SYDNEY GRAMMAR SCHOOL

2010 Assessment Examination

FORM VI
MATHEMATICS EXTENSION 1

Thursday 4th February 2010

General Instructions
- Writing time — Period 7.
- Write using black or blue pen.
- Board-approved calculators may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question on a new leaflet.

Structure of the paper
- Total marks — 36
- All three questions may be attempted.
- All three questions are of equal value.

Collection
- Write your name, class and master clearly on each leaflet.
- Hand in the three questions in a single well-ordered pile.
- Hand in a leaflet for each question, even if it has not been attempted.
- If you use a second leaflet for a question, place it inside the first.
- Write your name on the question paper and place it inside your leaflet for Question One.

6A: REP	6B: PKH	6C: BDD
6D: FMW	6E: KWM	6F: MLS
6G: SJE	6H: LYL	6I: RCF

Checklist
- Writing leaflets: 3 per boy.
- Candidature — 133 boys

Examiner
RCF

The following list of standard integrals may be used:

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\quad n \neq -1;\quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x,\quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax,\quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\quad a > 0,\quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\ x > 0$

QUESTION ONE (12 marks) Use a separate writing booklet. Marks

(a) (i) Expand $\cos(A - B)$. 1

(ii) Hence find the exact value of $\cos 15°$. 2

(b) Solve $\dfrac{3}{x+1} \leq 1$. 3

(c) Find the acute angle between the lines $y = 4x + 5$ and $6x + 3y = 7$. Give your answer correct to the nearest minute. 3

(d) Find $\lim\limits_{x \to 0} \dfrac{\sin 3x}{x}$. Show your working clearly. 1

(e) Let A be the point $(-5, 6)$ and B be the point $(0, -4)$. Find the co-ordinates of the point P which divides the interval AB internally in the ratio $3 : 2$. 2

QUESTION TWO (12 marks) Use a separate writing booklet. Marks

(a) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} 2\cos^2 x \, dx$. 3

(b) A spherical snowball is melting at a rate of $10\,\text{cm}^3/\text{min}$, whilst retaining its shape. [*The volume of a sphere is* $V = \frac{4}{3}\pi r^3$. *The surface area is* $S = 4\pi r^2$.]

(i) Find the rate at which the radius is decreasing at the instant when the radius is 4 cm. 3

(ii) What is the rate of change of the snowball's surface area at the same instant? 2

(c) Use mathematical induction to prove that 4

$$1 + r + r^2 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r}$$

for all positive integer values of n.

Exam continues next page ...

QUESTION THREE (12 marks) Use a separate writing booklet. Marks

(a) Consider the rational function $f(x) = \dfrac{4x^2 - 1}{x^2 - 9}$ and its graph.

(i) State the domain of the function. 1

(ii) Test the function for odd or even symmetry. 1

(iii) Find the co-ordinates of any intercepts with the axes. 2

(iv) Find the equations of any vertical or horizontal asymptotes of the graph. 2

(v) Sketch the graph of the function, indicating all the information detailed above. 2

(vi) State the range of the function. 1

(b) (i) Prove the identity 2

$$\tan\left(\tfrac{\pi}{4} + \alpha\right) - \tan\left(\tfrac{\pi}{4} - \alpha\right) = 2\tan 2\alpha$$

(ii) Hence or otherwise solve, for $0 \leq \alpha \leq 2\pi$, 1

$$\tan\left(\tfrac{\pi}{4} + \alpha\right) - \cot\left(\tfrac{\pi}{4} + \alpha\right) = 2\sqrt{3}.$$

END OF EXAMINATION

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1) a) (i) $\cos(A-B) = \cos A\cos B + \sin A \sin B$ ✓

(ii) $\cos 15°$ Let $A = 45°$ $B = 30°$

$\cos(45°-30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$

$= \frac{1}{\sqrt{2}} \times \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \times \frac{1}{2}$ ✓

