SYDNEY INSTITUTE OF TECHNOLOGY – RANDWICK CAMPUS

Student Name : ....

1999
SEMESTER 2 EXAMINATION

MATHEMATICS
2/3 UNIT (COMMON)

Time allowed - Two hours and a half
(Plus 5 minutes' reading time)

Directions to Candidates

- Attempt ALL questions.
- ALL questions are of equal value.
- All necessary working must be shown in every question.
- Marks may be deducted for careless or badly arranged work.
- Standard integrals are printed on page 10.
- Board-approved calculators may be used.
- Each question is to be started on a new page.
- This examination paper must NOT be removed from the examination room.

QUESTION 1. *(Start a new page)* **MARKS**

(a) Find $\sqrt[4]{2.8385}$ correct to 5 significant figures. **2**

(b) Simplify $\frac{3}{4}-\frac{2x-1}{3}$. **2**

(c) Differentiate $5x^8+14x-2$. **2**

(d) Factorise completely $4x^3-4x^2-9x+9$. **2**

(e) By rationalizing the denominator, express $\frac{3}{2+\sqrt{3}}$ in the form $c+d\sqrt{3}$. **2**

(f) Solve $2-5x>9$. **2**

QUESTION 2. *(Start a new page)*

(a) On a number plane, mark the points $A(-1,2)$ and $C(3,4)$. **1**

(b) Find the gradient of AC. **1**

(c) Show that the midpoint of AC has coordinates $M(1,3)$. **1**

(d) Find the distance AC. **1**

(e) Show that the line k, passing through M and perpendicular to AC has equation $2x+y-5=0$. **2**

(f) Line k intersects with the y-axis in point B. Find coordinates of point B. **1**

(g) Find the coordinates of the point D such that M is also the midpoint of BD. **1**

(h) Find the distance BD. **1**

(i) State what shape best describes the quadrilateral $ABCD$. **1**

(j) Hence calculate the area of the quadrilateral $ABCD$. **2**

QUESTION 3. *(Start a new page)* **MARKS**

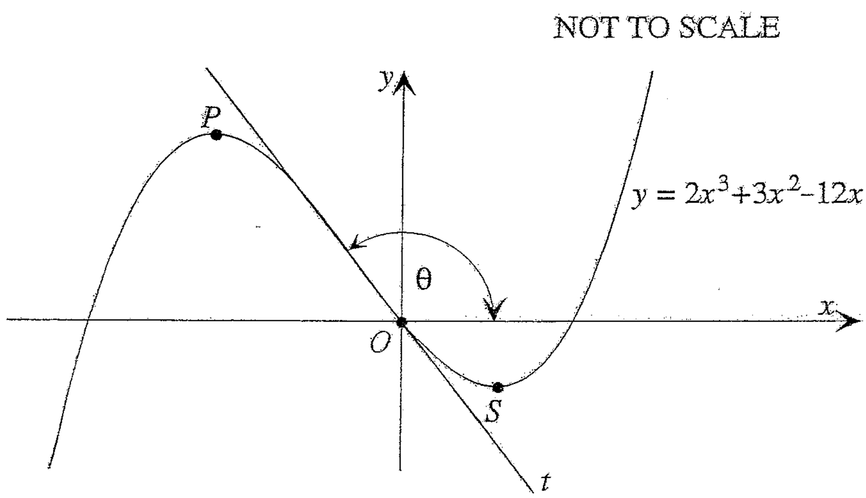
(a) Differentiate : **6**

(i) $y = \left(x^3 + 8x - 4\right)^{24}$

(ii) $y = \dfrac{7 - x}{3x + 1}$

(iii) $y = \sqrt[3]{2x + 5}$.

(b) **6**



The diagram shows a sketch of the curve $y = 2x^3 + 3x^2 - 12x$.
Tangents to the curve at P and S are parallel to the x axis.

(i) Find the co-ordinates of P and S.

(ii) Find the angle θ which the tangent t to the curve at the origin O makes with the positive part of the x axis.

