SYDNEY TECHNICAL HIGH SCHOOL
(Est 1911)

MATHEMATICS EXTENSION II

HSC ASSESSMENT TASK 1

MARCH 2003

Time allowed : 70 minutes (COMPLEX NUMBERS & CONICS)

Instructions :

- Show all necessary working in every question.
- Start each question on a new page.
- Attempt all questions.
- All questions are of equal value.
- Full marks may not be awarded for careless or badly arranged work.
- Approved calculators may be used.
- This test forms part of your HSC assessment.
- These questions are to be handed in with your answers.

Name : Tommy Lim.

Question 1	Question 2	Question 3	Total
16	17	17	50

Question 1 **Marks**

a) Write $\dfrac{3+2i}{1+i}$ in the form $x + iy$ **2**

b) $(3+2i)(d+i)$ is real. Find the value of d **2**

c) In answering questions about the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1\ (a>b)$ Ting gave the following answers. **3**

i) eccentricity $0 \le e \le 1$

ii) foci $(0, \pm ae)$

Are Ting's answers correct. Give explanations for your choices.

d) Find $|x+iy+2|$ **2**

e) If z is the complex number $x + iy$ simplify $(z-\bar{z})^2$ **3**

f) i) Sketch the locus of z defined by arg $(z+1)=\dfrac{\pi}{6}$ **2**

ii) Find z in the modulus argument form such that $|z|$ is a minimum **3**

Question 2 **Marks**

a) For the ellipse $4x^2 + 9y^2 = 36$ find

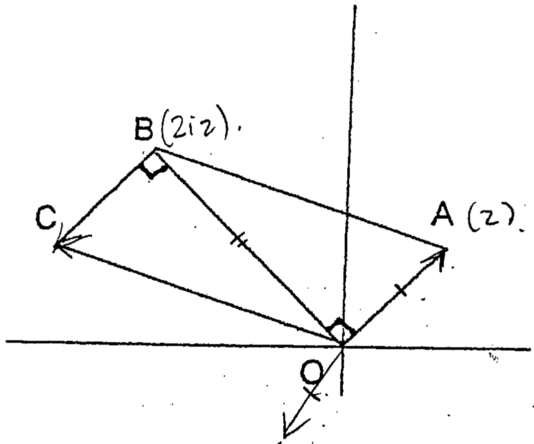
i) eccentricity **2**

ii) co-ordinates of foci **1**

iii) equation of directrices **1**

iv) sketch the ellipse marking all the necessary information **1**

b)



$2 + (-2)$.

$2i2 - 2$

$2(2i - 1)$.

In the diagram < CBO = < BOA = 90° and OB = 2OA. If A is the complex number z

i) Explain why B is the complex number 2iz **2**

ii) If OABC is a parallelogram find the complex number for C **2**

c) Find the equation and sketch the locus of z if $Im(z^2) = 2$ **2**

d) Let A and B be the complex numbers z_1 and z_2 satisfying $|z_1| = |z_2|$ rhombus.

i) Draw an Argand diagram showing the complex numbers z_1, z_2 and $z_1 + z_2$ (label C), z_1 and z_2 in the first quadrant. **2**

ii) What type of figure is OACB **1**

iii) Mark on your diagram the complex number $z_2 - z_1$ (label D) **1**

iv) Use your diagram or otherwise to show that $\frac{z_1 + z_2}{z_2 - z_1}$ is imaginary **2**

Question 3

Marks

a) The complex number z is given by $z = -\sqrt{3} + i$. Find

i) arg z **1**

ii) $|z|$ **1**

iii) z^7 in modulus argument form **2**

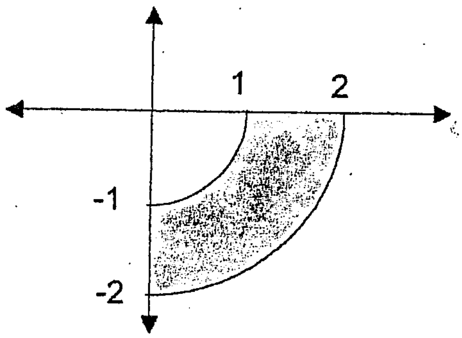
b) The roots of $z^6 - 1 = 0$ are $1, w, w^2, w^3, w^4, w^5$ $(z^4+1)(z-1)(z^4+w^2+1)$,

