Sydney Grammar School

2009 Half-Yearly Examination

FORM VI
MATHEMATICS 2 UNIT

Monday 9th March 2009

General Instructions

- Writing time — 2 Hours
- Write using black or blue pen.
- Board-approved calculators may be used.
- A list of standard integrals is provided at the end of the examination paper.
- All necessary working should be shown in every question.
- Start each question in a new book.

Structure of the paper

- Total marks — 96
- All six questions may be attempted.
- All six questions are of equal value.

Collection

- Write your candidate number clearly on each leaflet.
- Hand in the six questions in a single well-ordered pile.
- Hand in a leaflet for each question, even if it has not been attempted.
- If you use a second leaflet for a question, place it inside the first.
- The question papers will be collected separately.

Checklist

- Writing leaflets: 6 per boy.
- Candidature — 101 boys

Examiner
RCF

QUESTION ONE (16 marks) Use a separate writing booklet. Marks

(a) Simplify:

(i) $e^x \times e^{-2x}$ [1]

(ii) $(3e^{4x})^2$ [1]

(iii) $\log_e e^3$ [1]

(b) (i) Sketch the parabola with equation $x^2 = -4y$. [1]

(ii) What is its focal length? [1]

(c) Find the derivative of:

(i) $y = 5x + 2$ [1]

(ii) $y = 3\sqrt{x}$ [2]

(iii) $y = (x+3)^6$ [1]

(iv) $y = 5e^x$ [1]

(v) $y = \ln(2x+1)$ [2]

(d) Find a primitive of:

(i) $2x$ [1]

(ii) $\dfrac{1}{x^2}$ [2]

(iii) e^{2x} [1]

Exam continues next page ...

QUESTION TWO (16 marks) Use a separate writing booklet. **Marks**

(a) Evaluate $\int_{-1}^{2} 6x\,dx$ [2]

(b) A curve has gradient function $f'(x) = 5x + 3x^2$ and passes through the point $(0, 2)$. Find the equation of the curve. [2]

(c) Find the following indefinite integrals:

(i) $\int e^{-4x}\,dx$ [2]

(ii) $\int (5 + 2x)^3\,dx$ [2]

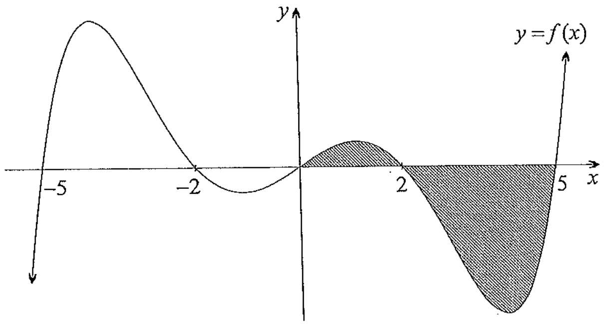
(d) The point $P(x, y)$ moves so that it is always 5 units from the point $A(2, -4)$. Write down the equation of the locus of P. [2]

(e) Consider the curve $y = e^{3x-4}$.

(i) Write down $\dfrac{dy}{dx}$. [1]

(ii) Find the equation of the tangent to the curve at the point where $x = 2$. [2]

(f)



In the diagram above, an ODD function $y = f(x)$ is graphed. You are given that:

$$\int_0^2 f(x)\,dx = 3$$

(i) Write down the value of $\int_{-2}^{2} f(x)\,dx$. [1]

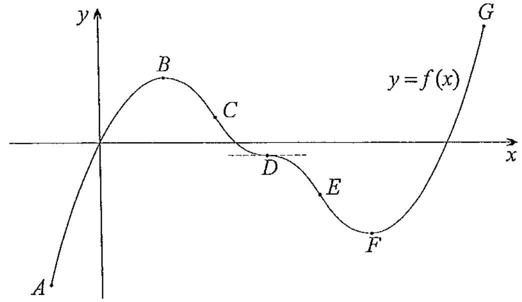
(ii) Given that $\int_{-5}^{-2} f(x)\,dx = 8$, what is the value of $\int_{2}^{5} f(x)\,dx$? [1]

(iii) What is the area of the shaded region? [1]

Exam continues overleaf ...

QUESTION THREE (16 marks) Use a separate writing booklet. **Marks**

(a)



In the diagram above, the function $y = f(x)$ is graphed. The endpoints, the stationary points and the points of inflection are marked by capital letters.