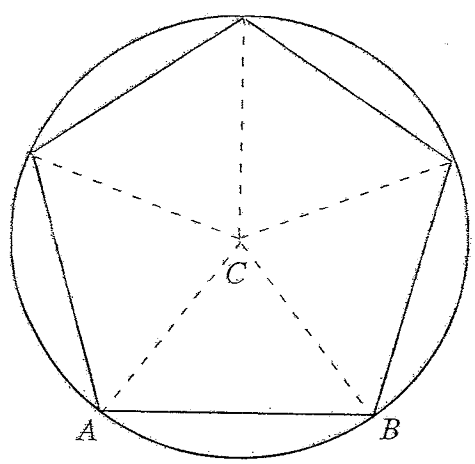
QUESTION 4. *(Start a new page)* **MARKS**

(a) If α and β are the roots of $5x^2 - 2x - 3 = 0$, find the value of : 3

(i) $\alpha + \beta$

(ii) $\frac{1}{\alpha} + \frac{1}{\beta}$.

(b) 5



A regular pentagon is inscribed in a circle with radius 4 cm.

(i) Find the perimeter of the pentagon .

(ii) Find the area of the pentagon.

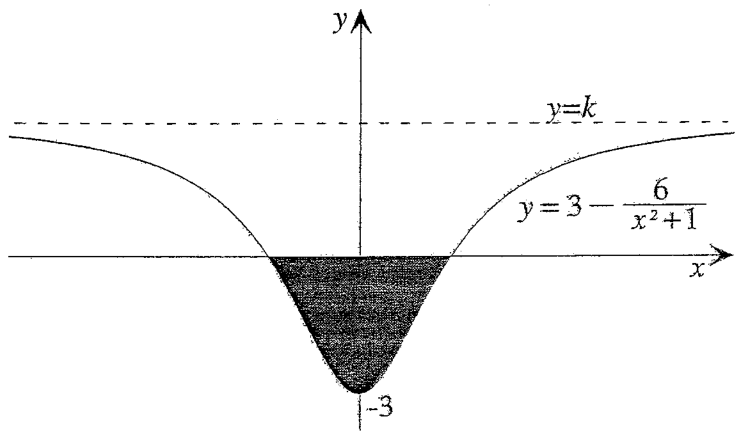
(c) (i) Sketch the graph of $y = 3 - x^2$ and label all intercepts with the axes. 4

(ii) On the same set of axes, sketch carefully the graph of $y = |2x|$.

(iii) Use your sketch to determine the coordinates of the two points P and Q where the graphs intersect. (Do not use algebraic methods.)

QUESTION 5. *(Start a new page)* **MARKS**

(a) Function $y = 3 - \dfrac{6}{x^2+1}$ is shown below **8**



(i) Find $\lim\limits_{x\to-\infty} f(x)$ and $\lim\limits_{x\to\infty} f(x)$.

(ii) Write down the domain and range of $y = 3 - \dfrac{6}{x^2+1}$.

(iii) Determine whether the function is even, odd or neither.

(iv) Write down a pair of inequalities which define the shaded region.

(b) The graph of a parabola $y = ax^2 + bx + c$ passes through the points **4**

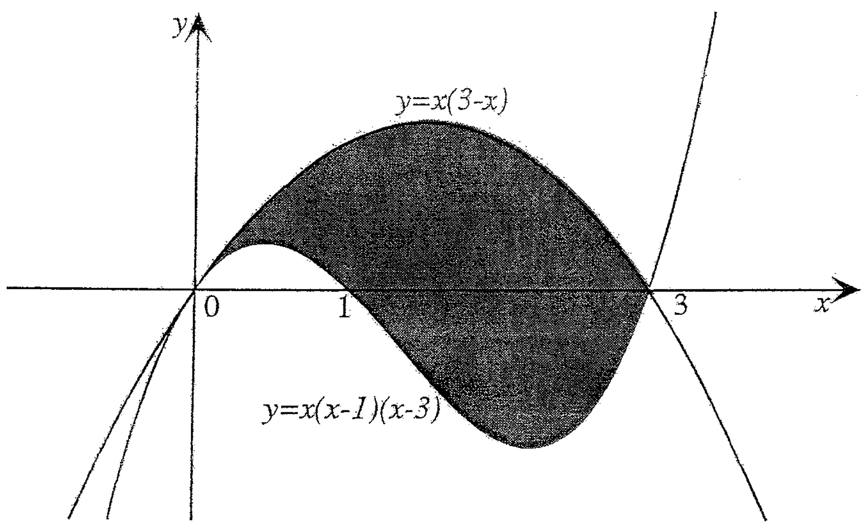
$P(0,-3)$, $Q(1,4)$ and $R(-1,-6)$. Find the values of a, b and c.