i) Find w (first complex root) in modulus argument form **1**

ii) Plot all the roots on an Argand diagram **2**

iii) By factorising or otherwise write down the equation whose roots are w, w^3 and w^5 **1**

c) Give the inequalities which describe this region in the complex number plane. **3**



d) i) Prove that the equation of the tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **3**

at the point $P(x_1\ y_1)$ *is* $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

ii) This ellipse meets the y axis at B *and* B^1. The tangents at B and B^1 to the ellipse meet the tangent at P at Q and Q^1 respectively. **3**

α) Draw a neat sketch labelling each of the points and showing the tangent.

β) Prove $BQ \times B^1Q^1 = a^2$

Teacher's Name: Mr Snider **Student's Name/N°:** Tommy Lim.

QUESTION 1

(a) $\frac{3+2i}{1+i} \times \frac{1-i}{1-i} = \cancel{\frac{3-i-2i^2}{1+1}}$

$\cancel{= \frac{5}{2} - \frac{i}{2}}$

$\frac{3-i-2i^2}{1+1}$

$= \frac{5}{2} - \frac{i}{2}$

2

(b) $(3+2i)(d+i) = 3d + 3i + 2di + 2i^2$

$= 3d - 2 + i(3+2d)$

$\therefore \text{Im}(z) = 0$

$3+2d = 0$

$d = -\frac{3}{2}$

2

(c) (i) ~~Correct~~ incorrect $\boxed{\text{answer is: } 0 < e < 1}$

if $e = 0 \Rightarrow$ circle.

$e = 1 \Rightarrow$ Parabola.

3

(ii) incorrect $\boxed{\text{answer is: } S(\pm ae, 0)}$

since $a > b$ foci lie on the x-axis.

(d) $|(x+2) + iy| = \sqrt{(x+2)^2 + y^2}$

$= \sqrt{x^2 + 4x + 4 + y^2}$

2

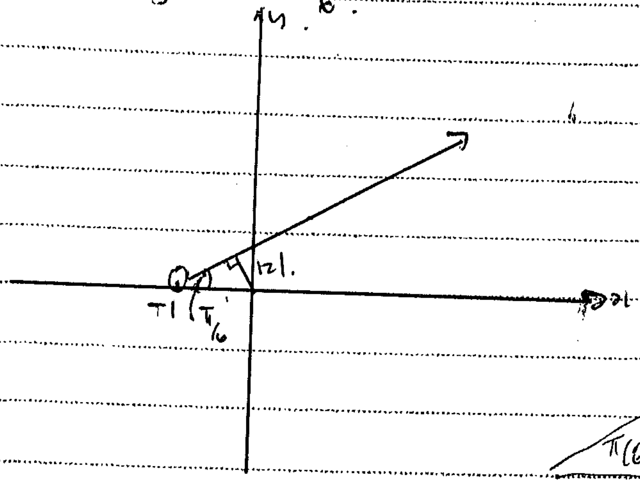
(e) $z - \bar{z} = 2\text{Im}(z) = 2y$

$\Rightarrow (2y)^2 = 4y^2$

2

Teacher's Name: Mr Sendr **Student's Name/Nº:** T. Lin.

(f). (i). $\arg(z+1) = \frac{\pi}{6}$.



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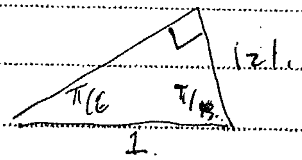


(ii). ~~cos~~ $\sin \frac{\pi}{6} = \frac{|z|}{1}$.

$$\boxed{\frac{1}{2} = |z|.}$$

$$\arg z = \cancel{\frac{\pi}{2} - \frac{\pi}{6}} \quad \pi - \left(\frac{\pi}{2} - \frac{\pi}{6}\right).$$

$$= \pi - \frac{\pi}{3} = \frac{2\pi}{3}.$$

$$\therefore z = \frac{1}{2}\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

3

16

Teacher's Name: Mr Sender   Student's Name/N°: Tommy Lim

QUESTION 2

(a) $\frac{4x^2}{36} + \frac{9y^2}{36} = 1$

$\frac{x^2}{9} + \frac{y^2}{4} = 1$ $a > b$.