$= \frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$ ✓

b) $\frac{3}{x+1} \leqslant 1 \quad (x \neq -1)$

$\times (x+1)^2 \quad \times (x+1)^2$ ✓

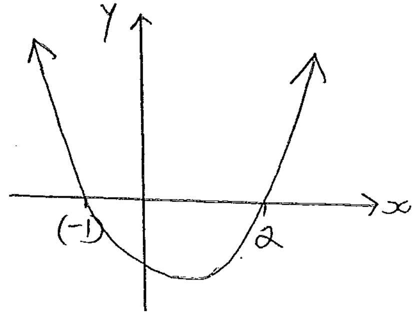
$3(x+1) \leqslant (x+1)^2$

$3x+3 \leqslant x^2+2x+1$

$0 \leqslant x^2 - x - 2$

$0 \leqslant (x-2)(x+1)$ ✓

$x \geqslant 2$ OR $x < (-1)$ ✓ (−1 mark if $x \leqslant -1$)



c) $l_1: y = 4x+5 \quad m_1 = 4$

$l_2: 6x+3y = 7$

$y = \frac{7-6x}{3} \quad m_2 = (-2)$ ✓

$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right| = \left|\frac{4-(-2)}{1+4(-2)}\right|$ ✓

$= 6/7$

$\theta \doteqdot 40°36'$ ✓ (do not penalise rounding)

d) $\lim_{x\to 0}\left(\frac{\sin 3x}{x}\right) = \lim_{x\to 0}\left(3 \times \frac{\sin 3x}{3x}\right)$

$= 3 \times 1$

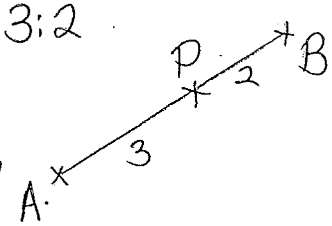
$= 3$ ✓ (must show working for mark)

e) $A(-5,6)$ $B(0,-4)$ Internal $3:2$

$P\left(\frac{kx_2+lx_1}{k+l}, \frac{ky_2+ly_1}{k+l}\right)$

$= P\left(\frac{3\times 0 + 2\times(-5)}{3+2}, \frac{3\times(-4)+2\times 6}{3+2}\right)$ ✓

$= P(-2, 0)$ ✓



② a) $\int_0^{\pi/2} 2\cos^2 x \, dx = \int_0^{\pi/2} 1+\cos 2x \, dx$ ✓

$= \left[x + \frac{1}{2}\sin 2x\right]_0^{\pi/2}$ ✓

$= \frac{\pi}{2} + \frac{1}{2}\sin\pi - 0$

$= \frac{\pi}{2}$ ✓

b) i) $\frac{dV}{dt} = -10 \text{ cm}^3/\text{min}$ $V = \frac{4}{3}\pi r^3$ $\therefore \frac{dV}{dr} = 4\pi r^2$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$ ✓

$-10 = 4\pi r^2 \times \frac{dr}{dt}$

$\frac{dr}{dt} = \frac{-5}{2\pi r^2}$ ✓

When $r = 4$cm $\frac{dr}{dt} = \frac{-5}{2\pi \times 4^2} = \frac{-5}{32\pi}$

hence radius is decreasing at a rate of $\frac{5}{32\pi}$ ✓ cm/min

(ii.) Surface Area $S = 4\pi r^2$ $\frac{dS}{dr} = 8\pi r$

$\frac{dS}{dt} = \frac{dS}{dr} \times \frac{dr}{dt}$

$= 8\pi r \times \left(\frac{-5}{2\pi r^2}\right)$ ✓

$= \frac{-20}{r}$ ✓

When $r = 4$cm $\frac{dS}{dt} = \frac{-20}{4} = -5 \text{ cm}^2/\text{min}$ ✓ (Penalise no units here)

c) $1 + r + r^2 + \ldots + r^{n-1} = \frac{1-r^n}{1-r}$

Test $n=1$

LHS $= r^0 = 1$

RHS $= \frac{1-r^1}{1-r} = 1$ $\therefore$ Conjecture true for $n=1$ ✓

Assume true for $n=k$ ie $1 + r + r^2 + \ldots + r^{k-1} = \frac{1-r^k}{1-r}$ ✓

R.T.P. for $n=k+1$ $1 + r + r^2 + \ldots + r^{k-1} + r^k = \frac{1-r^{k+1}}{1-r}$

LHS $= \frac{1-r^k}{1-r} + r^k$ ✓

$= \frac{1-r^k}{1-r} + \frac{r^k(1-r)}{(1-r)}$ ✓

$= \frac{1-r^k+r^k-r^{k+1}}{1-r}$

$= \frac{1-r^{k+1}}{1-r}$

$=$ RHS $\therefore$ If true for $n=k$, result also true for $n=k+1$

By mathematical induction the original conjecture is true for all positive integer n. ✓

