SYDNEY GRAMMAR SCHOOL

2009 Annual Examination

FORM V
MATHEMATICS EXTENSION 1

Thursday 10th September 2009

General Instructions

- Writing time — 3 hours
- Write using black or blue pen.
- Board-approved calculators may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question on a new leaflet.

Structure of the paper

- Total marks — 120
- All ten questions may be attempted.
- All ten questions are of equal value.

Collection

- Write your name, class and master clearly on each leaflet.
- Hand in the ten questions in a single well-ordered pile.
- Hand in a leaflet for each question, even if it has not been attempted.
- If you use a second leaflet for a question, place it inside the first.
- Write your name on the question paper and place it inside your leaflet for Question One.

Checklist

- Writing leaflets: 10 per boy.
- Candidature — 150 boys

Examiner
MLS

QUESTION ONE (12 marks) Use a separate writing booklet. **Marks**

(a) Factorise $9x^2 - 25$. [1]

(b) Solve $7 - 3x < -2$. [1]

(c) Simplify $\dfrac{x}{3} - \dfrac{x-2}{5}$. [2]

(d) Express $\dfrac{3}{\sqrt{5}-1}$ with a rational denominator. [2]

(e) Convert $\dfrac{3\pi}{5}$ radians to degrees. [1]

(f) If $f'(x) = 3x + 7$ find an expression for $f(x)$. [1]

(g) Differentiate:

(i) $y = 3\sin 2x$ [1]

(ii) $y = \sqrt{x}$ [2]

(h) Find $\displaystyle\int \frac{2}{x}\,dx$. [1]

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) Find the equation of the tangent to the curve $y = \log_e x$ at the point $(e, 1)$. [2]

(b) The third term of an arithmetic sequence is 7 and the seventh term is 31.

(i) Find the common difference. [1]

(ii) Find the eighteenth term. [2]

(iii) Find the sum of the first twenty terms. [1]

(c) Consider the function given by $y = x^3 - 3x^2 - 24x + 12$.

(i) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. [2]

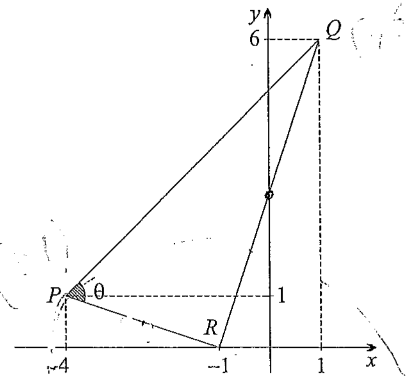
(ii) Show that there are stationary points at $x = -2$ and $x = 4$. [1]

(iii) Show that $(1, -14)$ is a point of inflexion. [2]

(iv) For what values of x is the gradient of this function positive? [1]

Exam continues next page ...

QUESTION THREE (12 marks) Use a separate writing booklet. Marks



The diagram shows the points $P(-4,1)$, $Q(1,6)$ and $R(-1,0)$. Angle QPR is θ.

(a) Show that the equation of QR is $3x - y + 3 = 0$. $\boxed{1}$

(b) Find the gradient of PR. $\boxed{1}$

(c) Show that PR and RQ are perpendicular. $\boxed{1}$

(d) Show that the length of PR is $\sqrt{10}$ units. $\boxed{1}$

(e) Find $\tan\theta$. $\boxed{2}$

(f) Write down the equation of the circle with centre R that passes through P. $\boxed{2}$

(g) Let T be the point on QR such that $RT = PR$. Find the coordinates of T. $\boxed{2}$

(h) Copy the diagram and shade the region where $y - 3x \leq 3$. $\boxed{2}$

Exam continues overleaf ...

QUESTION FOUR (12 marks) Use a separate writing booklet. Marks

(a) Differentiate :

(i) $y = (3x^2 - 5)^6$ $\boxed{2}$

(ii) $y = e^x \cos x$ $\boxed{2}$

(iii) $y = \dfrac{\tan x}{2x}$ $\boxed{2}$

(b) Find $\displaystyle\int \frac{x}{x^2+3}\,dx$. $\boxed{2}$

(c) Evaluate :

(i) $\displaystyle\int_0^3 e^{2x}\,dx$ $\boxed{2}$

(ii) $\displaystyle\int_1^2 \frac{1}{x^3}\,dx$ $\boxed{2}$

Exam continues next page ...

