SYDNEY TECHNICAL HIGH SCHOOL

YEAR 12 ASSESSMENT TASK 3

JUNE 2003

MATHEMATICS

Extension 1

Time Allowed: 70 minutes

Instructions:

- Attempt all questions
- Start each question on a new page
- Show all necessary working
- Marks may be deducted for careless or badly arranged work
- Approved calculators may be used
- Standard integrals are attached and may be removed for your convenience.

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Teacher: Mr Sender

Question 1	Question 2	Question 3	Question 4	Question 5	Question 6	Total
9	8	6	10	9	10	52

10 9 9 10 9 10

Question 1

a) Consider the function $f(x) = 2\tan^{-1} x$.

(i) Evaluate $f(\sqrt{3})$ **(1)**

(ii) Draw the graph of $y = f(x)$, labelling any key features **(2)**

(iii) Find the slope of the curve at the point where it cuts the y axis. **(2)**

b) (i) $\int \frac{dy}{\sqrt{16 - y^2}}$ **(1)**

(ii) $\int \frac{dt}{5 + t^2}$ **(2)**

c) By letting $A = \cos^{-1}(\frac{3}{7})$, find the exact value of sin A. **(2)**

Question 2

a) Differentiate

$e^{2x} \ln x$ **(2)**

b) Evaluate $\int_0^1 \frac{2x}{(2x+1)^2}\, dx$ by using the substitution $v = 2x + 1$ **(4)**

c) Solve for x in exact form:

$e^{2x} - 3e^x + 2 = 0$ **(3)**

Question 3

a) The polynomial $P(x) = x^4 - 3x^3 + ax^2 + bx - 6$ leaves a remainder of 8 when divided by $x + 1$. If $x - 3$ is a factor for $P(x)$, find the values of a and b. **(3)**

b) Two of the roots of the equation $x^3 + ax^2 + b = 0$ are reciprocals of each other (a, b both real).

(i) Show that the third root is equal to $-b$. **(1)**

(ii) Show that $a = b - \frac{1}{b}$ $\quad b^2 - 1 = ab.$ **(2)**

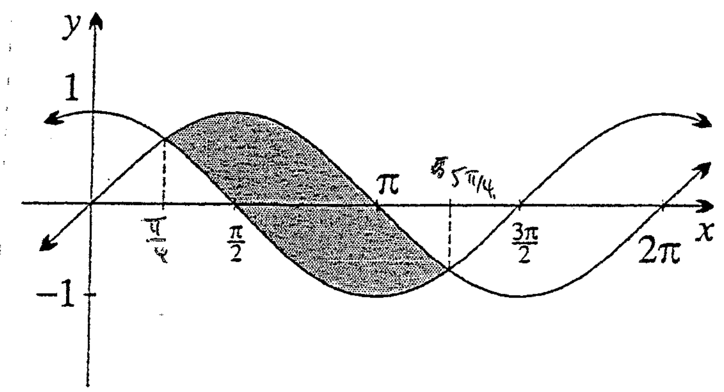
(iii) Show that the two roots, which are reciprocals, will be real if $-\frac{1}{2} \le b \le \frac{1}{2}$. $\quad b^2 - \frac{1}{4}.$ **(3)**

Question 4

a) Write a result for $\sin^2 3x$ in terms of $\cos 6x$. **(1)**

Hence evaluate $\int_0^{\frac{\pi}{2}} \sin^2 3x\,dx$ **(2)**

b) Find the area of the shaded region below: **(4)**



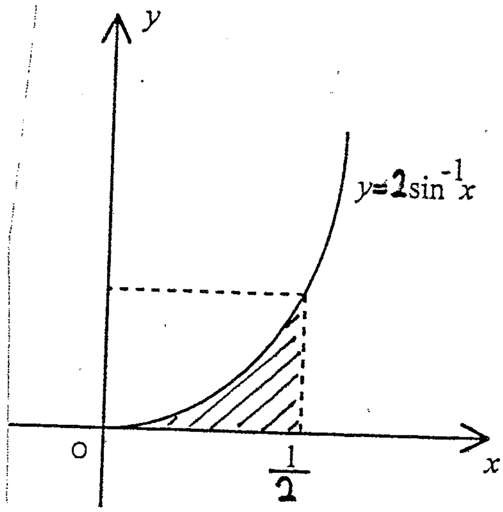
c) Solve $\log_e(x+1) + \log_e x = \log_e 2$ **(3)**