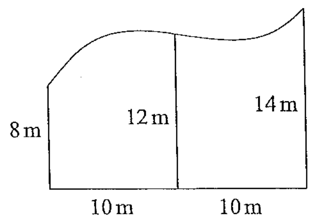
(i) At what point is the function at its minimum value? [1]

(ii) List all the stationary points of the function. [1]

(iii) Name the point for which $f'(x) = 0$ and $f''(x) > 0$. [1]

(iv) List all the points of inflection. [1]

(b)



The diagram above shows a waterfront building plot. Use the trapezoidal rule with three function values to estimate the area of the building plot. [2]

Exam continues next page ...

QUESTION THREE (Continued)

(c) A parabola has equation $(y+3)^2 = 4(x-2)$.

(i) Write down the coordinates of its vertex. [1]

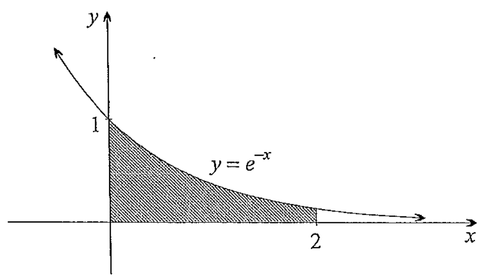
(ii) What are the co-ordinates of its focus? [1]

(iii) What is the equation of its directrix? [1]

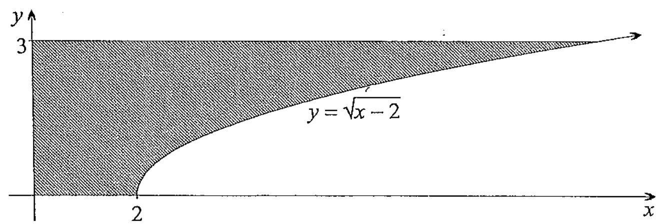
(iv) Sketch the parabola, clearly marking all these features. [2]

(d) Find the exact area of each shaded region in the diagrams given below.

(i) [2]



(ii) [3]



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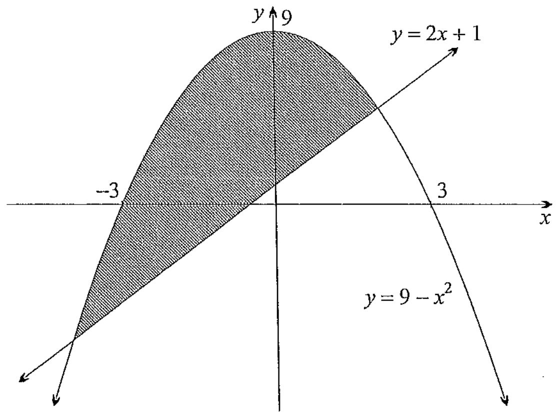
QUESTION FOUR (16 marks) Use a separate writing booklet. **Marks**

(a) (i) Given that $y = \ln x$, copy and complete the table of values below. Leave the y values in exact form. [1]

x	1	2	3	4	5
y					

(ii) Use Simpson's rule with these five function values to estimate the area between the curve $y = \ln x$ and the x-axis from $x = 1$ to $x = 5$. Give your answer rounded to two decimal places. [3]

(b)



The diagram above shows the parabola $y = 9 - x^2$ and the line $y = 2x+1$. The region enclosed between the parabola and the line is shaded.

(i) Find the co-ordinates of the points of intersection of the parabola and the line. [3]

(ii) Write down a definite integral which can be evaluated to find the area of the shaded region. [1]

(iii) Hence calculate the area of the shaded region. [2]

(c) Find $\displaystyle\int \frac{x^3 - 2x^2}{4x}\, dx$. [2]

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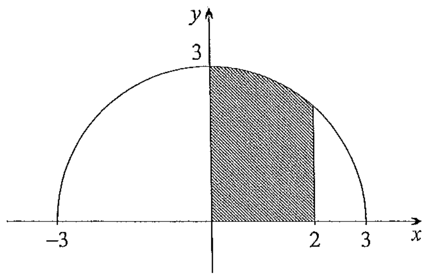
QUESTION FOUR (Continued)

(d)



The diagram above shows the region between the semicircle $y = \sqrt{9 - x^2}$ and the x-axis from $x = 0$ to $x = 2$. Find the exact volume of the solid generated when this region is rotated about the x-axis. 4

Exam continues overleaf ...