QUESTION FIVE (12 marks) Use a separate writing booklet. Marks

(a) (i) Show that the line $y = x + 4$ intersects the parabola $y = 10 - x^2$ at $x = -3$ and $x = 2$. 1

(ii) Find the area of the region enclosed by $y = x + 4$ and $y = 10 - x^2$. 2

(b) Let α and β be the roots of the equation $x^2 - 5x + 2 = 0$.
Without solving the equation, find the values of:

(i) $\alpha + \beta$ 1

(ii) $\alpha\beta$ 1

(iii) $(\alpha + 1)(\beta + 1)$ 1

(iv) $\alpha^2 + \beta^2$ 1

(c)

$time(min)$	0	1	2	3	4
$v(km/hr)$	0	25	34	30	40

The speed of a new Tangara train was recorded at intervals of one minute during peak hour. The times, in minutes, and the corresponding speeds v, in kilometres per hour, are listed in the table above.

(i) Explain carefully why the distance x kilometres travelled by the train in these four minutes is given by $x = \int_0^{\frac{1}{15}} v\,dt$. 2

(ii) Estimate x by using Simpson's rule with five function values. Give your answer correct to one decimal place. 3

QUESTION SIX (12 marks) Use a separate writing booklet. Marks

(a) A geometric series has first term 35 and common ratio 2^x.

(i) Find the values of x for which the series has a limiting sum. 1

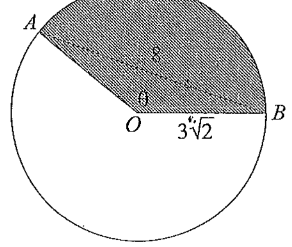
(ii) Find the value of x for which the limiting sum of the series is 40. 1

(b) (i) Write down the discriminant of $x^2 - (k + 5)x + 9$. 1

(ii) For what values of k does the equation $x^2 - (k + 5)x + 9 = 0$ have equal roots? 2

(iii) For what values of k is the expression $x^2 - (k + 5)x + 9$ positive definite? 2

(c)



The diagram above shows a circle with centre O and radius $3\sqrt{2}$ cm. A chord AB is 8 cm. Let $\angle AOB = \theta$.

(i) Show that $\cos\theta = -\dfrac{7}{9}$. 1

(ii) Hence find the exact value of $\sin\theta$. 1

(iii) Find the exact area of triangle AOB. 1

(iv) Find, correct to the nearest cm^2, the area of the shaded sector AOB. 2

QUESTION SEVEN (12 marks) Use a separate writing booklet. Marks

(a) (i) Simplify $\sec^2\theta - 1$. 1

(ii) Sketch the graph of $y = \tan 2x$, for $0 \le x \le \pi$. 2

(iii) Shade the region bounded by the x-axis, the curve and the line $x = \frac{\pi}{6}$. 1

(iv) Find the exact volume of the solid formed by rotating the shaded region about the x-axis. 4

(b) Gold is extracted from a mine in Kalgoorlie at a rate proportional to the amount of gold remaining in the mine. Hence the rate is given by

$$\frac{dG}{dt} = -kG$$

where G is the amount remaining after t years and k is a positive constant. After 20 years, 50% of the initial amount of gold remains.

(i) Show that $G = G_0 e^{-kt}$ is a solution to $\dfrac{dG}{dt} = -kG$, where G_0 is the initial amount of gold. 1

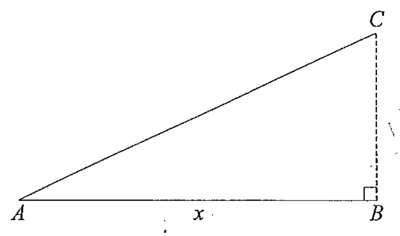
(ii) Find the exact value of k. 1

(iii) How many more years will elapse before only 20% of the original amount remains? Give your solution to the nearest year. 2

Exam continues overleaf ...

QUESTION EIGHT (12 marks) Use a separate writing booklet. Marks

(a)



A piece of wire of length 7 metres is bent to form the base and hypotenuse of a right-angled triangle ABC as shown in the diagram above. Let the length of the base AB be x metres.

(i) What is the length of the hypotenuse AC? 1

(ii) Show that the area of the triangle ABC is $\frac{1}{2}x\sqrt{49 - 14x}$. 1

(iii) What is the maximum possible area of the triangle? Give your answer to the nearest square metre. 4

(b) (i) If $y = \log_e\left(\dfrac{1+\cos x}{\sin x}\right)$, show that $\dfrac{dy}{dx} = -\operatorname{cosec} x$. 2

(ii) Hence evaluate $\displaystyle\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \operatorname{cosec} x\, dx$. 2

(c) (i) Show that $(n+1)^3 - n^3 = 3n^2 + 3n + 1$. 1

(ii) Hence evaluate $\displaystyle\sum_{1}^{10}(3n^2 + 3n + 1)$. 1

Exam continues next page ...

QUESTION NINE (12 marks) Use a separate writing booklet. **Marks**

(a) The acceleration of a particle after t seconds is given by $\ddot{x} = 6 - \dfrac{8}{(t+1)^2}$. [3]
The particle is initially at the origin with a velocity of 2 ms^{-1}. Find expressions for its velocity $\dot{x}$, and its displacement x.