Question 5

a) (i) Find the domain of $y = \sin\ (\sin^{-1} x)$ **(1)**

(ii) Hence sketch the function **(1)**

b)



The sketch shows the graph of the curve $y = f(x)$ where $f(x) = 2\sin^{-1} x$.

The area under the curve for $0 < x < \frac{1}{2}$ is shaded.

(i) Find $f(\frac{1}{2})$. **(1)**

(ii) Find $f^{-1}(x)$ **(1)**

(iii) Hence or otherwise, calculate the area of the shaded region. **(3)**

c) Give the general solution to the equation $\tan\vartheta = 1$ (ϑ in radians) **(2)**

Question 6

a) (i) Differentiate $y = \cot 2x$ **(2)**

(ii) Hence or otherwise find $\int \operatorname{cosec}^2 2x\, dx$ **(2)**

b) Consider the function $y = \cos x - \dfrac{1}{4\sqrt{3}\sin x}$

(i) Verify that $\dfrac{dy}{dx} = 0$ when $x = \dfrac{\pi}{6}$ **(3)**

(ii) Sketch the curve over the domain **(3)**

$0 < x \le \dfrac{\pi}{2}$, given that $\dfrac{d^2y}{dx^2} < 0$.

On your sketch, write the coordinates of the turning point in exact form and label the asymptote (the x intercepts are not required).

EXT 1 H.S.C. TASK 3 2003

Question 1

a) i) $f(\sqrt{3}) = 2\tan^{-1}\sqrt{3}$

$= 2 \times \frac{\pi}{3}$

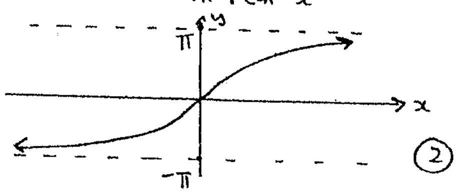
$= \frac{2\pi}{3}$ — 1

ii)

$-\frac{\pi}{2} < \frac{y}{2} < \frac{\pi}{2}$

∴ Range $-\pi < y < \pi$

Domain all real x



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iii) $f'(x) = 2\left(\frac{1}{1+x^2}\right)$ — 1

$y=0 \quad x=0$

$\therefore f'(0) = 2$

∴ slope at $(0,0)$ $m = 2$ — 1

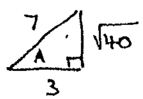
b)

i) $\int \frac{dy}{\sqrt{16-y^2}} = \sin^{-1}\frac{y}{4} + c$ — 1

ii) $\int \frac{dt}{5+t^2} = \frac{1}{\sqrt{5}}\tan^{-1}\frac{t}{\sqrt{5}} + c$ — 2

