SYDNEY GRAMMAR SCHOOL

2009 Half-Yearly Examination

FORM VI
MATHEMATICS EXTENSION 1

Monday 16th March 2009

General Instructions

- Writing time — 2 Hours
- Write using black or blue pen.
- Board-approved calculators may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question in a new book.

Structure of the paper

- Total marks — 84
- All seven questions may be attempted.
- All seven questions are of equal value.

Collection

- Write your candidate number clearly on each booklet.
- Hand in the seven questions in a single well-ordered pile.
- Hand in a booklet for each question, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- The question papers will be collected separately.

Checklist

- SGS booklets — 7 per boy
- Candidature — 111 boys

Examiner
FMW

QUESTION ONE (12 marks) Use a separate writing booklet. Marks

(a) Evaluate:

(i) $\cos^{-1}(\frac{1}{\sqrt{2}})$ 1

(ii) $e^{\log_e 4}$ 1

(b) Differentiate with respect to x:

(i) e^{3x^2} 1

(ii) $\tan(\frac{1}{4}x)$ 1

(iii) $\sin^{-1}(2x)$ 2

(c) Find:

(i) $\int \cos(2-3x)\,dx$ 1

(ii) $\int \frac{x}{x^2-3}\,dx$ 1

(iii) $\int \frac{1}{e^{2x}}\,dx$ 1

(d) Find the Cartesian equation of the curve with parametric equations $x = 6t$, $y = 10t^2$. 2

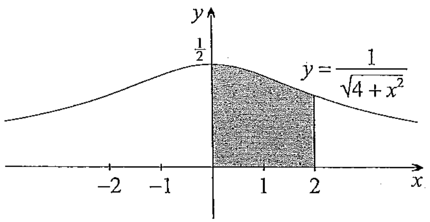
(e) Without evaluating the integral, explain why $\int_{-1}^{1} \tan^{-1} x\,dx$ is equal to zero. 1

Exam continues next page ...

QUESTION TWO (12 marks) Use a separate writing booklet. **Marks**

(a) Evaluate $\sin^{-1}(\sin \frac{2\pi}{3})$. [2]

(b) [3]



The diagram above shows the region bounded by the curve $y = \dfrac{1}{\sqrt{4+x^2}}$, the x-axis, the y-axis and the line $x = 2$. Find the exact volume of the solid generated when the shaded region is rotated about the x-axis.

(c) Write down the general solution of $\cos x = \frac{\sqrt{3}}{2}$. [2]

(d) Let $f(x) = 3\sin^{-1} x$.

(i) Sketch the graph of $y = f(x)$, indicating clearly the coordinates of the endpoints. [2]

(ii) Find the gradient of the tangent to the curve $y = f(x)$ at $x = \frac{1}{3}$. [2]

(iii) Write down the equation of the image of $y = f(x)$ after it has been reflected in the y-axis. [1]

Exam continues overleaf ...

QUESTION THREE (12 marks) Use a separate writing booklet. **Marks**

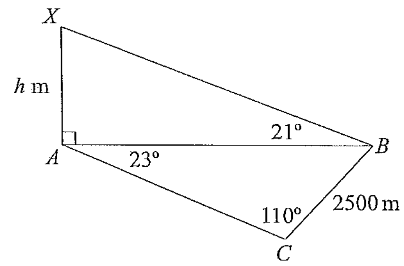
(a) Find, to the nearest minute, the acute angle between the lines $2x + y - 2 = 0$ and $x - y - 4 = 0$. [2]

(b) (i) Write $\sin x \cos x$ in terms of $\sin 2x$. [1]

(ii) Hence sketch the graph of $y = \sin x \cos x$ for $0 \le x \le 2\pi$, clearly showing the x-intercepts and the range of the function. [2]

(iii) Use your graph to write down the number of solutions of the equation $\sin x \cos x = \frac{1}{2}x$ in the domain $0 \le x \le 2\pi$. [1]

(c)



In the diagram above the points A, B and C lie in a horizontal plane. From B, the angle of elevation of the top of the tower AX, of height h metres, is 21°. The distance BC is 2500 m, $\angle BAC = 23°$ and $\angle ACB = 110°$.