(b) Consider the function $f(x) = xe^{-4x} + 1$.

(i) Show that $f''(x) = 16xe^{-4x} - 8e^{-4x}$. [1]

(ii) Find the value of x where $f(x)$ has a stationary point. [1]

(iii) Find the values of x where $f(x)$ is decreasing. [1]

(iv) Find any values of x where $f(x)$ has a point of inflexion. [2]

(v) Find where the graph of $f(x)$ is concave up. [1]

(vi) State the value of $\lim\limits_{x \to \infty} f(x)$. [1]

(vii) Sketch the curve $y = f(x)$, for $x \geq -1$. [2]

QUESTION TEN (12 marks) Use a separate writing booklet. **Marks**

(a) (i) Using the fact that $\sin x < x < \tan x$ for $0 < x < \frac{\pi}{2}$, explain why [1]

$$\int_0^{\frac{\pi}{6}} x^2 \sin x \, dx < \int_0^{\frac{\pi}{6}} x^3 \, dx < \int_0^{\frac{\pi}{6}} x^2 \tan x \, dx.$$

(ii) Hence show that $\displaystyle\int_0^{\frac{\pi}{6}} x^2 \sin x \, dx < \frac{\pi^4}{2^6 \times 3^4} < \int_0^{\frac{\pi}{6}} x^2 \tan x \, dx.$ [1]

(b) (i) Draw the graphs of $y = 4 \sin 2x$ and $y = 4 - 2x$, for $-\pi \leq x \leq \pi$, on the same set of axes. [2]

(ii) How many positive solutions are there to the equation $2 \sin 2x = 2 - x$? [1]

(c) Two particles P and Q are moving on the x-axis. The position of particle P at time t is given by

$$x_p = -2 \cos 2t$$

and the position of particle Q at time t is given by

$$x_q = -6 + 4t - t^2.$$

(i) Find expressions for the velocities of the two particles. [2]

(ii) Use part (b) to find the number of occasions when these two particles have the same velocity. [1]

(iii) Find the distance travelled by particle Q between $t = 0$ and $t = 3$. [2]

(iv) Show that the particles never meet. [2]

END OF EXAMINATION

Form V Maths Ext 1. 2009 Annual

Q1

(a) $9x^2-25=(3x-5)(3x+5)$ ✓

(b) $7-3x<-2$
$-3x<-9$
$x>3$ ✓

(c) $\frac{x}{3}-\frac{x-2}{5}$
$=\frac{5x-3x+6}{15}$ ✓ (lose ✓ for -6).
i.e $\frac{2x-6}{15}$
$=\frac{2x+6}{15}$ ✓

(d) $\frac{3}{\sqrt{5}-1}=\frac{3}{\sqrt{5}-1}\cdot\frac{\sqrt{5}+1}{\sqrt{5}+1}$ ✓
$=\frac{3(\sqrt{5}+1)}{4}$ ✓

(e) $\frac{3\pi}{5}$ rad $=\frac{3\pi}{5}\times\frac{180^\circ}{\pi}$
$=108^\circ$ ✓

(f) $f(x)=\int 3x+7\,dx$
$=\frac{3}{2}x^2+7x+c$ ✓ (don't worry about c)

(g) (i) $y=3\sin 2x$
$\frac{dy}{dx}=6\cos 2x$ ✓

(ii) $y=\sqrt{x}$
$=x^{\frac{1}{2}}$ ✓

$\frac{dy}{dx}=\frac{1}{2}x^{-\frac{1}{2}}$ ✓
$=\frac{1}{2\sqrt{x}}$

(h) $\int\frac{2}{x}\,dx=2\ln x+c$ ✓ (don't worry about c)

Q2.

(a) $y = \ln x$

$\frac{dy}{dx} = \frac{1}{x}$

$x = e, \quad m = \frac{1}{e}$

tangent is $y - 1 = \frac{1}{e}(x - e)$

$ey - e = x - e$

$ey = x \quad$ or $\quad y = \frac{1}{e}x$ or $\frac{x}{e}$

(b) $T_3 = a + 2d = 7$

$T_7 = a + 6d = 31$

(i) $T_7 - T_3 = 4d = 24$

$d = 6$

(ii) Find a.

