]**

Question Twelve (16 marks) Use a separate writing booklet. **Marks**

(a) Evaluate the following definite integrals.

(i) $\int_{-1}^{1} (6x - 2)\,dx$ **[2]**

(ii) $\int_1^2 \frac{1}{x^3}\,dx$ **[2]**

(b) By completing squares, find the centre and radius of the circle $x^2 + y^2 - 4x + 8y = 5$. **[2]**

(c) Given that $f'(x) = 2x^2 - 6$, find $f(x)$ if $f(1) = 0$. **[2]**

(d) Consider the function $y = x^3 - 6x^2 + 7$.

(i) Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$. **[2]**

(ii) Find the coordinates of any stationary points and determine their nature. **[2]**

(iii) Find the coordinates of the point of inflexion. You must show that it is a point of inflexion. **[2]**

(iv) Sketch the graph of the function, clearly showing all stationary and inflexion points. Do NOT attempt to find any x-intercepts. **[2]**

Question Thirteen (16 marks) Use a separate writing booklet. **Marks**

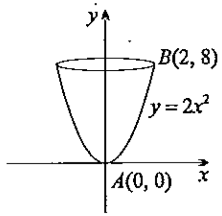
(a) Find the first and second derivatives of e^{x^2}. **[3]**

(b) Use the quotient rule to differentiate $y = \frac{2e^{2x+3}}{x+3}$. In your answer simplify the numerator as far as possible. **[2]**

(c) Use Simpson's Rule with three function values to approximate $\int_0^1 2^x\,dx$. Give your answer correct to two decimal places. **[2]**

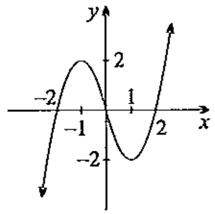
(d) Find the value of p if $\int_1^p (3x + 4)\,dx = 20$ and $p > 1$. **[3]**

(e) **[3]**



The diagram above shows a cup of height 8 cm whose width at the top is 4 cm. It is formed by rotating the arc AB of the parabola $y = 2x^2$ about the y-axis. Find the exact volume of the cup.

(f)



The graph of $y = f(x)$ is sketched above. Sketch on separate diagrams, clearly indicating any x-intercepts, possible graphs of:

(i) $y = f'(x)$ **[2]**

(ii) $y = f''(x)$ **[1]**

Question Fourteen (16 marks) Use a separate writing booklet. **Marks**

(a) Use the second derivative to explain why the graph of the function $y = e^{-2x}$ is always concave up. [1]

(b) Find the equation of the normal to the curve $y = x + e^x$ at the point where the curve cuts the y-axis. [3]

(c) Sketch a graph of the parabola $6x + y^2 = 18$ clearly indicating the vertex, focus, and directrix. [3]

(d) A car's velocity v in metres per second is recorded each second as it accelerates along a drag strip. The table below gives the results. [2]

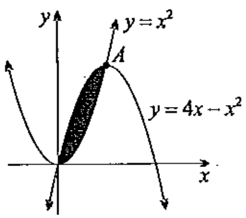
$t\,(s)$	0	1	2	3	4	5
$v\,(ms^{-1})$	0	15	31	48	64	83

Given that the distance travelled may be found by calculating the area under a velocity/time graph, use the trapezoidal rule to estimate the distance travelled by the car in the first five seconds.

(e) Solve for x: [3]

$$e^{2x} + e^x - 2 = 0$$

(f)



The diagram shows the curves $y = x^2$ and $y = 4x - x^2$ which intersect at the origin and at point A.

(i) Find the coordinates of point A. [2]

(ii) Hence find the area enclosed by the two parabolas. [2]

Question Fifteen (16 marks) Use a separate writing booklet. **Marks**

(a) A continuous function $y = f(x)$ satisfies all of the following conditions: [3]

$f(2) > 0$

$f(-4) < 0$

$f'(x) > 0$

$f''(x) < 0$

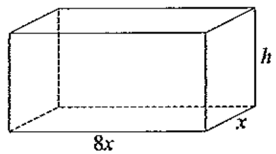
Draw a possible sketch of the function for $-4 \le x \le 2$.

(b) Suppose that $y = e^{kx}$.

(i) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. [2]

(ii) Find the value of k such that $y = 2\dfrac{dy}{dx} - \dfrac{d^2y}{dx^2}$. [2]

(c)



The diagram above shows the framework of a storage container which has been constructed in the shape of a rectangular prism. The container is eight times as long as it is wide, and has breadth x metres and height h metres.