QUESTION 6. *(Start a new page)* **MARKS**

(a) Prove that $\dfrac{\sec\theta}{\tan\theta + \cot\theta} = \sin\theta$. **3**

(b) Evaluate $\int_1^6 \sqrt{3x-2}\,dx$ **2**

(c) **4**



The diagram shows the graphs of functions $y = 3x - x^2$ and $y = x^3 - 4x^2 + 3x$.

The two graphs intersect at $x = 0$ and $x = 3$. Calculate the area of the shaded region.

(d) The region which lies between the x-axis and the portion of the curve $y = \sqrt[3]{x}$ **3**

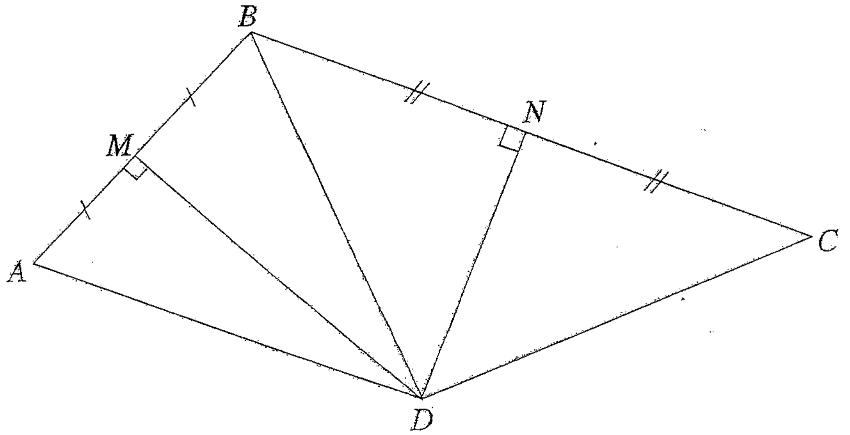
from $x = 1$ to $x = 8$ is rotated about the x-axis to form the solid.

Find the volume of the solid.

QUESTION 7. *(Start a new page)* **MARKS**

(a) Sketch the region defined by $\{(x, y) : y \leq 9 - x^2 \cap y \geq x + 3\}$ **3**

(b) **5**



In a quadrilateral $ABCD$, M and N are mid-points of AB and BC respectively. AB is perpendicular to DM and BC is perpendicular to DN.

Copy the diagram into your Writing Booklet.

(i) Show that the triangles ADM and BDM are congruent.

(ii) Show that $AD = BD$.

(iii) Show that D is the centre of a circle which passes through points A, B and C.

(c) George and Sam play a game in which they take turns in throwing two ordinary dice. The game is won by the player who first throws dice so that the difference of two numbers shown is 1. **4**

(i) Draw up a table of possible outcomes for one throw of two dice.