$a + 12 = 7$

$a = -5$

$T_{18} = -5 + 17 \times 6$

$= 97$

(iii) $S_{20} = 10(-10 + 19 \times 6) = 1040$

(c) $y = x^3 - 3x^2 - 24x + 12$

(i) $\frac{dy}{dx} = 3x^2 - 6x - 24$

$\frac{d^2y}{dx^2} = 6x - 6$

(ii) At a stationary point $\frac{dy}{dx} = 0$

$3x^2 - 6x - 24 = 0$

$x^2 - 2x - 8 = 0$

$(x - 4)(x + 2) = 0$

$x = 4$ or -2

(iii) At a point of inflexion $\frac{d^2y}{dx^2} = 0$ and concavity changes

$\frac{d^2y}{dx^2} = 6x - 6 = 0$

$x = 1$

$y = -14$

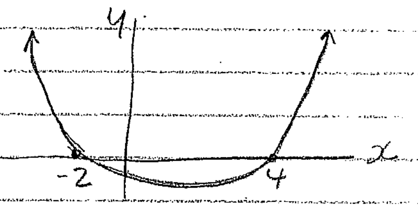
x	0	1	2
$\frac{d^2y}{dx^2}$	-6	0	6
	∩	–	∪

concavity changes

(iv) Need $3x^2 - 6x - 24 > 0$

$x^2 - 2x - 8 > 0$

$(x-4)(x+2) > 0$



$x < -2$ or $x > 4$

Q3

(a) gradient $= \frac{6-0}{1+1}$

$= 3$

QR is $y-6 = 3(x-1)$

$y-6 = 3x-3$

$3x - y + 3 = 0$

(b) gradient $= \frac{1-0}{-4+1} = -\frac{1}{3}$

(c) $m_{PR} \times m_{QR} = 3 \times -\frac{1}{3}$

$= -1$

so $PR \perp QR$

(d) $|PR| = \sqrt{(-4--1)^2 + (1-0)^2}$

$= \sqrt{1+9}$

$= \sqrt{10}$

(e) $\tan\theta = \frac{RQ}{RP}$ so we need RQ

$RQ = \sqrt{(1+1)^2 + (6)^2}$

$= \sqrt{4+36}$

$= \sqrt{40}$

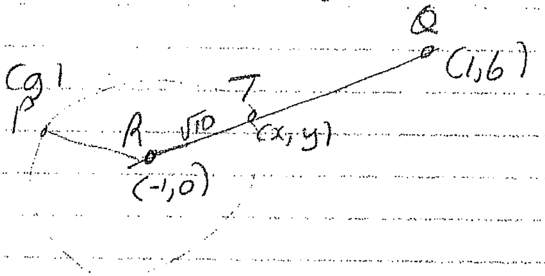
$= 2\sqrt{10}$

$\tan\theta = \frac{2\sqrt{10}}{\sqrt{10}} = 2$

(f) Circle has centre $(-1,0)$ & $r = \sqrt{10}$

$(x+1)^2 + y^2 = 10$

(g)



T is where the circle cuts RQ

$y = 3x+3$ and $(x+1)^2 + y^2 = 10$

$(x+1)^2 + (3x+3)^2 = 10$

$x^2 + 2x + 1 + 9x^2 + 18x + 9 = 10$

$10x^2 + 20x = 0$

$10x(x+2) = 0$

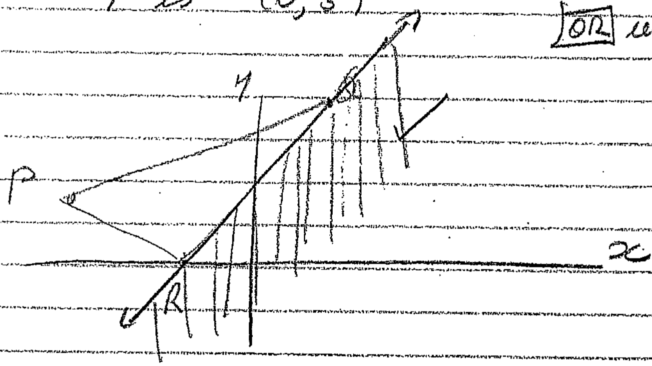
$x = 0$ or -2

But T lies between Q & R

so $x = 0$ and $y = 3$

T is $(0,3)$ $\boxed{\text{OR}}$ use rotation

(h)



$y - 3x \le 3$

$3x - y + 3 \ge 0$

test $(0,0)$

Q4.

a (i) $y = (3x^2-5)^6$

$\frac{dy}{dx} = 36x(3x^2-5)^5$

(ii) $y = e^x \cos x$

$\frac{dy}{dx} = -e^x \sin x + e^x \cos x$

$= e^x(\cos x - \sin x)$

✓ for an attempt to use product rule

(iii) $y = \frac{\tan x}{2x}$

$\frac{dy}{dx} = \frac{2x\sec^2 x - 2\tan x}{4x^2}$

$= \frac{x\sec^2 x - \tan x}{2x^2}$

✓ for one attempt at quot rule.