(i) Find, in terms of x and h, an expression for the total length L of steel required to construct the frame. [2]

(ii) The container has volume 2304 m^3.

(α) Show that $h = \dfrac{288}{x^2}$. [2]

(β) Show that $L = 36x + \dfrac{1152}{x^2}$. [1]

(γ) Find the dimensions of the container so that the minimum length of steel is used in the construction of the frame. [4]

End of Section II

END OF EXAMINATION

The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\ a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a > 0,\ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\ x > 0$

Tear-off pages follow ...

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- Record your multiple choice answers by filling in the circle corresponding to your choice for each question.
- Fill in the circle completely.
- Each question has only one correct answer.

CANDIDATE NUMBER: ....

Question One
A ○ B ○ C ○ D ●

Question Two
A ○ B ○ C ● D ○

Question Three
A ● B ○ C ○ D ○

Question Four
A ○ B ○ C ● D ○

Question Five
A ○ B ● C ○ D ○

Question Six
A ○ B ○ C ● D ○

Question Seven
A ○ B ○ C ○ D ●

Question Eight
A ● B ○ C ○ D ○

Question Nine
A ○ B ● C ○ D ○

Question Ten
A ○ B ○ C ○ D ●

MATHS 2-UNIT SOLUTIONS HALF-YEARLY 2012

QUESTION 11.

(a) 10·04

(b) $\frac{e^{4x}}{e^x} = e^{3x}$

(c) $x^2 = 8y$

(i) Vertex $V(0,0)$

(ii) Focus

$a = 2$ $S(0,2)$

(iii) Directrix $y = -2$

(d) (i) $\frac{d}{dx}\left(\frac{1}{2}x^4\right) = \frac{1}{2} \times 4x^3$

$= 2x^3$

(ii) $\frac{d}{dx}\left(3e^{2x}\right) = 6e^{2x}$

(iii) $\frac{d}{dx}\left((2x-1)^5\right) = 5 \times 2(2x-1)^4$

$= 10(2x-1)^4$

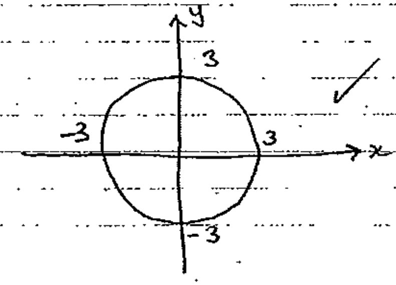
(e) (i) $\int x + 16\, dx = \frac{1}{2}x^2 + 16x + c$

(ii) $\int e^{4x+1}\, dx = \frac{1}{4}e^{4x+1} + c$

(iii) $\int x^{\frac{1}{2}}\, dx = \frac{2}{3}x^{\frac{3}{2}} + c$

$= \frac{2x\sqrt{x}}{3} + c$

(f)



