Due Friday 18 November.

1. Let $z = 3 + i$ and $w = 2 - 5i$. Find, in the form $x + iy$,
   (a) z^2
   (b) $\overline{z}w$
   (c) $\dfrac{w}{z}$

2. Solve the equation $z^2 - 6z + 10 = 0$.

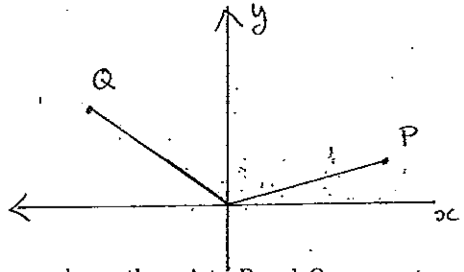
3. Find the two square roots of $24 - 10i$.

4. If $z = 5 - 5i$, express in modulus-argument form:
   (a) z
   (b) $\overline{z}$
   (c) z^2
   (d) $\dfrac{1}{\overline{z}}$

5. Given $z = 3 + 4i$, find the two possible values of w so that the points representing 0, z, and w form a right angled isosceles triangle whose:
   (a) right angle is at the origin,
   (b) right angle is at the point representing z.

6. Sketch the locus of a point z which satisfies:
   (a) $\arg(z+1) = \frac{\pi}{4}$
   $z-(-1)$
   (b) $\arg\left(\dfrac{z-1}{z+2}\right) = \frac{\pi}{4}$
   (c) $\arg(z+i) = \arg(z-1)$
   (d) $z\overline{z} + 2(z + \overline{z}) \leq 0$

7. Let $z = a + ib$ where $a^2 + b^2 \neq 0$.
   (a) Show that if $\mathrm{Im}(z) > 0$, then $\mathrm{Im}(\frac{1}{z}) < 0$.
   (b) Prove that $|\frac{1}{z}| = \dfrac{1}{|z|}$.

8. 

In the diagram above, the points P and Q represent complex numbers z and w respectively.
   (a) Use the diagram to show that $|z - w| \leq |z| + |w|$.
   (b) If R is the point representing $z + w$, what can be said about the quadrilateral $OPRQ$?
   (c) If $|z - w| = |z + w|$, what can be said about the complex number $\dfrac{w}{z}$?

9. Prove by induction that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ for all integers $n \geq 1$.

COMPLEX NUMBERS HWK 6D

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① $z = 3 + i$ and $w = 2 - 5i$

(a) $z^2 = (3+i)(3+i)$
$= 9 + 6i - 1$
$= 8 + 6i$

(b) $\bar{z} w = (3 - i)(2 - 5i)$
$= 6 - 15i - 2i - 5$
$= 1 - 17i$

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(c) $\dfrac{w}{z} \times \dfrac{\bar{z}}{\bar{z}} = \dfrac{w\bar{z}}{z\bar{z}}$
$= \dfrac{1 - 17i}{3^2 + 1^2}$ (from b))
$= \dfrac{1}{10} - \dfrac{17}{10} i$

② $z^2 - 6z + 10 = 0$

$\Delta = 6^2 - 4(1)(10)$
$= -4$
$= (2i)^2$

$z = \dfrac{6 + 2i}{2}$ or $\dfrac{6 - 2i}{2}$
$= 3 + i$ $\quad 3 - i$

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③ let $(x + iy)^2 = 24 - 10i$

$x^2 - y^2 = 24$
$xy = -5$

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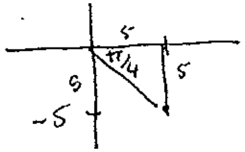
if $x = 5$, $y = -1$
if $x = -5$, $y = 1$

the square roots are $\pm(5 - i)$

④ $z = 5 - 5i$

(a) 

