SYDNEY GRAMMAR SCHOOL

2010 Assessment Examination

FORM VI
MATHEMATICS 2 UNIT

Wednesday 3rd February 2010

General Instructions

- Writing time — Period 3.
- Write using black or blue pen.
- Board-approved calculators may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question on a new leaflet.

Structure of the paper

- Total marks — 36
- All three questions may be attempted.
- All three questions are of equal value.

Collection

- Write your name, class and master clearly on each leaflet.
- Hand in the three questions in a single well-ordered pile.
- Hand in a leaflet for each question, even if it has not been attempted.
- If you use a second leaflet for a question, place it inside the first.
- Write your name on the question paper and place it inside your leaflet for Question One.

6G: SJE	6H: LYL	6I: RCF
6P: SO	6Q: MW	6R: DS

Checklist

- Writing leaflets: 3 per boy.
- Candidature — 84 boys

Examiner
DS

QUESTION ONE (12 marks) Use a separate writing booklet.

Marks

(a) Differentiate:

(i) $y = x^4 + 4$ [1]

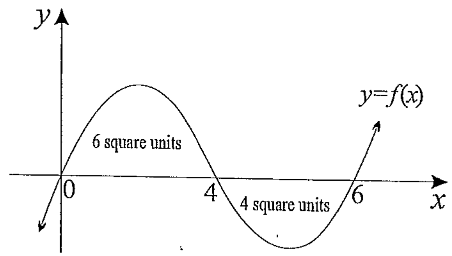
(ii) $y = (2x-1)^5$ [1]

(b) Find:

(i) $\int (x^4+4)\,dx$ [1]

(ii) $\int (2x-1)^5\,dx$ [1]

(c)



The diagram above shows the curve $y = f(x)$, and the areas of the two regions bounded by the curve and the x-axis.

Write down the value of:

(i) $\int_4^6 f(x)\,dx$ [1]

(ii) $\int_0^6 f(x)\,dx$ [1]

(d) Evaluate $\int_2^3 x(4-x)\,dx$. [2]

(e) The curve $y = f(x)$ has gradient function $\dfrac{dy}{dx} = 3x^2 - 6x + 4$, and it is known that the curve passes through the point $(0,2)$. Find the equation of the curve. [2]

(f) The curve $y = x^3 + kx$ has a stationary point at $x = -2$. Find the value of k. [2]

Exam continues next page ...

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) (i) Find the derivative of the function $y = \dfrac{1}{x}$. [1]

(ii) Hence explain why the function $y = \dfrac{1}{x}$ is decreasing throughout its domain. [1]

(b) Consider the curve with equation $y = x^4 - 18x^2 + 45$.

(i) Show that $y' = 4x(x+3)(x-3)$ and that $y'' = 12(x+\sqrt{3})(x-\sqrt{3})$. [2]

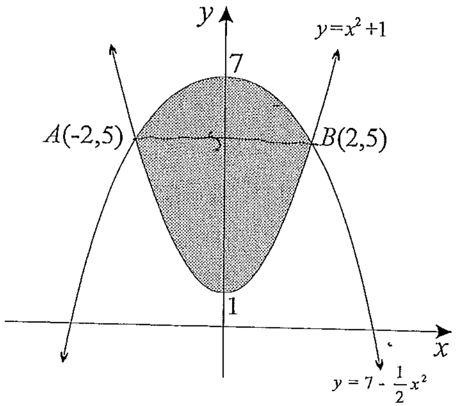
(ii) Find the three stationary points and classify them. [4]

(iii) Find any points of inflexion. [3]

(iv) Sketch the curve using the above information. [1]

QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

(a)

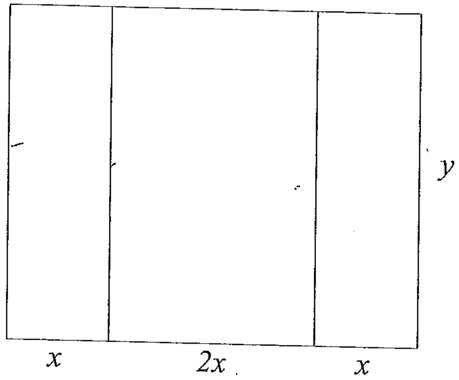


The diagram above shows the parabolas $y = x^2 + 1$ and $y = 7 - \frac{1}{2}x^2$. Their points of intersection are $A(-2,5)$ and $B(2,5)$. Find the area of the shaded region bounded by the two parabolas. [3]

Exam continues overleaf ...

QUESTION THREE (Continued)

(b)



Farmer Brown wishes to fence-off three rectangular paddocks, as shown in the diagram above. The combined area of the three paddocks is to be 45 000 square metres, and the larger paddock is to be twice as wide as the smaller paddocks.
Let y metres be the length of each paddock, and let x metres be the width of each of the smaller paddocks.