locus of P : $x^2 + y^2 = 9$

(g) (i) $\int_0^7 f(x)\, dx = \frac{1}{2} \times 1 \times 1$

$= \frac{1}{2}$

(ii) $A = \pi\left(\frac{3}{2}\right)^2 + \frac{1}{2}$

$= \frac{9\pi}{4} + \frac{1}{2}$ units^2

MATHS 2-UNIT SOLUTIONS

QUESTION 12

(a) (i) $\int_{-1}^{1}(6x-2)\, dx = \left[\frac{6x^2}{2} - 2x\right]_{-1}^{1}$

$= \left[3x^2 - 2x\right]_{-1}^{1}$

$= (3-2) - (3+2)$

$= -4$

(ii) $\int_1^2 \frac{1}{x^3}\, dx = \int_1^2 x^{-3}\, dx$

$= \left[\frac{x^{-2}}{-2}\right]_1^2$

$= \left[-\frac{1}{2x^2}\right]_1^2$

$= \left(-\frac{1}{8}\right) - \left(-\frac{1}{2}\right)$

$= \frac{3}{8}$

(b) $x^2 + y^2 - 4x + 8y = 5$

$x^2 - 4x + 4 + y^2 + 8y + 16 = 5 + 4 + 16$

$(x-2)^2 + (y+4)^2 = 25$

Centre $(2,-4)$
Radius 5 units

(c) $f'(x) = 2x^2 - 6$

$f(x) = \frac{2x^3}{3} - 6x + c$

But $f(1) = 0$

$0 = \frac{2}{3} - 6 + c$

$c = \frac{16}{3}$

$\therefore f(x) = \frac{2x^3}{3} - 6x + \frac{16}{3}$

(d) $y = x^3 - 6x^2 + 7$

(i) $\frac{dy}{dx} = 3x^2 - 12x$

$\frac{d^2y}{dx^2} = 6x - 12$

(ii) Stationary points when $\frac{dy}{dx} = 0$

$3x^2 - 12x = 0$

$3x(x-4) = 0$

$\therefore x = 0$ or 4

When $x = 0$ $y = 7$ and $\frac{d^2y}{dx^2} = -12 < 0$

So $(0,7)$ is a local maximum turning pt

When $x = 4$ $y = 4^3 - 6 \times 4^2 + 7$

$= -25$

$\frac{d^2y}{dx^2} = 12 > 0$ So $(4,-25)$ is a local minimum turning point

(iii) Possible pt. of inflexion when $\frac{d^2y}{dx^2} = 0$

$6x - 12 = 0$

$x = 2$

When $x = 2$ $y = 2^3 - 6 \times 2^2 + 7$

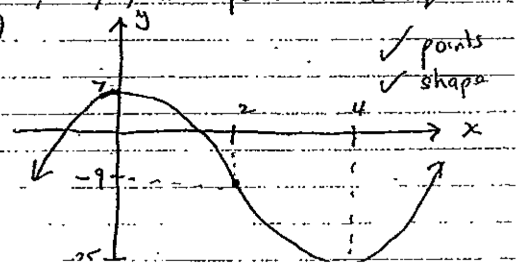
$= -9$

Check concavity changes

x	0	2	4
$\frac{d^2y}{dx^2}$	-12	0	+12
	concave down	•	concave up

So, $(2,-9)$ is a point of inflexion

(iv)



✓ points
✓ shape

QUESTION 13

(a) $y = e^{x^2}$

$\frac{dy}{dx} = 2xe^{x^2}$ ✓

$\frac{d^2y}{dx^2} = 2x \cdot 2xe^{x^2} + 2e^{x^2}$ ✓

$= 4x^2e^{x^2} + 2e^{x^2}$

$= 2e^{x^2}(2x^2+1)$ ✓

(b) $y' = \frac{(x+3)4e^{2x+3} - 2e^{2x+3}}{(x+3)^2}$ ✓

$= \frac{e^{2x+3}(4(x+3)-2)}{(x+3)^2}$

$= \frac{e^{2x+3}(4x+10)}{(x+3)^2}$ ✓

(c)

x	0	$\frac{1}{2}$	1
$f(x)$	1	$\sqrt{2}$	2

$\int_0^1 2^x\,dx \doteqdot \frac{1-0}{6}(1+4\sqrt{2}+2)$ ✓

$\doteqdot \frac{1}{6}(3+4\sqrt{2})$

$\doteqdot 1.44$ ✓

(d) $\int_1^p (3x+4)\,dx = \left[\frac{3x^2}{2} + 4x\right]_1^p$ ✓

$= 20$

$\frac{3p^2}{2} + 4p - \left(\frac{3}{2}+4\right) = 20$

$\frac{3p^2}{2} + 4p - \frac{11}{2} = 20$

$3p^2 + 8p - 11 = 40$ ✓

$3p^2 + 8p - 51 = 0$

$(3p+17)(p-3) = 0$

$\therefore p = 3 \quad (p > 1)$ ✓

(e) $V = \pi\int_0^8 x^2\,dy$

$= \pi\int_0^8 \frac{y}{2}\,dy$ ✓

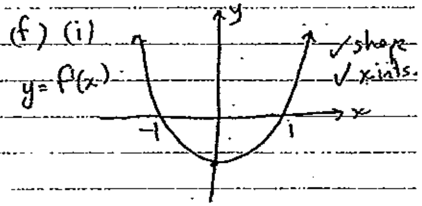
$= \frac{\pi}{2}\int_0^8 y\,dx$

$= \frac{\pi}{2}\left[\frac{y^2}{2}\right]_0^8$ ✓

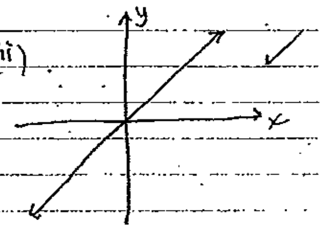
$= \frac{64\pi}{4}$

$= 16\pi$ units^3 ✓

(f) (i)



(ii)



✓

QUESTION 14

(a) $y = e^{-2x}$ $\quad y' = -2e^{-2x}$ $\quad y'' = 4e^{-2x}$

$\frac{4}{e^{2x}} > 0$ for all x, so $y = e^{-2x}$ is always concave up. ✓

(b) $y = x + e^x$

$\frac{dy}{dx} = 1 + e^x$ ✓

At $x = 0$

$\frac{dy}{dx} = 1+1$

$= 2$

So gradient of tangent is 2.