(i). $b^2 = a^2(1-e^2)$.

$4 = 9(1-e^2)$.

$\frac{4}{9} = 1 - e^2$.

$e^2 = 1 - \frac{4}{9} = \frac{5}{9}$.

$e = \frac{\sqrt{5}}{3}$.   2

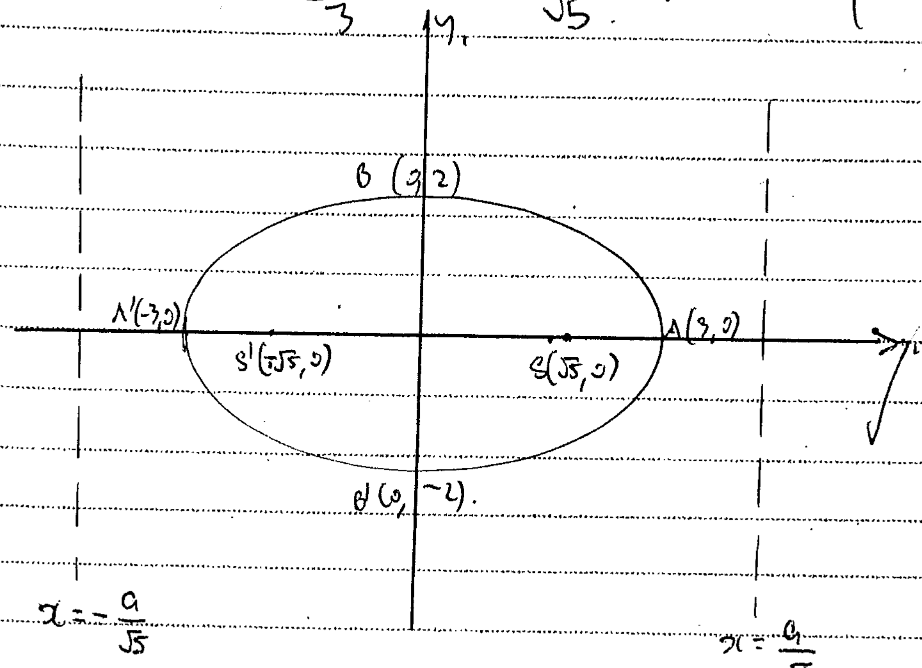
(ii). $S(\pm ae, 0)$, $a = 3$.

$S\left(\cancel{3} \times \frac{\sqrt{5}}{\cancel{3}}, 0\right)$ $S'\left(\cancel{3} \times -\frac{\sqrt{5}}{\cancel{3}}, 0\right)$

$S(\sqrt{5}, 0)$ $S'(-\sqrt{5}, 0)$.   1

(iii) $x = \pm \frac{a}{e} = \pm \frac{3}{\frac{\sqrt{5}}{3}} = \pm \frac{9}{\sqrt{5}}$.   1

(iv).



1

5

Teacher's Name: M sender **Student's Name/N°:** T. Lin

(b) (i) $A = z$

$|B| = 2|z|$ — B is twice the magnitude. or $\Rightarrow |B| = 2|A| = 2|z|$.

$\therefore |B| = |2z|$ $\underline{B = 2z \operatorname{cis}\theta}$. $\theta = \frac{\pi}{2}$

$\angle BOA = 90°$, multiplying z by i is equivalent to an anticlockwise rotation by $90°$. $\left[\operatorname{cis}\frac{\pi}{2} = i\right]$

Hence.