(i) Show that $h = \dfrac{2500 \sin 110° \tan 21°}{\sin 23°}$. [2]

(ii) Find the height of the tower, correct to the nearest metre. [1]

(d) An experiment is set up to grow a sample of bacteria in a petri dish. The number of bacteria, P, in the sample is increasing exponentially with time according to the equation $P = 200e^{\frac{4}{3}t}$, where t is measured in hours from the start of the experiment.

(i) How long will it take for there to be 18 000 bacteria in the sample? Answer in hours correct to 2 decimal places. [2]

(ii) At what rate is the number of bacteria increasing after 4 hours? [1]

Exam continues next page ...

QUESTION FOUR (12 marks) Use a separate writing booklet. **Marks**

(a) Differentiate with respect to x:

(i) $y = \dfrac{1}{1+x^2}$ $\boxed{1}$

(ii) $y = \tan^{-1}\frac{1}{x}$ $\boxed{2}$

(b) Find:

(i) $\displaystyle\int \cos^2 x \, dx$ $\boxed{2}$

(ii) $\displaystyle\int \frac{1}{\sqrt{9-4x^2}} \, dx$ $\boxed{2}$

(c) (i) Sketch the graph of $f(x) = \ln x - 1$. (Use the same scale on both axes.) $\boxed{2}$

(ii) Sketch the graph of $f^{-1}(x)$ on the same diagram. $\boxed{1}$

(iii) Find an expression for $f^{-1}(x)$. $\boxed{2}$

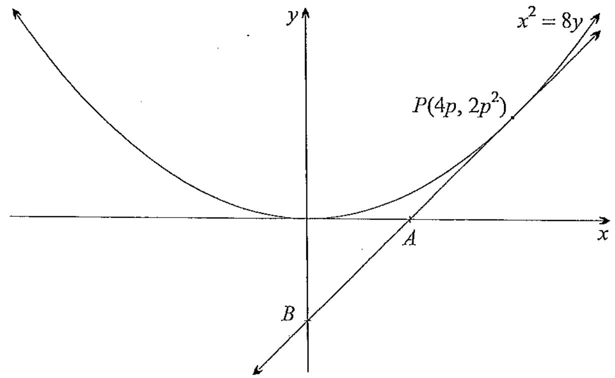
Exam continues overleaf ...

QUESTION FIVE (12 marks) Use a separate writing booklet. **Marks**

(a) Solve $\cos 2\theta = \cos\theta$ for $0 \le \theta \le \pi$. $\boxed{3}$

(b) Evaluate $\tan(2\sin^{-1}\frac{3}{5})$. $\boxed{3}$

(c)



The diagram above shows the graph of the parabola $x^2 = 8y$. The tangent to the parabola at $P(4p, 2p^2)$ cuts the x-axis at A and the y-axis at B.

(i) Show that the equation of the tangent at P is $px - y - 2p^2 = 0$. $\boxed{2}$

(ii) Show that A has coordinates $(2p, 0)$ and find the coordinates of B. $\boxed{2}$

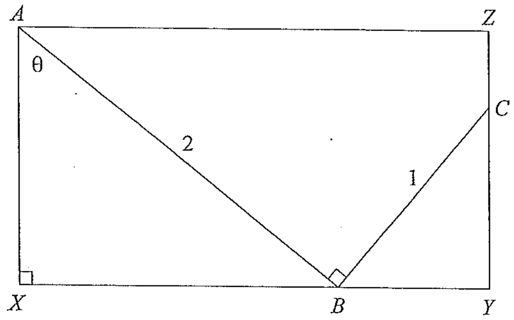
(iii) Let C be the midpoint of AB. Show that C has coordinates $(p, -p^2)$. $\boxed{1}$

(iv) Hence find the Cartesian equation of the locus of C. $\boxed{1}$

Exam continues next page ...