QUESTION FIVE (16 marks) Use a separate writing booklet. Marks

(a) Differentiate the following functions:

(i) $f(x) = xe^x$ (Write your answer in factored form) 2

(ii) $g(x) = \dfrac{x^2 + 1}{e^x}$ (Write your answer as a simplified fraction) 3

(iii) $h(x) = \ln\left(\dfrac{x+1}{2x-1}\right)$ (Write your answer as a single fraction) 3

(b) Evaluate:

(i) $\displaystyle\int_0^9 \frac{1}{\sqrt{x+3}}\,dx$ 3

(ii) $\displaystyle\int_0^{\ln 8} e^{\frac{x}{3}}\,dx$ 3

(c) (i) Differentiate $y = \log_e(2t^3 + t^2 + 4)$ with respect to t. 1

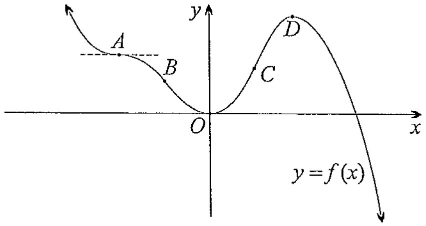
(ii) Hence find $\displaystyle\int \frac{12t^2 + 4t}{2t^3 + t^2 + 4}\,dt$. 1

Exam continues next page ...

QUESTION SIX (16 marks) Use a separate writing booklet. Marks

(a) Simplify $\log_e e^3 + e^{\frac{1}{2}\log_e 4}$. [1]

(b) [3]



The graph of a function $y = f(x)$ is drawn above.

Sketch the graph of $y = f'(x)$, where $f'(x)$ is the gradient function of $f(x)$. Mark clearly the points A', B', C', D' and O', whose respective x-values are the same as those of the original points A, B, C, D and O.

(c) Consider the function $f(x) = xe^{-x^2}$.

(i) State the domain. [1]

(ii) Show the function is odd. [1]

(iii) Find any intercepts with the co-ordinate axes. [1]

(iv) Show that $f'(x) = e^{-x^2}(1 - 2x^2)$. [1]

(v) Find the co-ordinates of any stationary points and determine their nature. [3]

(vi) Find the x co-ordinates of any points of inflection. [3]

(vii) Sketch the graph of $y = f(x)$, marking all the key points identified above. [2]
(You may use the fact that as $x \to \pm\infty$, $xe^{-x^2} \to 0$.)

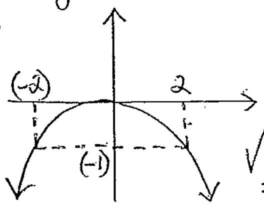
END OF EXAMINATION

MATHEMATICS HALF YEARLY 2004

① a) (i) $e^{x} \times e^{-2x} = e^{-x}$ ✓ (ii) $(3e^{4x})^2 = 9e^{8x}$ ✓

(iii) $\log_e e^3 = 3$ ✓

b) (i)



✓ SHAPE & ORIENTATION

(ii) $a = 1$ ✓

c) (i) $y = 5x + 2$, $\frac{dy}{dx} = 5$ ✓

(ii) $y = 3x^{\frac{1}{2}}$, $\frac{dy}{dx} = \frac{3}{2}x^{-\frac{1}{2}}$ ✓ $\left(\text{OR} = \frac{3}{2\sqrt{x}}\right)$

(iii) $y = (x+3)^6$, $\frac{dy}{dx} = 6(x+3)^5$ ✓

(iv) $y = 5e^{x}$, $\frac{dy}{dx} = 5e^{x}$ ✓

(v) $y = \ln(2x+1)$, $\frac{dy}{dx} = \frac{2}{2x+1}$ ✓✓ (Chain Rule reqd.)

d) (i) $f(x) = 2x$, $F(x) = x^2 + c$ ✓

(ii) $f(x) = x^{-2}$ ✓, $F(x) = \frac{x^{-1}}{-1} + c$ ✓ $\left(\text{OR} = -\frac{1}{x} + c\right)$