and $\therefore$ gradient of normal is $-\frac{1}{2}$ ✓

When $x=0$

$y = 0 + e^0$

$= 1$

$y - y_1 = m(x - x_1)$

$y - 1 = -\frac{1}{2}(x-0)$

$y - 1 = -\frac{1}{2}x$

$2y - 2 = -x$

$x + 2y - 2 = 0$ ✓

(c) $y^2 = -6x + 18$

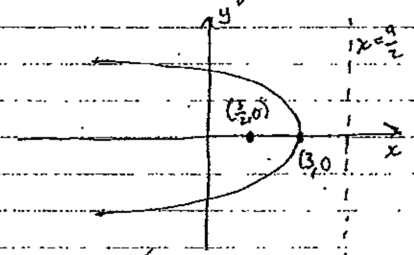
$y^2 = -6(x-3)$

$a = 1\frac{1}{2}$

Vertex $(3,0)$ ✓

Focus $(\frac{3}{2},0)$ ✓

Directrix $x = \frac{9}{2}$



✓

(d) $D \doteqdot \frac{1}{2}\left(0+2(15)+2(31)+2(48)+2(64)+83\right)$ ✓

$\doteqdot \frac{399}{2}$

$\doteqdot 199.5$ Metres ✓

(e) $e^{2x} + e^x - 2 = 0$

$(e^x - 1)(e^x + 2) = 0$ ✓

Either $e^x = 1$

$x = 0$ ✓

or $e^x = -2$

No solution, $e^x > 0$ ✓

(f) (i) $x^2 = 4x - x^2$ ✓

$2x^2 - 4x = 0$

$2x(x-2) = 0$

$x = 0$ or 2

When $x = 2$, $y = 4$

So A has coordinates $(2,4)$ ✓

(ii) $\int_0^2 (4x - x^2 - x^2)\,dx$ ✓

$= \int_0^2 (4x - 2x^2)\,dx$

$= \left[\frac{4x^2}{2} - \frac{2x^3}{3}\right]_0^2$

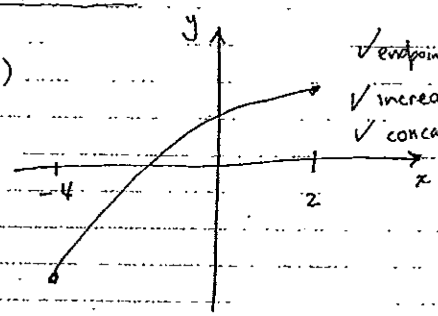
$= \left[2x^2 - \frac{2x^3}{3}\right]_0^2$

$= \left(8 - \frac{16}{3}\right) - 0$

$= \frac{8}{3}$ units^2 ✓

Question 15

(a)



✓ endpoint
✓ increasing
✓ concave down.

(b) (i) $\frac{dy}{dx} = ke^{kx}$ ✓

$\frac{d^2y}{dx^2} = k^2e^{kx}$ ✓

(ii) $e^{kx} = 2ke^{kx} - k^2e^{kx}$

$k^2e^{kx} - 2ke^{kx} + e^{kx} = 0$

$e^{kx}(k^2 - 2k + 1) = 0$ ✓

$e^{kx}(k-1)^2 = 0$

When $e^{kx} = 0$

No solution

When $(k-1) = 0$ ✓

$k = 1$

(c) (i) $L = 4(8x + x + h)$ ✓

$= 36x + 4h$ ✓

(ii) (α) $V = 8x \times x \times h$

$= 8x^2h$ ✓

$2304 = 8x^2h$ ✓

$h = \frac{2304}{8x^2}$

$= \frac{288}{x^2}$ ✓

(β) $L = 36x + 4h$

$= 36x + \frac{4 \times 288}{x^2}$

$L = 36x + \frac{1152}{x^2}$ ✓

(γ) Minimum occurs when

$\frac{dL}{dx} = 0$

✓ $36 - \frac{2304}{x^3} = 0$

$x^3 = \frac{2304}{36}$

$x^3 = 64$ ✓

So $x = 4$ ✓

Check that this is a minimum

$\frac{d^2L}{dx^2} = \frac{6912}{x^4}$ ✓

> 0

So when $x = 4$, $h = \frac{288}{16}$

$= 18$

(is a minimum)

Dimensions of container

$4m \times 32m \times 18m$ ✓