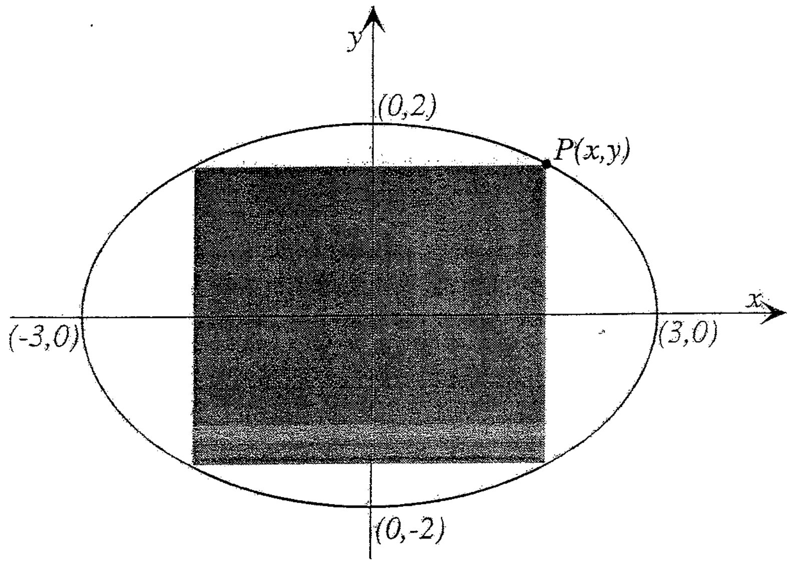
(ii) Given that George is to start the game :

(α) Calculate the probability that he wins on his first throw of the dice.

(β) What is the probability that he wins on his first, second or third throw ?

QUESTION 8. *(Start a new page)* **MARKS**

A rectangle with sides parallel to the x and y axes is inscribed inside the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ so that all its vertices are lying on the ellipse.



(a) Estimate the area of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ **5**

(correct to 2 decimal places) using :

(i) the trapezoidal Rule with 5 ordinates

(ii) Simpson's Rule with 5 ordinates .

(Note for comparison : exact area of this ellipse is $A = 6\pi$ units^2)

(b) (i) Show that the area of the rectangle inscribed inside the ellipse **7**

is given by $A=\frac{4x}{3}\sqrt{36-4x^2}$.

(ii) Find the maximum possible area of the rectangle.

STANDARD INTEGRALS

$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$

$\int \frac{1}{x} \, dx = \ln x + C, \quad x > 0$

$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C, \quad a \neq 0$

$\int \cos ax \, dx = \frac{1}{a} \sin ax + C, \quad a \neq 0$

$\int \sin ax \, dx = -\frac{1}{a} \cos ax + C, \quad a \neq 0$

$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C, \quad a \neq 0$

$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax + C, \quad a \neq 0$

$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C, \quad a \neq 0$

$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a} + C, \quad a > 0, \quad -a < x < a$

$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right) + C, \quad x > a > 0$

$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$

NOTE : $\ln x = \log_e x, \quad x > 0$

QUESTION 1.

(a) $\sqrt[4]{2.8385} = 1.29799262...$

≈ 1.2980 (to 5 sig. fig.) Aw. 2

(b) $\frac{3}{4} - \frac{2x-1}{3} = \frac{9-4(2x-1)}{12}$

$= \frac{13-8x}{12}$ Aw. 2

(c) $y = 5x^8 + 14x - 2$

$\frac{dy}{dx} = 40x^7 + 14$ Aw. 2

(d) $4x^3 - 4x^2 - 9x + 9$

$= 4x^2(x-1) - 9(x-1)$

$= (4x^2 - 9)(x-1)$

$= (2x-3)(2x+3)(x-1)$ Aw. 2

(e) $\frac{3}{2+\sqrt{3}} \times \frac{2-\sqrt{3}}{2-\sqrt{3}} = \frac{6-3\sqrt{3}}{4-3}$

$= 6 - 3\sqrt{3}$ Aw. 2

(f)

$2 - 5x > 9$

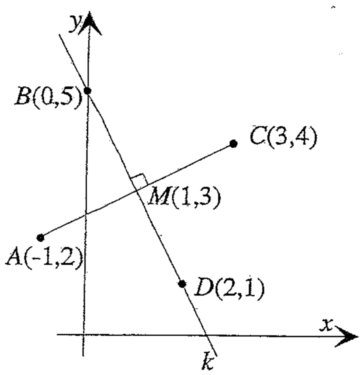
$-5x > 7$

$5x < -7$

$x < -\frac{7}{5}$ Aw. 2

QUESTION 2.