(iii) $f(x) = e^{2x}$, $F(x) = \frac{e^{2x}}{2} + c$ ✓

② a) $\int_{-1}^{2} 6x\,dx = \left[3x^2\right]_{-1}^{2}$ ✓

$= 3(2)^2 - 3(-1)^2$

$= 12 - 3$

$= 9$ ✓

b) $f'(x) = 5x + 3x^2$

$f(x) = \frac{5x^2}{2} + x^3 + c$ ✓

$f(0) = 2 \therefore 2 = c$

$\therefore f(x) = \frac{5x^2}{2} + x^3 + 2$ ✓

c) (i) $\int e^{-4x}\,dx = \frac{e^{-4x}}{-4} + c$ ✓ exponential ✓ $-\frac{1}{4}$ co-eff

$\left(\text{OR} = -\frac{1}{4e^{4x}} + c\right)$

(ii) $\int (5+2x)^3\,dx = \frac{(5+2x)^4}{2 \times 4} + c$ ✓ Numerator ✓ Denominator

$\left(\text{OR} = \frac{1}{8}(5+2x)^4 + c\right)$

d) $(x-2)^2 + (y+4)^2 = 25$ ✓ Correct LHS ✓ Correct RHS

e) (i) $y = e^{3x-4}$

$\frac{dy}{dx} = 3e^{3x-4}$ ✓

(ii) $\left(\frac{dy}{dx}\right)_{x=2} = 3e^2$ ✓ Eqn of tangent @ $(2, e^2)$

$y - e^2 = 3e^2(x-2)$

$y = 3e^2x - 5e^2$ ✓ OE

f) i) $\int_{-2}^{2} f(x)\,dx = 0$ ✓ (Odd function integrated over symmetrical interval across origin)

ii) $\int_{2}^{5} f(x)\,dx = -8$ ✓

iii) Shaded Area $= 3 + |-8| = 11u^2$ ✓

(3)

(i) pt A ✓

(ii) B, D & F ✓ (All three req'd)

(iii) F (Minimum turning point) ✓

(iv) C, D & E ✓ (All three req'd)

b) $A \doteqdot \frac{1}{2}\times 10(8+12) + \frac{1}{2}\times 10(12+14)$ ✓

$= 5(20) + 5(26)$

$= 5(46)$

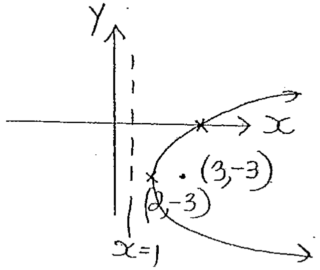
$= 230 u^2$ ✓ ✓

c) (i) $(2,-3)$ ✓

(ii) $a = 1$

$\therefore (3,-3)$ ✓

(iii) Directrix $x = 1$ ✓



✓ SHAPE ORIENTATION

✓ 3 FEATURES MARKED

d) (i) $A = \int_0^2 e^{-x}\,dx$

$= \left[\frac{e^{-x}}{-1}\right]_0^2$ ✓

$= \left[\frac{-1}{e^x}\right]_0^2$

$= \frac{-1}{e^2} - \left(-\frac{1}{1}\right)$

$= 1 - \frac{1}{e^2}\ u^2$ ✓

(ii) $A = \int_0^3 x\,dy$

$y = \sqrt{x-2}$

$y^2 = x-2$

$y^2 + 2 = x$ ✓

$= \int_0^3 y^2+2\,dy$

$= \left[\frac{y^3}{3} + 2y\right]_0^3$ ✓

$= (9+6) - 0$

$= 15 u^2$ ✓

(4)

a) (i)

x	1	2	3	4	5
y	0	ln2	ln3	ln4	ln5

✓

(ii) $A \doteqdot \frac{2}{6}(0 + 4\ln 2 + \ln 3)$ ✓ $+ \frac{2}{6}(\ln 3 + 4\ln 4 + \ln 5)$ ✓