$\Rightarrow B = 2iz$

2

(ii) $\vec{OC} = \vec{OB} + (-\vec{OA})$

$= 2iz - z$

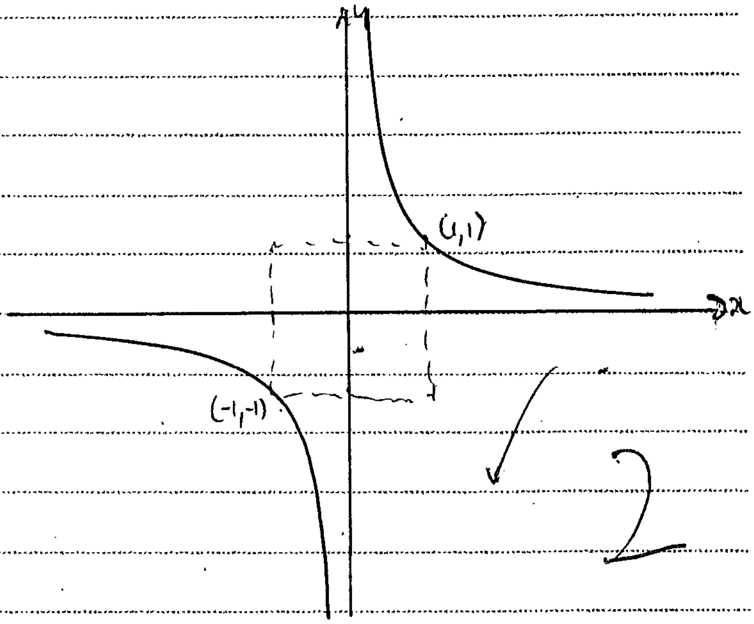
$= z(2i-1)$

2

(c) $z^2 = x^2 - y^2 + 2xyi$

$\Rightarrow \operatorname{Im}(z^2) = 2xy = 2$

$\underline{xy = 1}$



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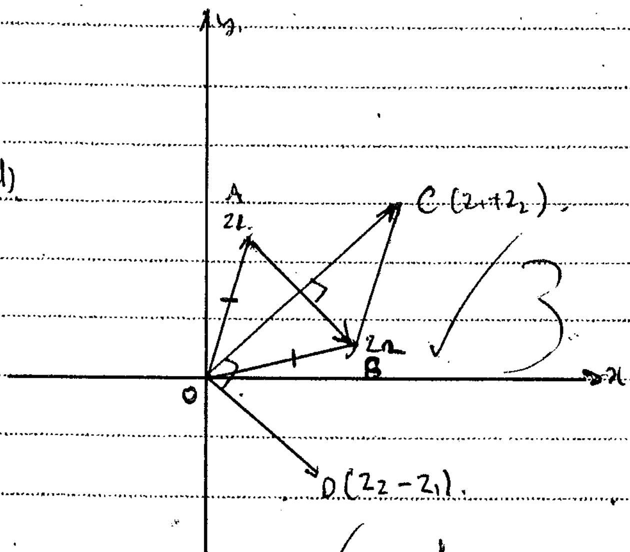
(d)



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(ii) Rhombus

(iii)

(iv) diagonals of a rhombus intersect at right $\angle$'s.

$\therefore$ transposing $z_2 - z_1$ to the origin we get:

$\angle COD = 90°$

$\Rightarrow z_2 - z_1$ needs to be rotated $90°$ anticlock. length

$\therefore$ ~~$z_2 - z_1 = ki(z_1$~~

$\Rightarrow z_1 + z_2 = ki(z_2 - z_1)$

$\therefore \dfrac{z_1+z_2}{z_2-z_1} = ki$

where k is a constant.

$\therefore ki$ is totally imaginary

hence.

$\Rightarrow \dfrac{z_1+z_2}{z_2-z_1}$ is totally imaginary.

2

Teacher's Name: Mr Sender **Student's Name/N°:** Tommy Lim

QUESTION 3

$z = -\sqrt{3} + i$.

(a)(i) $\arg z = \pi - \tan^{-1}\frac{1}{\sqrt{3}}$

$= \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

(ii). $|z| = \sqrt{1+3} = 2$.

(iii) $\Rightarrow z = 2\left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right)$.

$= 2\,\text{cis}\left(\frac{5\pi}{6}\right)$.

$\therefore z^7 = 128\,\text{cis}\left(\frac{35\pi}{6}\right)$.