$|z| = 5^2 + 5^2$
$= \sqrt{50}$
$= 5\sqrt{2}$

$z = 5\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)$

(b) $\bar{z} = 5\sqrt{2} \operatorname{cis}\left(\frac{\pi}{4}\right)$

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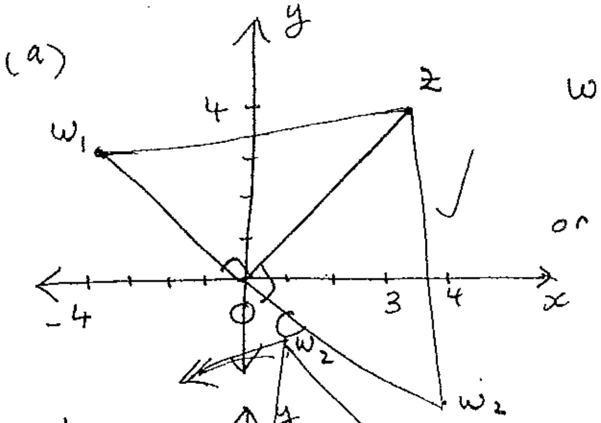
(c) $z^2 = (5\sqrt{2})^2 \operatorname{cis}\left(-\frac{\pi}{4} + -\frac{\pi}{4}\right)$
$= 50 \operatorname{cis}\left(-\frac{\pi}{2}\right)$

(d) $\dfrac{1}{\bar{z}} = \dfrac{1}{\bar{z}} \times \dfrac{z}{z}$
$= \dfrac{z}{|z|^2}$
$= \dfrac{5\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)}{50}$
$= \dfrac{\sqrt{2}}{10} \operatorname{cis}\left(-\frac{\pi}{4}\right)$

5) $z = 3 + 4i$

(a)

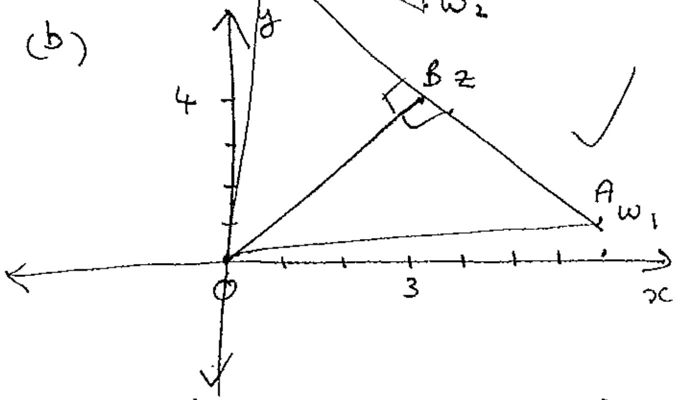


$w = iz$
$= i(3+4i)$
$= -4+3i$

or

$w = -i(z)$
$= -i(3+4i)$
$= 4-3i$

(b)



$\vec{BA} = \vec{BO} \times i$
$= i(-3-4i)$
$= 4-3i$

$\vec{BC} = \vec{BO} \times -i$
$= -i(-3-4i)$
$= -4+3i$

$w_1 = \vec{OA} = \vec{OB} + \vec{BA}$
$= 3+4i + 4-3i$
$= 7+i$

$w_2 = \vec{OC} = \vec{OB} + \vec{BC}$
$= 3+4i - 4+3i$
$= -1+7i$

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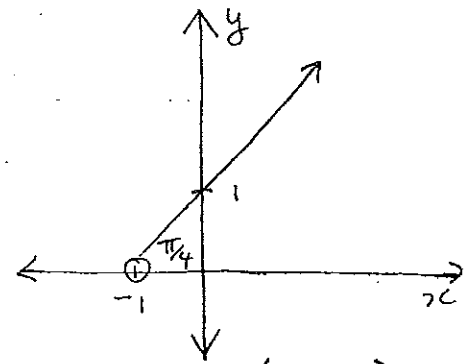
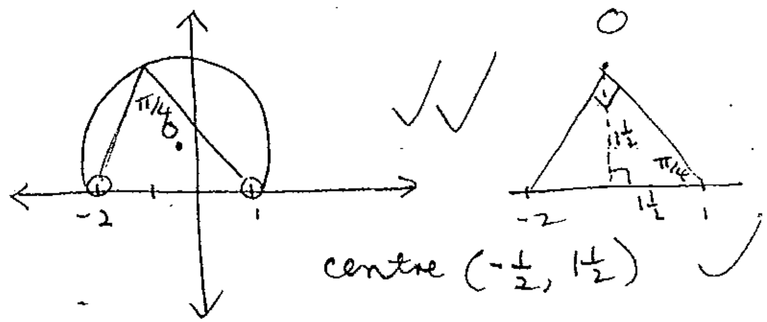
6) (a) $\arg(z+1) = \frac{\pi}{4}$