$= \frac{1}{3}(4\ln 2 + 2\ln 3 + 4\ln 4 + \ln 5)$

$= 4.04$ (2dp) ✓

b) (i) $y = 9 - x^2 \qquad y = 2x+1$

$\therefore 9 - x^2 = 2x + 1$ ✓

$x^2 + 2x - 8 = 0$

$(x+4)(x-2) = 0$ ✓

$x = (-4)$ OR $x = 2$ ✓ $\therefore$ Pts of intersection $(2,5)$ and $(-4,-7)$ ✓

(ii) $\int_{-4}^{2} (9-x^2) - (2x+1)\,dx$

$= \int_{-4}^{2} (8 - x^2 - 2x)\,dx$ ✓

(iii) $= \left[8x - \frac{x^3}{3} - x^2\right]_{-4}^{2}$ ✓

$= \left(16 - \frac{8}{3} - 4\right) - \left(-32 + \frac{64}{3} - 16\right)$

$= 60 - \frac{72}{3}$

$= 60 - 24$

$= 36 u^2$ ✓

c) $\int \frac{x^3 - 2x^2}{4x}\,dx = \int \frac{x^2}{4} - \frac{x}{2}\,dx$ ✓

$= \frac{x^3}{12} - \frac{x^2}{4} + C$ ✓

d) $V=\int_0^2 \pi y^2\,dx = \pi\int_0^2 9-x^2\,dx$ ✓

$= \pi\left[9x-\frac{x^3}{3}\right]_0^2$ ✓

$= \pi\left[\left(18-\frac{8}{3}\right)-0\right]$ ✓

$= \pi \times 15\frac{1}{3}$

$= \frac{46\pi}{3}\ u^3$ ✓

5) a) (i) $f(x)=xe^x$ PRODUCT RULE

$f'(x)=e^x\times(1)+x\times e^x$ ✓

$= e^x[1+x]$ ✓

(ii) $g(x)=\frac{x^2+1}{e^x}$ QUOTIENT RULE

$g'(x)=\frac{e^x(2x)-(x^2+1)e^x}{(e^x)^2}$ ✓

$= \frac{e^x[2x-x^2-1]}{e^{2x}}$ ✓

$= \frac{[2x-x^2-1]}{e^x}$ ✓

$\left(\text{OR } = \frac{-[x^2-2x+1]}{e^x}\right)$

$\left(\text{OR } = \frac{-(x-1)^2}{e^x}\right)$

(iii) $h(x)=\ln\left(\frac{x+1}{2x-1}\right)$

$= \ln(x+1)-\ln(2x-1)$ ✓

$h'(x)=\frac{1}{x+1}-\frac{1\times 2}{2x-1}$ ✓

$= \frac{(2x-1)-2(x+1)}{(x+1)(2x-1)}$

$= \frac{-3}{(x+1)(2x-1)}$ ✓ DENOMINATOR CAN BE FACTORED OR EXPANDED

b) (i) $\int_0^9 \frac{1}{\sqrt{x+3}}\,dx = \int_0^9 (x+3)^{-\frac{1}{2}}\,dx$ ✓

$= \left[\frac{(x+3)^{\frac{1}{2}}}{\frac{1}{2}}\right]_0^9$ ✓

$= \left[2\sqrt{x+3}\right]_0^9$

$= 2\sqrt{12}-2\sqrt{3}$

$= 4\sqrt{3}-2\sqrt{3}$

$= 2\sqrt{3}$ ✓

(ii) $\int_0^{\ln 8} e^{x/3}\,dx$

$= \left[\frac{e^{x/3}}{1/3}\right]_0^{\ln 8}$ ✓

$= 3\left[e^{\frac{1}{3}\ln 8}-e^0\right]$

$= 3\left[e^{\ln 2}-e^0\right]$ ✓

$= 3[2-1]$

$= 3$ ✓

c) $y=\log_e(2t^3+t^2+4)$

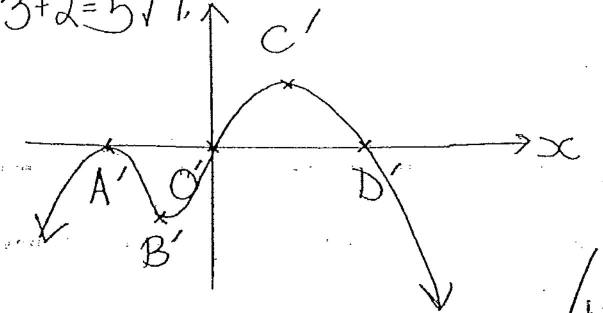
$\frac{dy}{dx}=\frac{1}{2t^3+t^2+4}\times(6t^2+2t)$

$= \frac{6t^2+2t}{2t^3+t^2+4}$ ✓

(ii) $\int \frac{2(6t^2+2t)}{4t^3+2t^2+8}\,dt = 2\log_e(2t^3+t^2+4)+C$ ✓

⑥ a) $3+2=5$ ✓ Y,

b)



✓ max t.p. &
A′ – x intercept
O′ – x intercept
D′ – x intercept
B′ – min t.p.
C′ – max t.p.