3 a) $f(x)=\dfrac{4x^2-1}{x^2-9}$

$=\dfrac{(2x+1)(2x-1)}{(x+3)(x-3)}$

(i) $x\in\mathbb{R},\ x\neq\pm3$ ✓

(ii) $f(-x)=\dfrac{4(-x)^2-1}{(-x)^2-9}$

$=\dfrac{4x^2-1}{x^2-9}$ ✓

$=f(x)$ $\therefore$ Even function

(iii) $x=0$ $f(0)=\dfrac{-1}{-9}=\frac{1}{9}$ $\therefore (0,\frac{1}{9})$ ✓

$f(x)=0 \Rightarrow 4x^2-1=0$

$x=\pm\frac{1}{2}$ $\therefore(\frac{1}{2},0)\ (-\frac{1}{2},0)$

intercepts are $(0,\frac{1}{9}),\ (\frac{1}{2},0)$ & $(-\frac{1}{2},0)$

(iv) Vertical Asymptotes $x=3$ and $x=(-3)$ ✓

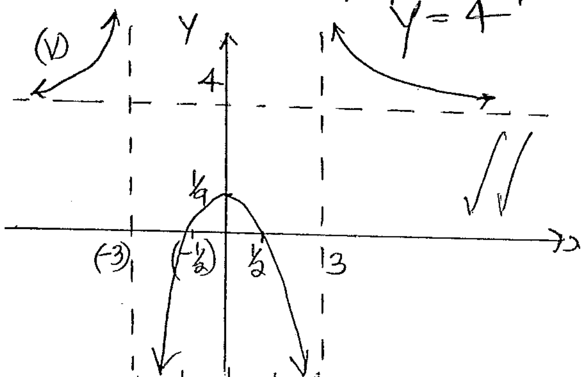
$f(x)=\dfrac{4-\frac{1}{x^2}}{1-\frac{9}{x^2}}$ as $x\to\pm\infty$

$f(x)\to4$

$\therefore$ Horizontal Asymptote

$y=4$ ✓

(v)



x	-4	-3	-2	$-\frac{1}{2}$	$-\frac{1}{4}$
$f(x)$	$\frac{63}{7}$	*	$\frac{15}{-5}$	0	$\frac{3/4}{-8\,15/16}$
	+		−		+

Then symmetric in y axis.

(vi) Range $f(x)\leq\frac{1}{9}$ or $f(x)>$ ✓

b)(i) $\tan\left(\frac{\pi}{4}+\alpha\right)-\tan\left(\frac{\pi}{4}-\alpha\right)=2\tan2\alpha$

LHS $=\tan\left(\frac{\pi}{4}+\alpha\right)-\tan\left(\frac{\pi}{4}-\alpha\right)$

$=\left(\dfrac{\tan\frac{\pi}{4}+\tan\alpha}{1-\tan\frac{\pi}{4}\tan\alpha}\right)-\left(\dfrac{\tan\frac{\pi}{4}-\tan\alpha}{1+\tan\frac{\pi}{4}\tan\alpha}\right)$

$=\dfrac{1+\tan\alpha}{1-\tan\alpha}-\left(\dfrac{1-\tan\alpha}{1+\tan\alpha}\right)$ ✓

$=\dfrac{(1+\tan\alpha)^2-(1-\tan\alpha)^2}{1-\tan^2\alpha}$

$=\dfrac{2\tan\alpha+2\tan\alpha}{1-\tan^2\alpha}$ ✓

$=2\left(\dfrac{\tan\alpha+\tan\alpha}{1-\tan\alpha\tan\alpha}\right)$

$=2\tan2\alpha$

$=$ RHS.

(ii) Hence $\tan\left(\frac{\pi}{4}+\alpha\right)-\cot\left(\frac{\pi}{4}+\alpha\right)=2\tan2\alpha$

since $\cot\theta=\tan\left(\frac{\pi}{2}-\theta\right)$

LHS $=\tan\left(\frac{\pi}{4}+\alpha\right)-\tan\left(\frac{\pi}{4}-\alpha\right)$

$=2\tan2\alpha$

$\therefore 2\tan2\alpha=2\sqrt{3}$ $\quad 0^\circ\leq\alpha\leq360^\circ$ (2π)

$\tan2\alpha=\sqrt{3}$ $\quad 0^\circ\leq2\alpha\leq4\pi$

$2\alpha=\frac{\pi}{3},\frac{4\pi}{3},\frac{7\pi}{3},\frac{10\pi}{3}$ ✓

$\alpha=\frac{\pi}{6},\frac{2\pi}{3},\frac{7\pi}{6},\frac{5\pi}{3}$ ✓