QUESTION SIX (12 marks) Use a separate writing booklet. **Marks**

(a)



The diagram above shows rectangle $AXYZ$. B and C are respective points on XY and YZ such that $AB = 2$, $BC = 1$ and $\angle ABC = 90°$. Let $\angle XAB = \theta$.

(i) Show that the perimeter of the rectangle is $P = 6\cos\theta + 4\sin\theta$. [3]

(ii) Express P in the form $R\cos(\theta - \alpha)$ where $R > 0$ and $0° < \alpha < 90°$. Write α correct to the nearest minute. [3]

(iii) Hence write down the maximum value of the perimeter of the rectangle, and the value of θ for which it occurs. [2]

(b) (i) Find the equation of the normal to the curve $y = \ln x$ at the point A on the curve where $x = a$. [2]

(ii) If the normal passes through the origin, show that A lies on the curve $y = -x^2$. [2]

Exam continues overleaf ...

QUESTION SEVEN (12 marks) Use a separate writing booklet. **Marks**

(a) (i) Find $\dfrac{d}{dx}(x\cos^{-1}x)$. [1]

(ii) Hence find $\displaystyle\int \cos^{-1}x\,dx$. [2]

(b) Consider the curve $y = \ln(\sin x) - \cot x - x$.

(i) Write down the domain of the function. [1]

(ii) Find the x coordinates of any stationary points on the curve in the interval $0 \le x \le 2\pi$. [3]

(c) The normal at the point $P(2ap, ap^2)$ on $x^2 = 4ay$ meets the curve again at $Q(2aq, aq^2)$. The equation of the normal at P is $x + py = 2ap + ap^3$. Note that $p \ne q$.

(i) Prove that $p^2 + pq + 2 = 0$. [2]

(ii) Find all possible points P and Q if the y-axis divides PQ in the ratio $1:3$. [3]

END OF EXAMINATION

Q1

(a) (i) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$ ✓

(ii) $e^{\log_e 4} = 4$ ✓

(b) (i) $\frac{d}{dx}\left(e^{3x^2}\right) = 6x e^{3x^2}$ ✓

(ii) $\frac{d}{dx}\left(\tan \frac{1}{4}x\right) = \frac{1}{4}\sec^2 \frac{1}{4}x$ ✓

(iii) $\frac{d}{dx}\left(\sin^{-1}(2x)\right) = \frac{1}{\sqrt{1-(2x)^2}} \times 2$ ✓

$= \frac{2}{\sqrt{1-4x^2}}$ ✓

(c) (i) $\int \cos(2-3x)\,dx = -\frac{1}{3}\sin(2-3x) + C$ ✓

ii) $\int \frac{x}{x^2-3}\,dx = \frac{1}{2}\int \frac{2x}{x^2-3}\,dx$

$= \frac{1}{2}\ln(x^2-3) + C$ ✓

iii) $\int \frac{1}{e^{2x}}\,dx = \int e^{-2x}\,dx$

$= -\frac{1}{2e^{2x}} + C$ ✓

d) $x = 6t$

$t = \frac{x}{6}$ ✓

$y = 10 \times \left(\frac{x}{6}\right)^2$

$= 10 \times \frac{x^2}{36}$

$= \frac{5x^2}{18}$ ✓

(e) $y = \tan^{-1}x$ is an odd function, has point symmetry across the origin, as the limits are opposite, the areas 'cancel each other out' ✓

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Q2

(a) $\sin^{-1}\left(\sin \frac{2\pi}{3}\right) = \sin^{-1}\left(\frac{\sqrt{3}}{2}\right)$ ✓