(b) $\int \frac{x}{x^2+3}\,dx = \frac{1}{2}\int \frac{2x}{x^2+3}\,dx$

$= \frac{1}{2}\ln(x^2+3) + c$

(c) (i) $\int_0^3 e^{2x}\,dx = \frac{1}{2}e^{2x}\Big]_0^3$

$= \frac{1}{2}[e^6 - e^0]$

✓ need to show $e^0=1$

$= \frac{e^6-1}{2}$

(ii) $\int_1^2 \frac{1}{x^3}\,dx = \int_1^2 x^{-3}\,dx$

$= \left[-\frac{1}{2}x^{-2}\right]_1^2 = \frac{3}{8}$

Q5.

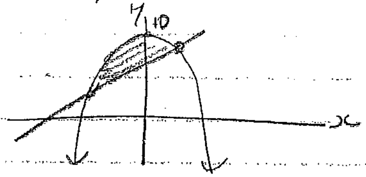
(a) (i) $x=-3$, $y = x+4 = 1$

$y = 10 - x^2 = 1$

$x = 2$, $y = x+4 = 6$

$y = 10-x^2 = 6$

(ii)



$A = \int_{-3}^{2} (10-x^2) - (x+4)\,dx$

$= \int_{-3}^{2} 10 - x^2 - x - 4\,dx$

$= \int_{-3}^{2} 6 - x^2 - x\,dx$

$= 6x - \frac{1}{3}x^3 - \frac{1}{2}x^2\Big]_{-3}^{2}$

$= \left(12 - \frac{8}{3} - 2\right) - \left(-18 + 9 - \frac{9}{2}\right)$

$= \frac{125}{6}$

(b) i) $\alpha + \beta = 5$

ii) $\alpha\beta = 2$

iii) $(\alpha+1)(\beta+1) = \alpha\beta + \alpha + \beta + 1$

$= \alpha\beta + (\alpha+\beta) + 1$

$= 2 + 5 + 1$

$= 8$

iv) $\alpha^2 + \beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$

$= 25 - 4 = 21$

c.

(i) $v = \frac{dx}{dt}$

so $x = \int_a^b v\,dt$

Since the initial time is 0, $a=0$.
The units for velocity are km/hr,
The final time is 4 mins or $\frac{1}{15}$ hours
so $b = \frac{1}{15}$

(ii) $x = \int_0^{1/15} v\,dt$, $h = \frac{1}{4}(\frac{2}{15} - 0) = \frac{2}{60} = \frac{1}{30}$

$\simeq \frac{1}{6} \times \frac{1}{30}(0 + 4(25) + 2(34) + 4(30) + 40)$

$\simeq \frac{1}{180}(328)$

$\simeq 1.8$ km

Q6

a) $T_1 = a = 35$, $r = 2^x$

i) Need $-1 < r < 1$

Now $2^x > 0$, so we want

$0 < 2^x < 1 = 2^0$

so $x < 0$

ii) $S_\infty = \frac{a}{1-r} = 40$

$\frac{35}{1-2^x} = 40$

$40 - 40 \times 2^x = 35$

$40 \times 2^x = 5$

$2^x = \frac{5}{40} = \frac{1}{8} = 2^{-3}$

$x = -3$

(b) (i) $(k+5)^2 - 36$

(ii) For equal roots $\Delta = 0$

$(k+5)^2 = 36$

$k+5 = 6$ or -6

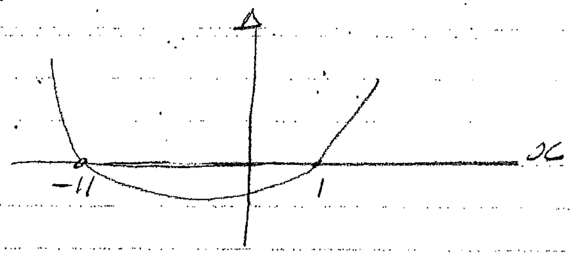
$k = 1$ or -11

(iii) Coeff of x^2 is positive so we need $\Delta < 0$

$(k+5)^2 - 36 < 0$

$(k+11)(k-1) < 0$

over



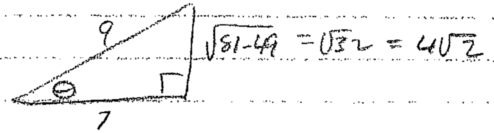
$x < -11$ or $x > 1$

C. (i) $8^2 = (3\sqrt{2})^2 + (3\sqrt{2})^2 - 2(3\sqrt{2})^2\cos\theta$

$\cos\theta = \dfrac{36 - 64}{36}$

$= \dfrac{-28}{36}$

$= \dfrac{-7}{9}$

(ii) 

$\sin\theta = \dfrac{4\sqrt{2}}{9}$

(iii) $A = \frac{1}{2} \times (3\sqrt{2})^2 \times \dfrac{4\sqrt{2}}{9}$

$= \frac{1}{\cancel{2}} \times \cancel{18}^{2} \times \dfrac{4\sqrt{2}}{\cancel{9}}$