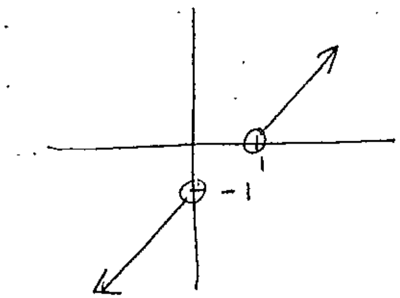


(b) $\arg\left(\frac{z-1}{z+2}\right) = \frac{\pi}{4}$



centre $(-\frac{1}{2}, 1\frac{1}{2})$

(c) $\arg(z+i) = \arg(z-1)$
$\arg(z+i) - \arg(z-1) = 0$

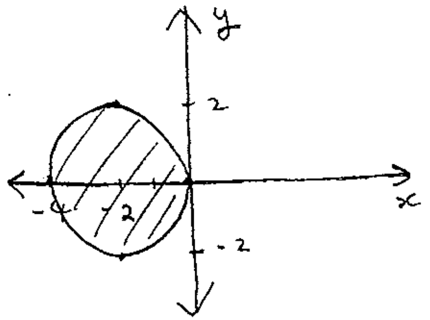


(d) $z\bar{z} + 2(z+\bar{z}) \le 0$

let $z = x + iy$

$x^2 + y^2 + 4x \le 0$

$(x+2)^2 + y^2 \le 4$



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(7) $z = a + ib$

(a) $\frac{1}{z} = \frac{1}{z} \times \frac{\bar{z}}{\bar{z}}$

$= \frac{1}{a^2+b^2}(a - ib)$

if $b > 0$, then $\text{Im}\left(\frac{1}{z}\right) = \frac{-b}{a^2+b^2}$

< 0 as $a^2+b^2 > 0$

(b) $\left|\frac{1}{z}\right|^2 = \left|\frac{a-ib}{a^2+b^2}\right|^2$

$= \left(\frac{a}{a^2+b^2}\right)^2 + \left(\frac{b}{a^2+b^2}\right)^2$

$= \frac{a^2+b^2}{(a^2+b)^2}$

$= \frac{1}{a^2+b^2}$

$\left|\frac{1}{z}\right| = \frac{1}{\sqrt{a^2+b^2}}$

$= \frac{1}{|z|}$

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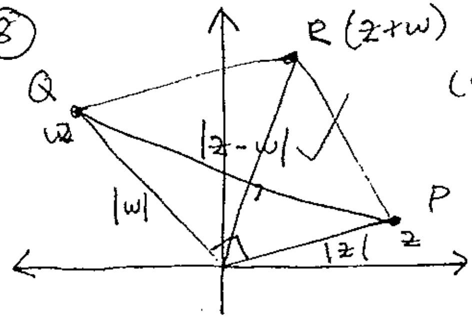
(8)



(a) using the triangle inequality $|z-w| \le |z| + |w|$

(b) quadrilateral OPRQ is a parallelogram

(c) If $|z-w| = |z+w|$ the diagonals are equal & we have a rectangle (or a square)

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so $\frac{w}{z}$ has argument $\frac{\pi}{2}$

& it is therefore purely imaginary

(9) $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
$n \geq 1$

Step 1: let $n = 1$

$LHS = (\cos\theta + i\sin\theta)^1$
$= \cos\theta + i\sin\theta$

$RHS = \cos\theta + i\sin\theta$
$= LHS$ ✓

So the result is true for $n=1$

Step 2: suppose k is a positive integer for which the result is true

that is, $(\cos\theta + i\sin\theta)^k = \cos k\theta + i\sin k\theta$ ✓ $(*)$

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We prove the result is true for $n=k+1$

That is, we prove that
$(\cos\theta + i\sin\theta)^{k+1} = \cos(k+1)\theta + i\sin(k+1)\theta$ ✓

$LHS = (\cos\theta + i\sin\theta)^{k+1}$
$= (\cos\theta + i\sin\theta)^k(\cos\theta + i\sin\theta)^1$
$= (\cos k\theta + i\sin k\theta)(\cos\theta + i\sin\theta)$
by the induction hypothesis $(*)$

$= \cos k\theta\cos\theta + \cos k\theta\, i\sin\theta + i\sin k\theta\cos\theta - \sin k\theta\sin\theta$ ✓

note: using $\mathrm{cis}\,k\theta \times \mathrm{cis}\,\theta$ is easier here

$= \cos(k\theta + \theta) + i\sin(k\theta+\theta)$
$= \cos(k+1)\theta + i\sin(k+1)\theta$
$= RHS$

Step 3: it follows from steps 1 + 2 by mathematical induction that the result is true for all positive integers n.