$= \frac{\pi}{3}$ ✓

(b) $V = \pi \int_0^2 \left(\frac{1}{\sqrt{4+x^2}}\right)^2 dx$ ✓

$= \pi \int_0^2 \frac{1}{4+x^2}\,dx$

$= \frac{\pi}{2}\left[\tan^{-1}\frac{x}{2}\right]_0^2$ ✓

$= \frac{\pi}{2}\left(\tan^{-1}1 - \tan^{-1}0\right)$

$= \frac{\pi}{2} \times \frac{\pi}{4}$

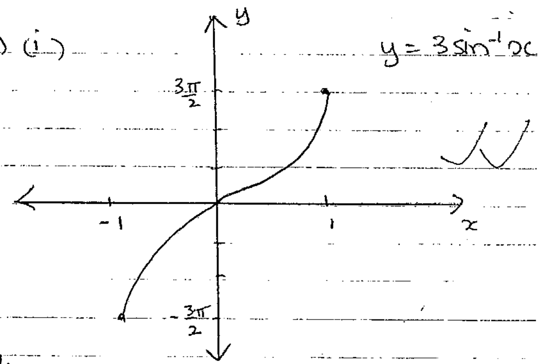
$= \frac{\pi^2}{8}$ cubic units ✓

(c) $\cos x = \frac{\sqrt{3}}{2}$

$x = \frac{\pi}{6} + 2n\pi$ ✓✓

or $x = -\frac{\pi}{6} + 2n\pi$ where n is an integer

(d) (i)



✓✓

ii) $f(x) = 3\sin^{-1}x$

$f'(x) = \frac{3}{\sqrt{1-x^2}}$ ✓

at $x = \frac{1}{3}$

$f'(x) = \frac{3}{\sqrt{1-\left(\frac{1}{3}\right)^2}}$

$= \frac{3}{\sqrt{\frac{8}{9}}}$

$= \frac{9}{2\sqrt{2}}$ ✓

$= \frac{9\sqrt{2}}{4}$

iii) $y = 3\sin^{-1}(-x)$ ✓

($= -3\sin^{-1}(x)$ as the function is odd)

/12

Q3

(a) $m_1 = -2$, $m_2 = 1$

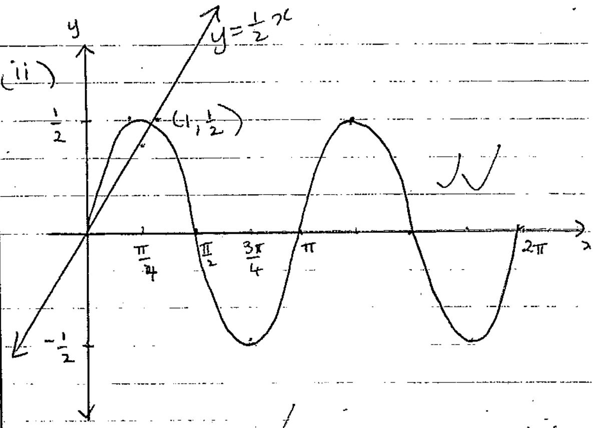
$\tan\theta = \left|\frac{1-(-2)}{1-2(1)}\right|$ ✓

$= 3$

$\theta \doteqdot 71°34'$ ✓

(b) (i) $\sin x \cos x = \frac{1}{2}\sin 2x$ ✓

(ii)



✓✓

(iii) 2 ✓

(c) (i) In $\triangle ABC$

$\frac{AB}{\sin 110°} = \frac{2500}{\sin 23°}$ ✓

$AB = \frac{2500 \sin 110°}{\sin 23°}$

In $\triangle ABX$

$\tan 21° = \frac{h}{AB}$ ✓

$h = AB \tan 21°$

$= \frac{2500 \sin 110° \tan 21°}{\sin 23°}$

(ii) $h = 2307.94\ldots$

$\doteqdot 2308$ m ✓

(d) (i) $18000 = 200e^{\frac{4}{3}t}$ ✓

$e^{\frac{4}{3}t} = 90$

$\log_e e^{\frac{4}{3}t} = \log_e 90$

$\frac{4}{3}t = \log_e 90$

$t = \frac{3\log_e 90}{4}$ ✓

$= 3.37$ h

(ii) $\frac{dP}{dt} = \frac{800}{3}e^{\frac{4}{3}t}$ ✓

at $t = 4$, $\frac{dP}{dt} = \frac{800}{3}e^{\frac{16}{3}}$

$\doteqdot 55234$ bacteria/h

/12

(4)