$= 4\sqrt{2}$ cm^2

(iv) Now θ is obtuse and $\sin^{-1}\frac{4\sqrt{2}}{9} = 0.6792$

so $\theta = \pi - 0.6792$

$= 2.4619$

next page

Area $= \frac{1}{2}r^2\theta$

$= \frac{1}{2} \times (3\sqrt{2})^2 \times 2.4619$

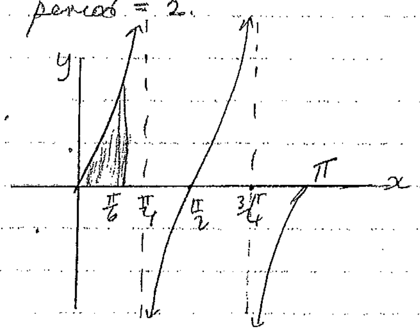
≈ 22 cm^2

Q7:

a) $\tan^2\theta$ ✓

b)

(i) period $= \frac{\pi}{2}$.



✓ for asymptotes

✓ period and slope

(ii) above ✓

(iii) $V = \pi\int_0^{\frac{\pi}{6}} y^2 \, dx$

$= \pi\int_0^{\frac{\pi}{6}} \tan^2 2x \, dx$ ✓

$= \pi\int_0^{\frac{\pi}{6}} (\sec^2 2x - 1) \, dx$ ✓

$= \pi\left[\frac{1}{2}\tan 2x - x\right]_0^{\frac{\pi}{6}}$ ✓

$= \pi\left[\left(\frac{1}{2}\tan\frac{\pi}{3} - \frac{\pi}{6}\right) - \left(\frac{1}{2}\tan 0 - 0\right)\right]$

$= \pi\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right) \quad u^3$ ✓

$= \frac{\pi}{6}\left(3\sqrt{3} - \pi\right)$

c.

(i) $R = R_0 e^{-kt}$

$\frac{dR}{dt} = -k R_0 e^{-kt}$ ✓

$= -kR$

(ii) $t = 20$, $\frac{1}{2} = e^{-20k}$

$\ln\frac{1}{2} = -20k$

$k = -\frac{\ln\frac{1}{2}}{20}$ ✓

$= \frac{\ln 2}{20}$

(iii) $\frac{1}{5} = e^{-kt}$

$\ln\frac{1}{5} = -kt$ ✓

$t = \ln\frac{1}{5} \div -\frac{\ln 2}{20}$

$= \ln\frac{1}{5} \times \frac{20}{-\ln 2}$

≈ 46 years ✓ mark here

so remaining time is 26 years

Q8.

(a) (i) $7-x$

(ii) $BC^2 = AC^2 - AB^2$, by Pythagoras
$= (7-x)^2 - x^2$
$= 49 - 14x + x^2 - x^2$
$BC = \sqrt{49-14x}$

area $\triangle ABC = \frac{1}{2} \times \sqrt{49-14x} \times x$
$= \frac{1}{2}x\sqrt{49-14x}$

(iii) $A = \frac{\sqrt{2}}{2}x\,(7-2x)^{\frac{1}{2}}$

$\frac{dA}{dx} = \frac{\sqrt{2}}{2}x\,\frac{1}{2}(-2)(7-2x)^{-\frac{1}{2}} + \frac{\sqrt{2}}{2}(7-2x)^{\frac{1}{2}} = 0$

at ~~stat~~ pt

$\frac{x}{\sqrt{7-2x}} = \sqrt{7-2x}$

$x = 7-2x$
$3x = 7$
$x = 7/3$

$\frac{dA}{dx} = \frac{\sqrt{2}}{2}\left(-x(7-2x)^{-\frac{1}{2}} + (7-2x)^{\frac{1}{2}}\right)$

x	2	7/3	3
$\frac{dA}{dx}$	3.81	0	−2.6

so we have a max at $x = 7/3$.

$x = \frac{7}{3}$, $A = \frac{\sqrt{2}}{2} \times \frac{7}{3} \times \left(7 - \frac{14}{3}\right)^{\frac{1}{2}}$
$\simeq 4.72$
$\simeq 5\ m^2$.