(i) Show that the required length, L metres, of fencing is given by the formula [2]

$$L = 8x + \frac{45\,000}{x}.$$

(ii) Hence use calculus to find the minimum length of fencing that the farmer must purchase in order to fence-off the three paddocks. [4]

(c) Find $\displaystyle\int \frac{x^2 - 2}{\sqrt{x}}\,dx$. [3]

END OF EXAMINATION

2 Unit Assessment, February 2010

TOTAL 36

(1)(a)(i) $y = x^4 + 4$

$y' = 4x^3$ ✓

(ii) $y = (2x-1)^5$

$y' = 10(2x-1)^4$ ✓

(b)(i) $\int (x^4+4)\,dx = \frac{x^5}{5} + 4x + c$ ✓ (no penalty for omission of c)

(ii) $\int (2x-1)^5\,dx = \frac{(2x-1)^6}{12} + c$ ✓

(c)(i) $\int_4^6 f(x)\,dx = -4$ ✓

(ii) $\int_0^6 f(x)\,dx = 6 - 4$

$= 2$ ✓

(d) $\int_2^3 x(4-x)\,dx = \int_2^3 (4x - x^2)\,dx$

$= \left[2x^2 - \frac{x^3}{3}\right]_2^3$ ✓

$= (18 - 9) - \left(8 - \frac{8}{3}\right)$

$= 1 + 2\frac{2}{3}$

$= 3\frac{2}{3}$ ✓

(e) $\frac{dy}{dx} = 3x^2 - 6x + 4$

$\therefore y = x^3 - 3x^2 + 4x + c$ ✓

When $x = 0$, $y = 2$.

$\therefore c = 2$ ✓

So the curve has equation $y = x^3 - 3x^2 + 4x + 2$.

(f) $y = x^3 + kx$

$y' = 3x^2 + k$ ✓

When $x = -2$, $y' = 0$.

$\therefore 0 = 12 + k$

$\therefore k = -12$ ✓

(2)(a)(i) $y = x$

$\therefore y' = -x^{-2}$

$= -\dfrac{1}{x^2}$

(ii) $-\dfrac{1}{x^2} < 0$ for all non-zero values of x,

so the {function / curve} is decreasing throughout its domain.

(b)(i) $y = x^4 - 18x^2 + 45$

$\therefore y' = 4x^3 - 36x$

$= 4x(x^2 - 9)$

$= 4x(x+3)(x-3)$

and

$y'' = 12x^2 - 36$

$= 12(x^2 - 3)$

$= 12(x + \sqrt{3})(x - \sqrt{3})$

(ii) Let $y' = 0$ for stationary points.

$\therefore 4x(x+3)(x-3) = 0$

$\therefore x = -3, 0$ or 3

When $x = -3$ or 3,

$y = 81 - 162 + 45$

$= -36$.

When $x = 0$,

$y = 45$.

So $(-3,-36)$, $(0,45)$ and $(3,-36)$ are stationary points.

x	-3	0	3
y''	72	-36	72
	↑	↓	↑

$(\pm 3, -36)$ are minimum turning points, and $(0,45)$ is a maximum turning point.

(iii) Let $y'' = 0$ for possible points of inflexion.

$\therefore 12(x+\sqrt{3})(x-\sqrt{3}) = 0$

$\therefore x = -\sqrt{3}$ or $\sqrt{3}$

When $x = \pm\sqrt{3}$,

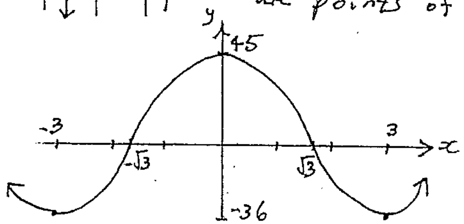
$y = 9 - 54 + 45$

$= 0$

x	-2	$-\sqrt{3}$	0	$\sqrt{3}$	2
y''	12	0	-36	0	12
	↑		↓		↑

Concavity changes, so $(-\sqrt{3}, 0)$ and $(\sqrt{3}, 0)$ are points of inflexion.

(iv)



(3)(a) Area $= 2\int_0^2 \left(7 - \frac{1}{2}x^2 - (x^2+1)\right) dx$

$= 2\int_0^2 \left(6 - \frac{3}{2}x^2\right) dx$

$= 2\left[6x - \frac{x^3}{2}\right]_0^2$

$= 2(12 - 4)$

$= 16 \ u^2$

(b)(i) $45000 = $ area

$\therefore 45000 = 4xy$

$\therefore y = \dfrac{11250}{x}$ (1)

$L = 8x + 4y$

$= 8x + \dfrac{4(11250)}{x}$

$= 8x + \dfrac{45000}{x}$, as required.

(ii) $L = 8x + 45000x^{-1}$

$\therefore \dfrac{dL}{dx} = 8 - 45000x^{-2}$

$= 8 - \dfrac{45000}{x^2}$

Let $\dfrac{dL}{dx} = 0$.

$\therefore 8 - \dfrac{45000}{x^2} = 0$

$x^2 = 5625$

$x = 75 \quad (x > 0)$

$\dfrac{d^2L}{dx^2} = \dfrac{90000}{x^3} > 0$ when $x = 75$,

so $x = 75$ minimises L.

So the minimum length of fencing is

$8(75) + \dfrac{45000}{75} = 1200$ m.

(c) $\displaystyle\int \frac{x^2 - 2}{x^{\frac{1}{2}}}\, dx = \int \left(x^{\frac{3}{2}} - 2x^{-\frac{1}{2}}\right) dx$

$= \frac{2}{5}x^{\frac{5}{2}} - 4x^{\frac{1}{2}} + c$