a) (i) $y = (1+x^2)^{-1}$

$y' = -1(1+x^2)^{-2} \times 2x$

$= \dfrac{-2x}{(1+x^2)^2}$ ✓

(ii) $y = \tan^{-1}\left(\frac{1}{x}\right)$

$y' = \dfrac{1}{1+\left(\frac{1}{x}\right)^2} \times -\dfrac{1}{x^2}$ ✓

$= \dfrac{-1}{\frac{x^2+1}{x^2}} \times \dfrac{1}{x^2}$

$= \dfrac{-1}{x^2+1}$

(iii) let $y = \ln x - 1$

the inverse is

$x = \ln y - 1$ ✓

$\ln y = x + 1$

$y = e^{x+1}$ ✓

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b) (i) $\int \cos^2 x\,dx = \frac{1}{2}\int 1 + \cos 2x\,dx$ ✓

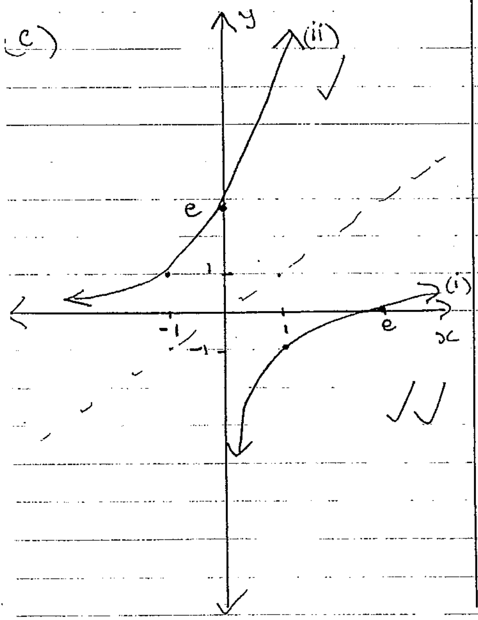
$= \frac{1}{2}\left(x + \frac{1}{2}\sin 2x\right) + C$ ✓

(ii) $\int \dfrac{1}{\sqrt{9-4x^2}}\,dx = \int \dfrac{1}{\sqrt{4\left(\frac{9}{4}-x^2\right)}}\,dx$ ✓

$= \frac{1}{2}\int \dfrac{1}{\sqrt{\frac{9}{4}-x^2}}\,dx$

$= \frac{1}{2}\sin^{-1}\dfrac{2x}{3} + C$ ✓

c)



✓✓

5)

a) $\cos 2\theta = \cos\theta, \quad 0 \le \theta \le \pi$

$2\cos^2\theta - 1 = \cos\theta$ ✓

$2\cos^2\theta - \cos\theta - 1 = 0$

$(2\cos\theta + 1)(\cos\theta - 1) = 0$

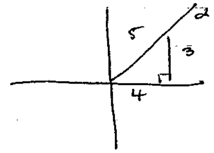
$\cos\theta = -\frac{1}{2}$ or $\cos\theta = 1$

$\theta = \frac{2\pi}{3}$ or $\theta = 0$ ✓

(b) $\tan\left(2\sin^{-1}\frac{3}{5}\right)$

let $\alpha = \sin^{-1}\frac{3}{5}$

$\sin\alpha = \frac{3}{5}, \quad -\frac{\pi}{2} \le \alpha \le \frac{\pi}{2}$ ✓



$\tan\alpha = \frac{3}{4}$

$\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$

$= \dfrac{2 \times \frac{3}{4}}{1-\left(\frac{3}{4}\right)^2}$ ✓

$= \frac{3}{2} \times \frac{16}{7}$

$= \frac{24}{7}$ ✓

$= 3\frac{3}{7}$

(c) (i) $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dp}}{\frac{dx}{dp}}$