c) $A = \cos^{-1}\frac{3}{7}$

$\cos A = \frac{3}{7}$



$\therefore \sin A = \frac{\sqrt{40}}{7}$ 2

Question 2

a) Let $u = e^{2x}$ $\quad v = \ln x$

$u' = 2e^{2x}$ $\quad v' = \frac{1}{x}$

$\therefore \frac{d}{dx}(e^{2x}.\ln x) = 2e^{2x}\ln x + \frac{e^{2x}}{x}$

$= e^{2x}\left(2\ln x + \frac{1}{x}\right)$

b) $v = 2x+1$

∴ if $x = 1$ $\quad v = 3$

$x = 0$ $\quad v = 1$

$\frac{dv}{dx} = 2$ $\quad dv = 2\,dx$

$\therefore dx = \frac{dv}{2}$

$\int_0^1 \frac{2x}{(2x+1)^2}\,dx = \int_1^3 \frac{v-1}{v^2}\cdot\frac{dv}{2}$

$= \frac{1}{2}\int_1^3 (v-1)v^{-2}\,dv$

$= \frac{1}{2}\int_1^3 (v^{-1} - v^{-2})\,dv$

$= \frac{1}{2}\left[\ln v + \frac{1}{v}\right]_1^3$

$= \frac{1}{2}\left[\ln 3 + \frac{1}{3} - \cancel{\ln 1} - 1\right]$

$= \frac{1}{2}\left[\ln 3 - \frac{2}{3}\right]$

c) Let $u = e^x$

$u^2 - 3u + 2 = 0$

$(u-2)(u-1) = 0$

$u = 2 \quad u = 1$

$e^x = 2 \quad e^x = 1$

$x = \ln 2 \quad x = 0$

Question 3

a) $P(-1) = 8$ ∴ $1 + 3 + a - b - 6 = 8$

$P(3) = 0$ ∴ $\cancel{81} - \cancel{81} + 9a + 3b - 6 = 0$

$a - b = 10$ — 1

$9a + 3b = 6$ — 2

$\therefore 3a + b = 2$ — 3

add 1, 3 $\quad 4a = 12$

$a = 3 \quad \therefore b = -7$

b) Root $\alpha, \frac{1}{\alpha}, \beta$, coef are

$A = 1 \quad B = a \quad C = 0 \quad D = b$

i) Roots 3 at time $\alpha.\frac{1}{\alpha}.\beta = -\frac{b}{1}$

∴ third root $\beta = -b$

ii) ∴ Roots now $\alpha, \frac{1}{\alpha}, -b$

Sum roots 2 at time

$1 - \alpha b - \frac{b}{\alpha} = 0$ — 1

$1 - b(\alpha + \frac{1}{\alpha}) = 0$ *

Sum roots 1 at time

$\alpha + \frac{1}{\alpha} - b = -a$ — 2

$\alpha + \frac{1}{\alpha} = b - a$ sub into *

$1 - b(b-a) = 0$

$1 - b^2 + ab = 0$

$ab = b^2 - 1$

ie $a = b - \frac{1}{b}$

iii) $-b$ a root ∴ $(x+b)$ factor

$\therefore x^3 + ax^2 + b = (x+b)(x^2 + Px + Q)$

$= x^3 + Px^2 + xQ + bx^2 + bPx + bQ$

equate coef $\quad b = bQ \quad \therefore Q = 1$ (const)

$Q + Pb = 0 \quad \therefore P = -\frac{1}{b}$ (coef x)

∴ α and $\frac{1}{\alpha}$ are roots of $x^2 - \frac{1}{b}x + 1 = 0$

$\Delta = \frac{1}{b^2} - 4$ if roots real $\Delta \geq 0$

$\therefore -1/2 \leq b \leq 1/2$

Question 4

a) $\sin^2 3x = \frac{1}{2}(1 - \cos 6x)$

$\int_0^{\pi/2} \sin^2 3x\,dx = \int_0^{\pi/2} \frac{1}{2}(1 - \cos 6x)\,dx$

$= \frac{1}{2}\left[x - \frac{1}{6}\sin 6x\right]_0^{\pi/2}$

$= \frac{1}{2}\left[\frac{\pi}{2} - \frac{1}{6}\cancel{\sin 3\pi}\right]$

$= \frac{\pi}{4}$

b) $A = \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx$

$A = \left[-\cos x - \sin x\right]_{\pi/4}^{5\pi/4}$

$= \left[-\cos\frac{5\pi}{4} - \sin\frac{5\pi}{4}\right] - \left[-\cos\frac{\pi}{4} - \sin\frac{\pi}{4}\right]$

$= \left[\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right] - \left[-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right]$