✓ both ends decreasing
✓ three x intercepts
✓ B′ min t.p. & C′ max t.p.

c) $f(x) = xe^{-x^2}$

(i) x – all real numbers ($x \in \mathbb{R}$) ✓

(ii) $f(-x) = (-x)e^{-(-x)^2}$

$= -xe^{-x^2}$

$= -f(x)$ ∴ Odd Function ✓

(iii) $x=0$ $f(0)=0$ ∴ Intercept $(0,0)$ ✓

$f(x)=0$ $0 = xe^{-x^2}$ $e^{-x^2} > 0$ ∴ no additional x-int.

Consider sign $f(1) = 1 \times e^{-1} = \frac{1}{e} > 0$ ∴ First quadrant

$f(-1) = -1 \times e^{-1} = -\frac{1}{e} < 0$ ∴ Third quadrant ✓

(iv) $f'(x) = e^{-x^2}(1) + x(-2xe^{-x^2})$ ✓ PRODUCT RULE.

$= e^{-x^2}[1-2x^2]$

(v) Stat pts $f'(x)=0$ $e^{-x^2}(1-2x^2)=0$

$e^{-x^2} \neq 0$ ∴ $(1+\sqrt{2}x)(1-\sqrt{2}x)=0$ ✓

$x = -\frac{1}{\sqrt{2}}$ or $\frac{1}{\sqrt{2}}$ ✓

$y = -\frac{1}{\sqrt{2}}e^{-\frac{1}{2}}$ or $\frac{1}{\sqrt{2}}e^{-\frac{1}{2}}$

Stat pts @ $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2e}}\right)$ & $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2e}}\right)$ ✓

Nature:

x	-1	$-\frac{1}{\sqrt{2}}$	0	$\frac{1}{\sqrt{2}}$	1
$f'(x)$	$-e^{-1}$	0	$e^{0}=1$	0	$-e^{-1}$
	\	—	/	—	\

Hence $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2e}}\right)$ Min turning point ✓

$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2e}}\right)$ Max turning point

(vi) Points of inflection $f''(x)=0$

$f'(x) = e^{-x^2}(1-2x^2)$

$f''(x) = e^{-x^2}(-4x) + (1-2x^2)(-2xe^{-x^2})$

$= e^{-x^2}[-4x-2x+4x^3]$

$= xe^{-x^2}[4x^2-6]$

$= 2xe^{-x^2}[2x^2-3]$ ✓

$= 2xe^{-x^2}(\sqrt{2}x+\sqrt{3})(\sqrt{2}x-\sqrt{3})$

∴ $f''(x)=0$

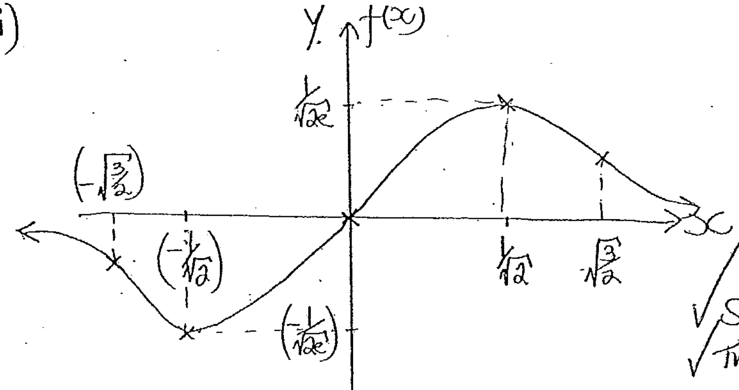
$e^{-x^2} \neq 0$ ∴ $x=0$ $x = -\frac{\sqrt{3}}{\sqrt{2}}$ or $\frac{\sqrt{3}}{\sqrt{2}}$ ✓

x	-2	$-\sqrt{\frac{3}{2}}$	-1	0	1	$\sqrt{\frac{3}{2}}$	2
$f''(x)$	$-\frac{20}{e^4}$	0	$\frac{2}{e}$	0	$-\frac{2}{e}$	0	$\frac{20}{e^4}$
	∩	•	∪	•	∩	•	∪

✓

∴ Pts of inflection @ $x = -\sqrt{\frac{3}{2}}, 0$ & $\sqrt{\frac{3}{2}}$

(vii)



✓ Shape
✓ Turning Pts.