(b) (i) $y = \ln\left(\frac{1+\cos x}{\sin x}\right)$

$= \ln(1+\cos x) - \ln \sin x$ ✓

$\frac{dy}{dx} = \frac{-\sin x}{1+\cos x} - \frac{\cos x}{\sin x}$

$= \frac{-\sin^2 x - \cos x(1+\cos x)}{\sin x(1+\cos x)}$

$= \frac{-\sin^2 x - \cos x - \cos^2 x}{\sin x(1+\cos x)}$

$= \frac{-1-\cos x}{\sin x(1+\cos x)}$ ✓

$= \frac{-1}{\sin x}$

$= -\operatorname{cosec} x$

(ii) from (i) $\int -\operatorname{cosec} x \, dx = \ln\left(\frac{1+\cos x}{\sin x}\right)$

so $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \operatorname{cosec} x \, dx = \ln\left(\frac{\sin x}{1+\cos x}\right)\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$ ✓

or $-\ln \frac{1+\cos x}{\sin x}\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$

$= \ln\frac{1}{1} - \ln\frac{\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}}$

$= 0 - \ln(\sqrt{2}-1)$ ✓

$= -\ln(\sqrt{2}-1) = (0.8813)$

$= \ln(\sqrt{2}+1)$

$= \frac{\frac{1}{\sqrt{2}}}{1+\frac{1}{\sqrt{2}}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \frac{1}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}$

$= \frac{\sqrt{2}-1}{1}$

$\frac{1}{(\sqrt{2}-1)} \times \frac{\sqrt{2}+1}{\sqrt{2}+1}$

c (i) $(n+1)^3 - n^3 = n^3 + 3n^2 + 3n + 1 - n^3$ ✓

$= 3n^2 + 3n + 1$

(ii) $\sum_1^{10} 3n^3 + 3n + 1 = \sum_1^{10} (n+1)^3 - n^3$

$= (2^3 - 1^3) + (3^3 - 2^3) + \dots + (11^3 - 10^3)$ ✓

$= 11^3 - 1$

Q9.

(a) $\ddot{x} = 6 - \frac{8}{(t+1)^2}$, $\boxed{t=0,\ v=2\ \ x=0}$

$\dot{x} = \int 6 - 8(t+1)^{-2}\,dt$

$= 6t + 8(t+1)^{-1} + c_1$

$t=0$, $v=2$ so $c_1 = -6$.

$\dot{x} = 6t + \frac{8}{t+1} - 6$

$x = \int 6t + \frac{8}{t+1} - 6 \; dt$

$= 3t^2 + 8\ln(t+1) - 6t + c_2$

$t=0$, $x=0$ so $0 = 0 + 8\ln 1 + 0 + c_2$

so $c_2 = 0$

and $x = 3t^2 + 8\ln(t+1) - 6t$

(only lose 1 mk if they forget to calculate either or both c_i).

(b) $f(x) = xe^{-4x} + 1$

(i) $f'(x) = -4e^{-4x}x + e^{-4x}$

$f''(x) = -4e^{-4x} + 16xe^{-4x} - 4e^{-4x}$

$= 16xe^{-4x} - 8e^{-4x}$

(ii) $e^{-4x} - 4e^{-4x}x = 0$

$e^{-4x}(1-4x) = 0$

$x = \frac{1}{4}$

(iii) $-4e^{-4x}x + e^{-4x} < 0$

$e^{-4x}(-4x+1) < 0$

now $e^{-4x} > 0$ for all x

so look at $1 - 4x < 0$

$-4x < -1$

$4x > 1$

$x > \frac{1}{4}$

(iv) $16xe^{-4x} - 8e^{-4x} = 0$

$8e^{-4x}(2x-1) = 0$

$x = \frac{1}{2}$

x	0	$\frac{1}{2}$	1
$f''(x)$	-8	0	$8e^{-4}\times 1 > 0$
	$\cap$		$\cup$

at $x = \frac{1}{2}$, $f''(x) = 0$ and concavity changes so we have a point of inflexion v)

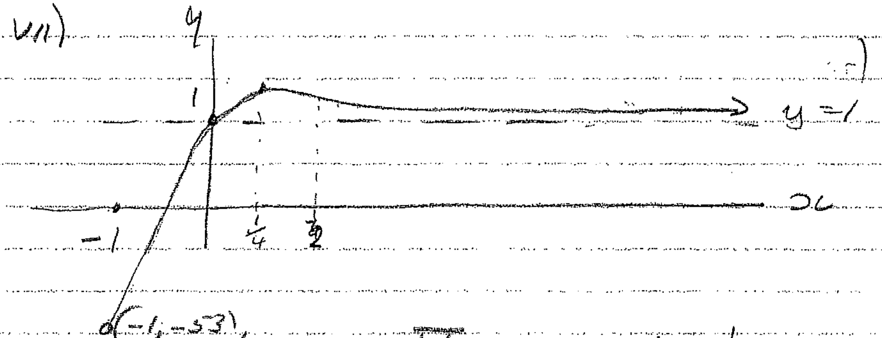
$$\delta e^{-4x}(2x-1) > 0$$
$$2x-1 > 0$$
$$x > \tfrac{1}{2}$$

vi)

$$\lim_{x\to\infty} e^{-4x} = 0$$

So $\lim_{x\to\infty}(xe^{-4x}+1) = 1$

vii)



for asymptote y = 1, max at $x = \tfrac{1}{2}$, ~~infl at x = 1/4~~

for general shape y = 1 intercept

(don't worry about end pt - they were just there to help)