$= \cancel{\text{cis}}\ 128\,\text{cis}\left(-\frac{\pi}{6}\right)$.

(b)(i) $z^6 = 1$

$= \cancel{\cos(2k\pi)}\ \text{cis}(2k\pi)$.

$z_k = \text{cis}\left(\frac{k\pi}{3}\right)$ where $k = 0, 1, 2, 3, 4, 5$.

$z_0 = \text{cis}(0) = 1$.

$z_1 = \text{cis}\left(\frac{\pi}{3}\right)$.

$z_2 = \text{cis}\left(\frac{2\pi}{3}\right)$

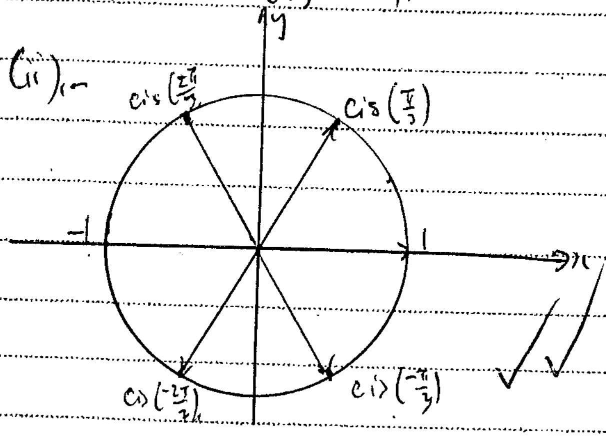
$z_3 = \text{cis}(\pi) = -1$.

$z_4 = \text{cis}\left(\frac{4\pi}{3}\right) = \text{cis}\left(\frac{-2\pi}{3}\right) = \overline{z_2}$.

$z_5 = \text{cis}\left(\frac{5\pi}{3}\right) = \text{cis}\left(\frac{-\pi}{3}\right) = \overline{z_1}$.

$\Rightarrow$ (i). $w = \text{cis}\frac{\pi}{3}$.

(ii).



(iii). $w^3 = \text{cis}\,\pi = -1$

$w^5 = \text{cis}\left(\frac{5\pi}{3}\right) = \text{cis}\left(\frac{-\pi}{3}\right) = \overline{z_1}$.

$\Rightarrow (z+1)\left(z^2 - (z_1 + \overline{z_1})z + z_1\overline{z_1}\right)$.

$\Rightarrow (z+1)\left(z^2 - 2\cos\frac{\pi}{3}z + 1\right)$

$(z+1)(z^2 - z + 1)$

Teacher's Name: **Student's Name/N°:** T. Lim.

(c) $|z| \geq 1$ & $|z| \leq 2$ poor

$Re(z) \geq 0$ $Im(z) \leq 0$.

(d)(i) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$ $P(x_1, y_1)$.

diff w.r.t x.

$$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0.$$

$$\frac{\cancel{2}y}{b^2} \cdot \frac{dy}{dx} = -\frac{\cancel{2}x}{a^2}.$$

$$\Rightarrow \frac{dy}{dx} = \frac{-x}{y}\frac{b^2}{a^2}.$$

at P $m = \frac{-x_1 b^2}{y_1 a^2}.$

$$\Rightarrow y - y_1 = \frac{-x_1 b^2}{y_1 a^2}(x - x_1).$$

$$y_1 a^2 y - y_1^2 a^2 = -x x_1 b^2 + x_1^2 b^2.$$

$$x x_1 b^2 + y y_1 a^2 = x_1^2 b^2 + y_1^2 a^2.$$

divide throughout by $a^2 \times b^2$.