$= \dfrac{4p}{4}$ ✓

$= p$

equation is

$y - 2p^2 = p(x - 4p)$ ✓

$y - 2p^2 = px - 4p^2$

$px - y - 2p^2 = 0$

as required

(ii) if $y = 0$, $px - 2p^2 = 0$

$px = 2p^2$

$x = 2p$ ✓

so A is $(2p, 0)$

if $x = 0$, $-y - 2p^2 = 0$

$y = -2p^2$ ✓

so B is $(0, -2p^2)$

(iii) $C = \left(\dfrac{2p+0}{2}, \dfrac{0-2p^2}{2}\right)$ ✓

$= (p, -p^2)$

(iv) $x = p$

$y = -p^2$

$= -x^2$ ✓

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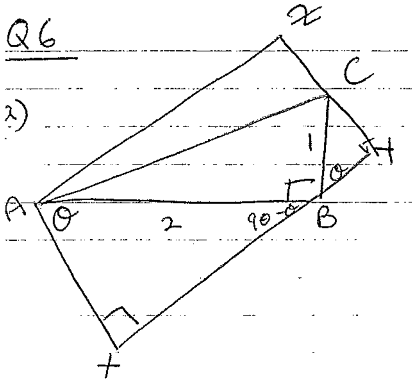
Q6

a)



(i) $\cos\theta = \frac{AX}{2}$

$AX = 2\cos\theta$

$\sin\theta = \frac{BX}{2}$

$BX = 2\sin\theta$

$\angle ABX = 90-\theta$ (angle sum ΔAXB)

$\angle CBY = \theta$ (straight angle)

$\cos\theta = \frac{BY}{1}$

$BY = \cos\theta$

$P = 2\times 2\cos\theta + 2(2\sin\theta + \cos\theta)$

$= 6\cos\theta + 4\sin\theta$

(ii) let $6\cos\theta + 4\sin\theta = R\cos(\theta-\alpha)$

RHS $= R\cos\theta\cos\alpha + R\sin\theta\sin\alpha$

so $R\cos\alpha = 6$ & $R\sin\alpha = 4$

squaring & adding,

$R^2 = 36+16$

$= 52$

$R = \sqrt{52}$

$= 2\sqrt{13}$

α is acute, $\cos\alpha = \frac{6}{\sqrt{52}}$

$\alpha \doteqdot 33°41'$

so $6\cos\theta + 4\sin\theta = \sqrt{52}\cos(\theta - 33°41')$

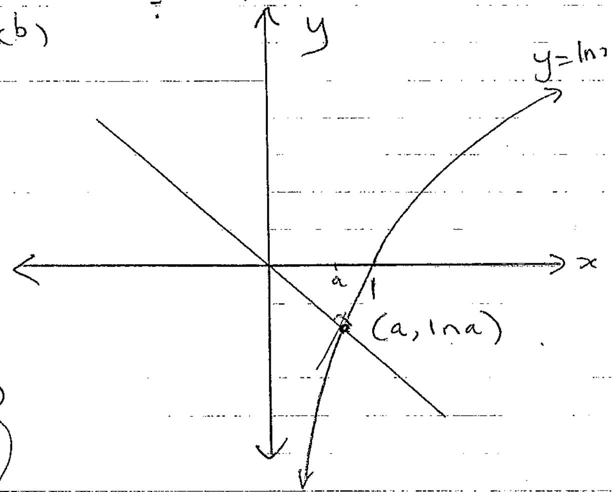
(iii) Maximum value of P is $\sqrt{52} = 2\sqrt{13}$ units

and this occurs when

$\theta = \alpha$

$\doteqdot 33°41'$

(b)



(i) $y = \ln x$

$y' = \frac{1}{x}$

at $x = a$, $y' = \frac{1}{a}$, $y = \ln a$

normal has equation

$y - \ln a = -a(x-a)$

$y = -ax + a^2 + \ln a$

(ii) if $(0,0)$ is on the equation

$a^2 + \ln a = 0$

$a^2 = -\ln a$

$-a^2 = \ln a$

as A is the point $(a, \ln a) = (a, -a^2)$

A lies on the curve

$y = -x^2$

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Q7

(a) $\frac{d}{dx}\left(x\cos^{-1}x\right)$

(i)

$= x \times \frac{-1}{\sqrt{1-x^2}} + \cos^{-1}x$

(ii) $\int \cos^{-1}x\,dx$ ~~+ x cos⁻¹x +~~

$= x\cos^{-1}x + \int \frac{x}{\sqrt{1-x^2}}\,dx$

$= x\cos^{-1}x - \frac{1}{2}\int \frac{-2x}{\sqrt{1-x^2}}\,dx$

$= x\cos^{-1}x \cdot \sqrt{1-x^2} + C$

(b)

(i) as $\sin x > 0$, $\cot x$ exists

$0 < x < \pi$, $2\pi < x < 3\pi$, etc.