Q10

a)

(i) $\sin x < x < \tan x$ for $0 < x < \frac{\pi}{2}$

so $x^2 \sin x < x^3 < x^2 \tan x$,

$x^2 > 0$ and all functions here are increasing on the interval

so $\int_0^{\frac{\pi}{6}} x^2 \sin x \, dx < \int_0^{\frac{\pi}{6}} x^3 \, dx < \int_0^{\frac{\pi}{6}} x^2 \tan x \, dx$

(ii) now $\int_0^{\frac{\pi}{6}} x^3 \, dx = \left.\frac{x^4}{4}\right]_0^{\frac{\pi}{6}}$

$$= \frac{\left(\frac{\pi}{6}\right)^4}{4}$$

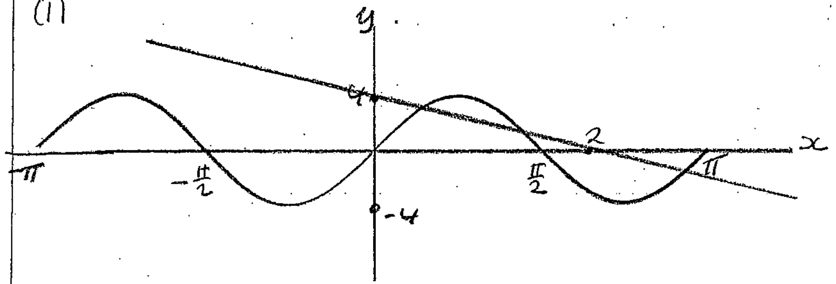
$$= \frac{\pi^4}{2^2 \times 6^4}$$

$$= \frac{\pi^4}{2^6 \times 3^4}$$

so we get the result.

(b)

(i)



✓ for sine curve, amplitude, period

✓ for line (0,4) & (2,0).

(ii) $4\sin 2x = 4 - 2x$ ✓

$2\sin 2x = 2 - x$ has 3 solutions

c)

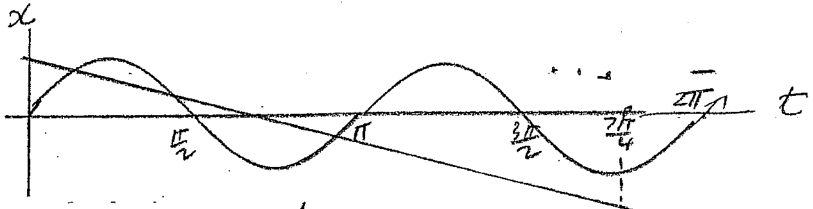
(i) P: $x = -2\cos 2t$

$v = \dot{x} = 4\sin 2t$ ✓

Q: $x = -6 + 4t - t^2$

$v = \dot{x} = 4 - 2t$ ✓

(ii)



From (ii) they have the same velocity on 3 occasions for $0 < t < \pi$.

Now $t = \frac{7\pi}{4}$,

Now for $t = \frac{7\pi}{4}$, $\dot{x}_P = 4\sin\frac{7\pi}{2}$

$= -4$.

$\dot{x}_Q = 4 - \frac{7\pi}{2}$

$= -6.99$

So the line is below the curve at $t = \frac{7\pi}{4}$

So we have only 3 pts of intersection ✓

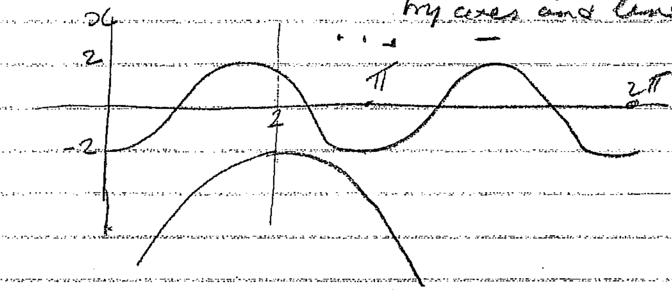
(iii) $x = \int_0^2 (4-2t)\,dt + \left|\int_2^3 (4-2t)\,dt\right|$ ✓

$= [4t - t^2]_0^2 + \left|[4t - t^2]_2^3\right|$

$= (8-4) + |(3-4)|$ ✓

$= 5$ OR area of 2 triangles formed by axes and line.

(iv)



Q: $x = -6 + 4t - t^2$

$= -2 - (t-2)^2$

So the vertex of x_Q is $(2,-2)$

The maximum value is -2 at $t=2$.

The minimum value of x_P is -2
but this does not occur at $x=2$.
So they never meet.