$$\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}.$$

$$\frac{x x_1}{a^2} + \frac{y y_1}{b^2} = 1.$$

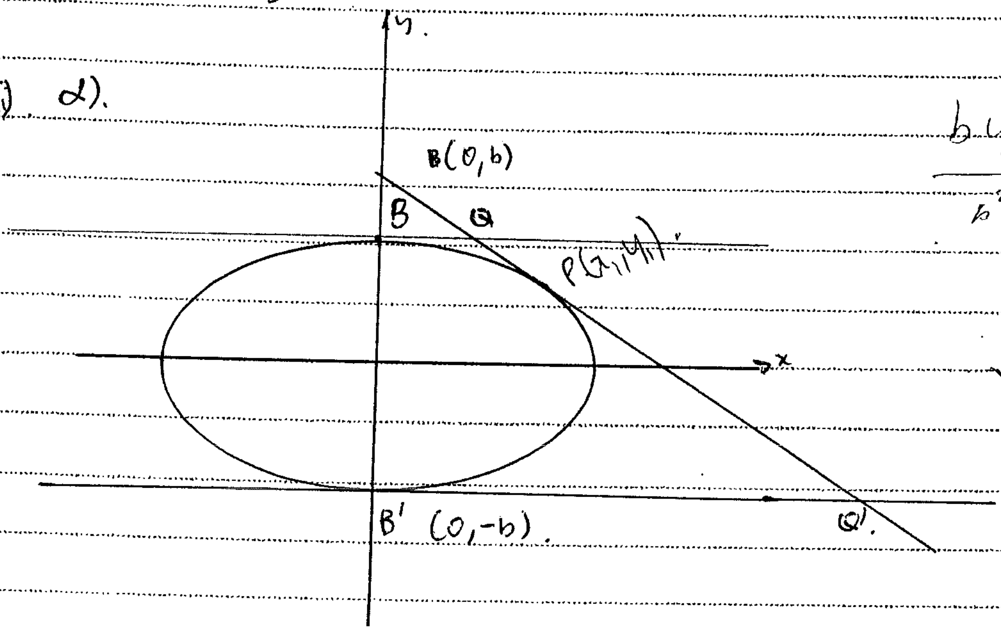
since $P(x_1, y_1)$ lies on the ellipse,

$$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1,$$

Teacher's Name: Mr Seider   Student's Name/N°: Tommy Lin

(ii) $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$

(i) d).



$\frac{by_1}{b^2}$

at Q, $y = b$; $\frac{xx_1}{a^2} + \frac{y_1}{b} = 1$.

$$\frac{xx_1}{a^2} = 1 - \frac{y_1}{b} = \frac{b - y_1}{b}$$

$$x = \frac{a^2(b - y_1)}{x_1 b}$$

$\Rightarrow BQ = \frac{a^2(b - y_1)}{x_1 b}$

at Q', $y = -b$, $\frac{xx_1}{a^2} - \frac{y_1}{b} = 1$.

$$\frac{xx_1}{a^2} = 1 + \frac{y_1}{b} = \frac{b + y_1}{b}$$

B'Q': $x = \frac{a^2(b + y_1)}{bx_1}$

$\Rightarrow BQ \times B'Q' = \frac{a^4(b - y_1)(b + y_1)}{b^2 x_1^2}$

$= \frac{a^4(b^2 - y_1^2)}{b^2 x_1^2}$ but, $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$.

PTO.

Teacher's Name: Mr Sender. **Student's Name/N°:** T. Lim

but $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1.$

$$\frac{x_1^2}{a^2} = \frac{b^2 - y_1^2}{b^2}.$$

$$b^2 x_1^2 = a^2(b^2 - y_1^2).$$

Hence $\dfrac{a^2 \times a^2(b^2 - y_1^2)}{b^2 x_1^2}$

$$= \frac{a^2 \times \cancel{b^2 x_1^2}}{\cancel{b^2 x_1^2}} = a^2.$$

$\Rightarrow BQ \times BQ' = a^2$ as req'd.

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but $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1.$

$$\frac{x_1^2}{a^2} = \frac{b^2 - y_1^2}{b^2}.$$

$$\underline{x_1^2} = \frac{a^2(b^2 - y_1^2)}{b^2}.$$

$$b^2x^2 = a^2(b^2 - y_1^2).$$

Hence. $\frac{a^2 \times a^2 (b^2 - y_1^2)}{b^2 x_1^2}$

$$= \frac{a^2 \cancel{\times b^2 x^2}}{\cancel{b^2 x_1^2}}.$$

$BQ \times BQ' = \underline{\underline{a^2}}.$ as req'd