(a)



Aw. 1

(b) $m_{AC} = \frac{4-2}{3+1} = \frac{1}{2}$ Aw. 1

(c) Mid-point of AC : $M(x, y)$

$x = \frac{x_A + x_C}{2} = \frac{-1+3}{2} = 1$

$y = \frac{y_A + y_C}{2} = \frac{2+4}{2} = 3$ Aw. 1

(d) $AC = \sqrt{(x_C - x_A)^2 + (y_C - y_A)^2}$

$= \sqrt{(3+1)^2 + (4-2)^2}$

$= \sqrt{20}$ units. Aw. 1

(e) Gradient of k: $m_2 = \frac{-1}{1/2} = -2$

Equation of k:

$y - y_M = m_2(x - x_M)$

$y - 3 = -2(x-1)$

$y - 3 = -2x + 2$ Aw. 2

$2x + y - 5 = 0$

(f) $x = 0$ and $2x + y - 5 = 0$

$\therefore y - 5 = 0 \Rightarrow y = 5$

Hence $B(0,5)$ Aw. 1

(g) $x = \frac{x_B + x_D}{2}$ $y = \frac{y_B + y_D}{2}$

$1 = \frac{0 + x_D}{2}$ $3 = \frac{5 + y_D}{2}$

$x_D = 2$ $y_D = 1$ hence $D(2,1)$ Aw. 1

(h) $BD = \sqrt{(x_D - x_B)^2 + (y_D - y_B)^2}$

$= \sqrt{(2-0)^2 + (1-5)^2}$

$= \sqrt{20}$ units . Aw. 1

(i) Since diagonals of $ABCD$ are equal length and are bisecting each other at right angles, $ABCD$ is a square. Aw. 1

(j) Area $ABCD = \frac{AC \times BD}{2}$

$= \frac{\sqrt{20} \times \sqrt{20}}{2}$

$= 10$ units^2. Aw. 2

QUESTION 3.

(a) (i) $y = \left(x^3+8x-4\right)^{24}$

$\frac{dy}{dx} = 24\left(x^3+8x-4\right)^{23}\left(3x^2+8\right)$ Aw. 2

(ii) $y = \frac{7-x}{3x+1}$

$$\frac{dy}{dx} = \frac{(3x+1)\cdot\frac{d}{dx}(7-x)-(7-x)\cdot\frac{d}{dx}(3x+1)}{(3x+1)^2}$$
$$= \frac{(3x+1)\cdot(-1)-(7-x)\cdot(3x+1)}{(3x+1)^2}$$
$$= \frac{-3x-1-21+3x}{9x^2+6x+1}$$
$$= \frac{-22}{(3x+1)^2}$$ Aw. 2

(iii) $y = (2x+5)^{1/2}$

$$\frac{dy}{dx} = \frac{1}{2}(2x+5)^{-1/2}\times 2$$
$$= (2x+5)^{-1/2}$$
$$= \frac{1}{\sqrt{2x+5}}$$ Aw. 2

(b) (i) $f(x) = 2x^3+3x^2-12x$

$f'(x) = 6x^2+6x-12$

For tangents to be parallel to the x axis :

$$f'(x) = 0$$
$$6x^2+6x-12 = 0$$
$$x^2+x-2 = 0$$
$$(x+2)(x-1) = 0$$ Aw. 2

Case 1 : Point P :

$x=-2,\ y=-16+12+24=20$

$\therefore P(-2,20)$

Case 2 : Point S :

$x=1,\ y=2+3-12=-7$

$\therefore S(1,-7)$ Aw. 2

(ii) $\tan\theta = m = f'(0)$

$m = 0+0-12=-12$

$\tan\theta = -12$

$\theta = 94^\circ 46'$ Aw. 2

QUESTION 4.