(ii) $y = \ln(\sin x) - \cot x - x$

$= \ln(\sin x) - \frac{1}{\tan x} - x$

$y' = \frac{\cos x}{\sin x} + \frac{\sec^2 x}{\tan^2 x} - 1$

$= \cot x + \operatorname{cosec}^2 x - 1$

$= \cot x + \cot^2 x + 1 - 1$

$= \cot x + \cot^2 x$

$= \cot x(1+\cot x)$

$y' = 0$

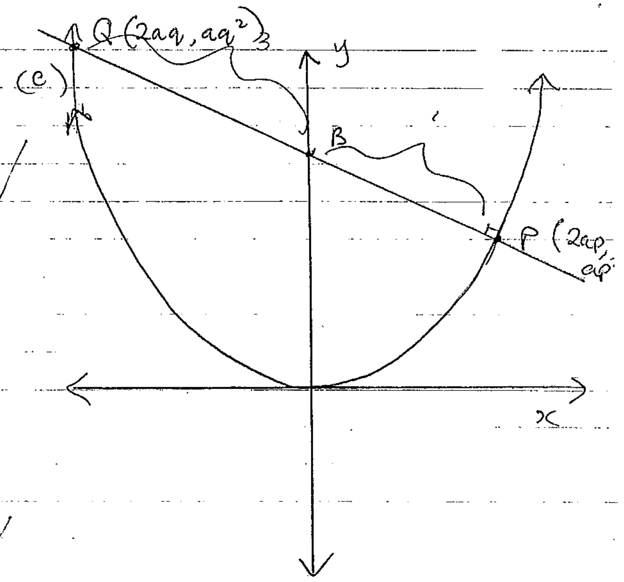
at $\cot x = 0$ or $\cot x = -1$

$x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$ $\quad x = \frac{3\pi}{4}$ or $\frac{7\pi}{4}$

checking the domain

from (i), $x = \frac{\pi}{2}$ or $\frac{3\pi}{4}$

(c)



(i) the normal has equation

$x + py = 2ap + ap^3$

Q satisfies the equation of the normal so

$2aq + p(aq^2) = 2ap + ap^3$

$2q + pq^2 = 2p + p^3$

$2q - 2p = p^3 - pq^2$

$2(q-p) = p(p^2 - q^2)$

$2(q-p) = p(p-q)(p+q)$

$(q \neq p)$ $\quad 2 = -p^2 - pq$

$2 + p^2 + pq = 0$

(ii) let B be the ~~y intercept~~ pt where the normal cuts the y-axis,

using ratio division:

$\frac{3(2ap) + 1(2aq)}{3+1} = 0$

$6ap + 2aq = 0$

$3p + q = 0$

$q = -3p$

using (i)

$2 + p^2 + p \times (-3p) = 0$

$2 - 2p^2 = 0$

$2p^2 = 2$

$p^2 = 1$

$p = 1$ or -1

$q = -3$ or 3

so $P = (2a, a)$, $Q = (-6a, 9a)$

or $P = (-2a, a)$, $Q = (6a, 9a)$

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alternatively, using ratio division with the y-co-ordinates

$$\frac{3(ap^2) + 1(aq^2)}{3+1} = 2a + ap^2$$

$3ap^2 + aq^2 = 8a + 4ap^2$

$q^2 = 8 + p^2$

using $q = -3p$,

$9p^2 = 8 + p^2$

$8p^2 = 8$

$p^2 = 1$ etc.