$= \frac{4}{\sqrt{2}}$ unit^2

c) $\log_e(x+1) + \log_e x = \log_e 2$

$\log_e(x(x+1)) = \log_e 2$

$x^2 + x = 2$

$x^2 + x - 2 = 0$

$(x-1)(x+2) = 0$

$x = 1$ only solution

since domain $x > 0$

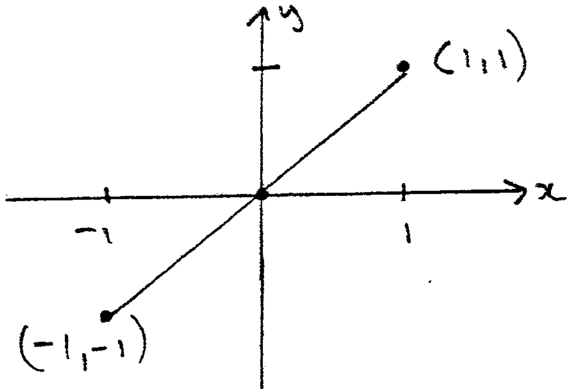
Question 5

a) i) $y = \sin(\sin^{-1} x)$

$-\pi/2 \le \sin^{-1} x \le \pi/2$

Domain: $-1 \le x \le 1$

ii)

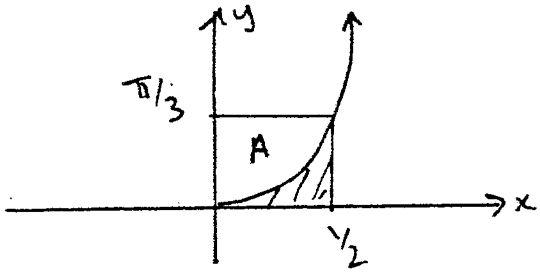


b) i) $f(\tfrac{1}{2}) = 2\sin^{-1}(\tfrac{1}{2})$

$= \dfrac{\pi}{3}$

ii) $f'(x) = \dfrac{2}{\sqrt{1-x^2}}$

$f'(\tfrac{1}{2}) = \dfrac{2}{\sqrt{1-\frac{1}{4}}} = \dfrac{4}{\sqrt{3}}$



Required area is

$= \text{Rect}\left(\tfrac{1}{2} \times \tfrac{\pi}{3}\right) - \text{area } A$

$$A = \int_0^{\pi/3} \sin\frac{y}{2}\, dy$$

$$A = \left[-2\cos\frac{y}{2}\right]_0^{\pi/3}$$

$$A = \left[-2\cos\frac{\pi}{6}\right] - \left[-2 \times 1\right]$$

$= (-\sqrt{3} + 2)$ unit^2

∴ Required Area

$= \dfrac{\pi}{6} - (-\sqrt{3} + 2)$

$= \dfrac{\pi}{6} + \sqrt{3} - 2$

c) $\tan\theta = 1$

$\therefore \theta = n\pi + \tan^{-1}(1)$

$\theta = n\pi + \dfrac{\pi}{4}$

n an integer

Question 6

a) i) $y = \cot 2x = \dfrac{\cos 2x}{\sin 2x}$

$u = \cos 2x \qquad v = \sin 2x$

$u' = -2\sin 2x \qquad v' = 2\cos 2x$

$\dfrac{d}{dx}(\cot 2x) = \dfrac{-2\sin^2 2x - 2\cos^2 2x}{\sin^2 2x}$

$= -2\,\text{cosec}^2 2x$

$\therefore \int \text{cosec}^2 2x\, dx = -\dfrac{1}{2}\cot 2x + c$

b) $y = \cos x - \dfrac{1}{4\sqrt{3}}(\sin x)^{-1}$

$y' = -\sin x + \dfrac{1}{4\sqrt{3}}(\sin x)^{-2}.\cos x$

if $x = \dfrac{\pi}{6}$ $\quad y' = -\sin\dfrac{\pi}{6} + \dfrac{1}{4\sqrt{3}}\left(\sin\dfrac{\pi}{6}\right)^{-2} \times \cos\dfrac{\pi}{6}$

$y' = -\dfrac{1}{2} + \dfrac{1}{2} = 0$

as required