(a) For $5x^2-2x-3=0$:

(i) $\alpha+\beta = -\frac{b}{a} = \frac{2}{5}$ Aw. 1

(ii) $\frac{1}{\alpha}+\frac{1}{\beta} = -\frac{\beta+\alpha}{\alpha\beta}$

$$= \frac{-\frac{b}{a}}{\frac{c}{a}} = \frac{\frac{2}{5}}{-\frac{3}{5}} = -\frac{2}{3}$$ Aw. 2

(b)

(i) Cosine Rule :

$c^2 = 4^2+4^2-2\times4\times4\times\cos72^\circ$

$c^2 = 22.111$

$c = 4.70$ cm

Perimeter of the pentagon

$P = 5\times4.70 = 23.5$ cm Aw. 3

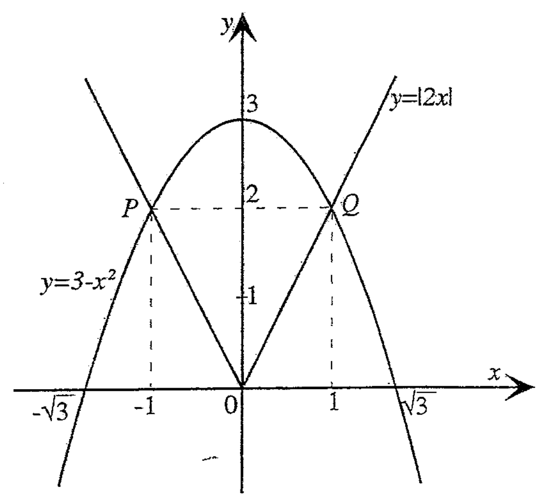
(ii) Area of $\Delta ABC = \frac{1}{2}\times4\times4\times\sin72^\circ$

$= 7.60845\ \text{cm}^2$

Area of the pentagon

$A = 5\times7.60845 = 38.04\ \text{cm}^2$ Aw. 2

(c) (i), (ii)



Aw. 2 + 1

(iii) From the sketch : $P(-1,2)$ & $Q(1,2)$ Aw. 1

QUESTION 5.

(a) $y = 3 - \dfrac{6}{x^2+1}$

(i) $\lim\limits_{x \to -\infty}\left(3 - \dfrac{6}{x^2+1}\right) = 3 - 0 = 3$

$\lim\limits_{x \to \infty}\left(3 - \dfrac{6}{x^2+1}\right) = 3 - 0 = 3$ Aw. 2

(ii) $D = \{x : \text{all real } x\}$

$R = \{y : -3 \le y < 3\}$ Aw. 2

(iii) $f(x) = 3 - \dfrac{6}{x^2+1}$

$f(-x) = 3 - \dfrac{6}{(-x)^2+1} = 3 - \dfrac{6}{x^2+1}$

Since $f(x) = f(-x)$ the function is even. Aw. 2

(iv) The shaded region is defined by :

$\left\{(x, y) : y \ge 3 - \dfrac{6}{x^2+1} \cap y \le 0\right\}$ Aw. 2

(b) $y = ax^2 + bx + c$

For $P(0,-3)$: $-3 = 0 + 0 + c$ - ①

For $Q(1,4)$: $4 = a + b + c$ - ②

For $R(-1.-6)$: $-6 = a - b + c$ - ③

From ① : $c = -3$ - ④

④ into ② : $a + b - 3 = 4$ - ⑤

④ into ③ : $a - b - 3 = -6$ - ⑥

⑤ : $a + b = 7$

⑥ : $a - b = -3$

⑤ + ⑥ : $2a = 4$

$a = 2$

⑤ - ⑥ : $2b = 10$

$b = 5$ Aw. 4

Hence the quadratic polynomial passing through the points P, Q and R has equation $y = 2x^2 + 5x - 3$.

QUESTION 6.

(a) LHS $= \dfrac{\sec\theta}{\tan\theta + \cot\theta}$

$$= \frac{\frac{1}{\cos\theta}}{\frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}}$$

$$= \frac{\frac{1}{\cos\theta}}{\frac{\sin^2\theta + \cos^2\theta}{\sin\theta\cos\theta}}$$

$$= \frac{\frac{1}{\cos\theta}}{\frac{1}{\sin\theta\cos\theta}}$$

$$= \frac{\sin\theta\cos\theta}{\cos\theta}$$

$= \sin\theta$ = RHS Aw. 3

(b) $\displaystyle\int_1^5 \sqrt{3x-2}\,dx = \int_1^5 (3x-2)^{1/2}\,dx$

$$= \left[\frac{(3x-2)^{3/2}}{3 \times \frac{3}{2}}\right]_1^6$$

$$= \frac{2}{9}\left(16^{3/2} - 1^{3/2}\right)$$

$$= \frac{2}{9}(64 - 1)$$

$= 14$ Aw. 2

(c) $A = \displaystyle\int_0^3 \left[(3x - x^2) - (x^3 - 4x^2 + 3x)\right]dx$

$$= \int_0^3 \left[-x^3 + 3x^2\right]dx$$

$$= \left[-\frac{x^4}{4} + x^3\right]_0^3$$

$$= \left(-\frac{81}{4} + 27\right) - \left(\frac{0}{4} + 0\right)$$

$= \dfrac{27}{4}$ units^2 Aw. 4

(d)

$$V = \pi\int_1^8 \left(x^{\frac{1}{3}}\right)^2 dx$$

$$= \pi\int_1^8 x^{\frac{2}{3}}\, dx$$

$$= \pi\left[\frac{x^{\frac{5}{3}}}{\frac{5}{3}}\right]_1^8$$

$$= \frac{3\pi}{5}\left[\sqrt[3]{x^5}\right]_1^8$$

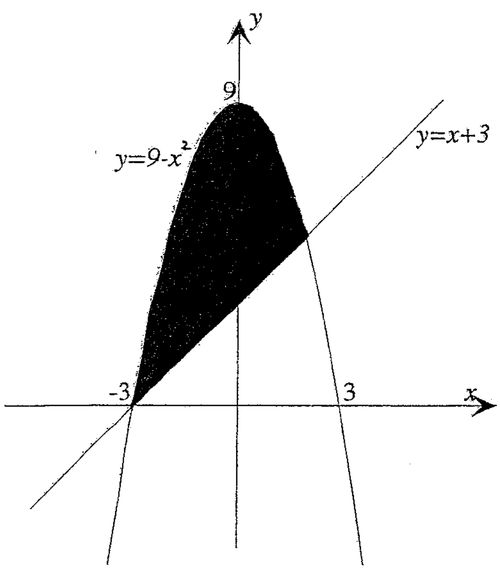
$$= \frac{3\pi}{5}\left(2^5 - 1^5\right)$$

$$= \frac{93\pi}{5} \text{ units}^3$$

Aw. 3

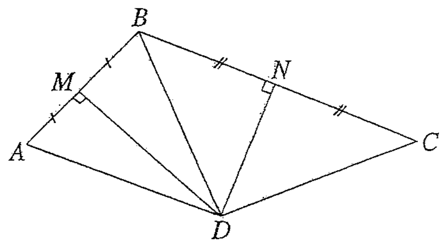
QUESTION 7.

(a)



Aw. 3

(b)



(i) $AM = BM$ (given)

$\angle AMD = \angle BMD = 90^\circ$ (given)

MD is common to both triangles

$\therefore \Delta ADM \equiv \Delta BDM$ (SAS)

(ii) $AD = BD$ (corresponding sides in congruent triangles)

(iii) Similarly :

$\Delta BND \equiv \Delta CND$ (SAS)

$BD = CD$ (corresponding sides in congruent triangles)

Hence $AD=BD=CD$ indicates that a circle with the centre at D passes through the points A, B and C.

Aw. 5

(c) (i) Sample space

	1	2	3	4	5	6
1	0	1	2	3	4	5
2	1	0	1	2	3	4
3	2	1	0	1	2	3
4	3	2	1	0	1	2
5	4	3	2	1	0	1
6	5	4	3	2	1	0

Aw. 1

(ii) (α) $P(G) = \frac{10}{36} = \frac{5}{18}$

Aw. 1

(β) $P(G) + P(\overline{G}, \overline{S}, G) + P(\overline{G}, \overline{S}, \overline{G}, \overline{S}, G)$

$$= \frac{5}{18} + \left(\frac{13}{18}\right)^2 \times \frac{5}{18} + \left(\frac{13}{18}\right)^4 \times \frac{5}{18}$$

≈ 0.4982

Aw. 2

QUESTION 8.

(a)

x	-3	-1.5	0	1.5	3
y	0	1.732	2	1.732	0

Aw. 1

(i) Trapezoidal Rule :

$$I \approx \frac{h}{2}\left[f(-3)+2f(-1.5)+2f(0)+2f(1.5)+f(3)\right]$$

$$\approx \frac{1.5}{2}\left[0+2\times1.73+2\times2+2\times1.73+0\right]$$

$$\approx 8.19$$

Area of ellipse $\approx 2\times I \approx 16.38$ units^2 Aw. 2

(ii) Simpson's Rule :

$$I \approx \frac{h}{3}\left[f(-3)+4f(-1.5)+2f(0)+4f(1.5)+f(3)\right]$$

$$\approx \frac{1.5}{3}\left[0+4\times1.73+2\times2+4\times1.73+0\right]$$

$$\approx 8.92$$

Area of ellipse $\approx 2\times I \approx 17.84$ units^2 Aw. 2

(b) (i)

$$\frac{x^2}{9}+\frac{y^2}{4} = 1$$

$$\frac{y^2}{4} = 1-\frac{x^2}{9}$$

$$y = \pm\sqrt{4-\frac{4x^2}{9}}$$

Area

$$A = 2x\cdot 2y$$

$$= 4xy$$

$$= 4x\sqrt{4-\frac{4x^2}{9}}$$

$$= 4x\sqrt{\frac{36-4x^2}{9}}$$

$$= \frac{4x}{3}\sqrt{36-4x^2}$$

Aw. 2

(ii) $A = \frac{4x}{3}\cdot\left(36-4x^2\right)^{\frac{1}{2}}$

$$\frac{dA}{dx} = \left(36-4x^2\right)^{\frac{1}{2}}\cdot\frac{d}{dx}\left(\frac{4x}{3}\right)+\frac{4x}{3}\cdot\frac{d}{dx}\left[\left(36-4x^2\right)^{\frac{1}{2}}\right]$$

$$= \left(36-4x^2\right)^{\frac{1}{2}}\cdot\left(\frac{4}{3}\right)+\left(\frac{4x}{3}\right)\cdot\left[\frac{1}{2}\left(36-4x^2\right)^{-\frac{1}{2}}\cdot(-8x)\right]$$

$$= \left(\frac{4}{3}\right)\sqrt{36-4x^2}+\left(\frac{4x}{3}\right)\cdot\left(\frac{-4x}{\sqrt{36-4x^2}}\right)$$

$$= \frac{4\sqrt{36-4x^2}}{3}-\frac{16x^2}{3\sqrt{36-4x^2}}$$

For stationary points $\frac{dA}{dx} = 0$:

$$\frac{4\sqrt{36-4x^2}}{3}-\frac{16x^2}{3\sqrt{36-4x^2}} = 0$$

$$\frac{4\sqrt{36-4x^2}}{3} = \frac{16x^2}{3\sqrt{36-4x^2}}$$

$$48x^2 = 12\left(36-4x^2\right)$$

$$x^2 = \frac{9}{2}$$

$$x = \frac{3}{\sqrt{2}}$$

Nature of the stationary point at $x = \frac{3}{\sqrt{2}}$:

When $x < \frac{3}{\sqrt{2}}$, say 1, $\frac{dA}{dx} = 6.6 > 0$

When $x > \frac{3}{\sqrt{2}}$, say $\frac{5}{2}$, $\frac{dA}{dx} = -5.628 < 0$

$\therefore$ a maximum stationary point when $x = \frac{3}{\sqrt{2}}$.

Hence maximum area :

$$A_{max} = \frac{4\times\frac{3}{\sqrt{2}}}{3}\times\sqrt{36-4\times\left(\frac{9}{2}\right)}$$

$$A_{max} = \frac{4}{\sqrt{2}}\times\sqrt{18}$$

$$A_{max} = 12 \text{ units}^2$$

Aw. 5