STUDENT NAME: ____________________

STUDENT NUMBER: ____________________

Circle your teacher's name Peter Mayne Wayne Lagesen

TAYLORS COLLEGE SYDNEY CAMPUS

HSC ASSESSMENT TASK

MATHEMATICS EXTENSION 1

JUNE 2006

Time Allowed: 1 ½ hours

Weighting: $\frac{15}{50}$

INSTRUCTIONS

- START EACH QUESTION IN A NEW ANSWER BOOKLET
- CIRCLE YOUR TEACHER'S NAME ON THE QUESTION PAPER
- WRITE YOUR NAME, STUDENT NUMBER AND TEACHER'S NAME AT THE TOP OF EACH ANSWER BOOKLET
- SHOW ALL NECESSARY WORKING
- APPROVED TEMPLATES AND CALCULATORS MAY BE USED

QUESTION 1	Marks

(a) Differentiate $y = \sin^{-1} 3x$ — 2

(b) Write down the general solution of $2\cos x = -\sqrt{3}$ — 2

(c) Use the Table of Standard Integrals to find the exact value of $\int_0^3 \frac{1}{\sqrt{x^2+4}}\,dx$ — 3

(d) Write down the first 3 terms (in ascending powers of x) of $(2-x)^5$ in simplest form. — 2

(e) Find the size of the acute angle between the line $3x + y + 2 = 0$ and the $y-$axis. Give your answer to the nearest degree. — 2

QUESTION 2 (START A NEW ANSWER BOOKLET)

(a) Prove $\cot A - \tan A = 2\cot 2A$ — 2

(b) (i) Sketch $y = \cos^{-1}\frac{x}{3}$ showing all main features — 2

(ii) Show that $\cos^{-1}\frac{x}{3} = x$ has a solution between $x = 1$ and $x = 2$ — 2

(iii) Taking $x = 1.5$ as a first approximation, use one application of Newton's Method to find a better approximation — 3

(c) (i) Use mathematical induction to prove — 4

$$\sum_{r=1}^{n} r \times r! = (n+1)! - 1$$

(ii) Find the smallest value of n such that $1\times 1! + 2\times 2! + 3\times 3! \ldots\ldots + n\times n! > 10^{21}$ — 1

QUESTION 3 (START A NEW ANSWER BOOKLET) **Marks**

(a) (α) Decide whether the statements below are: 4

ALWAYS TRUE, NEVER TRUE, SOMETIMES TRUE

For example,

$x+1>x$ is ALWAYS TRUE

$x+1<x$ is NEVER TRUE

$x^2>x$ is SOMETIMES TRUE (TRUE if $x=2$, FALSE if $x=\frac{1}{2}$)

If you think STATEMENT 1 below is ALWAYS TRUE write (i) = ALWAYS TRUE
ON YOUR ANSWER SHEET

A (i) STATEMENT 1

$(x+\frac{1}{x})^{2n}$ (for integer $n \geq 1$) has a term independent of x

S (ii) STATEMENT 2

A polynomial of odd degree is an odd function

S (iii) STATEMENT 3

The greatest coefficient in the expansion of $(1+2x)^n$ (for integer $n \geq 1$) is the coefficient of x^n

S (iv) STATEMENT 4

A polynomial (with real coefficients) of degree n has n real zeros

(β) Choose ONE of the Statements from above and JUSTIFY your answer. 2
That is, if you think ALWAYS TRUE or NEVER TRUE you must PROVE your result.
If you think SOMETIMES TRUE give an example where the statement is TRUE and an example where the statement is FALSE

(b) When a dishwasher has completed its cycle, and the contents removed, metallic utensils cool such that $\frac{dT}{dt}=-k(T-M)$ where T is the temperature of the utensils, M is the temperature of the room, and t is the time in minutes after the utensils have been removed from the dishwasher. **Marks**

(i) Show that $T=M+Ae^{-kt}$ is a solution of $\frac{dT}{dt}=-k(T-M)$ 1

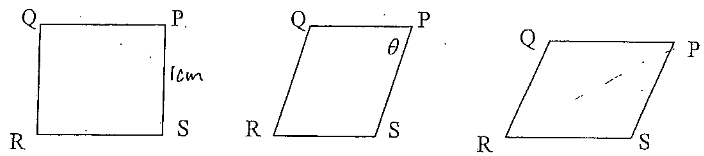
(ii) The utensils cool from 90° to 60° in 3 minutes in a room of temperature 18°. Find A and find k correct to 2 decimal places. 3

(iii) If the utensils can be safely handled when their temperature is 50° find how long it is (to the nearest minute) after the utensils are taken from the dishwasher that they can be safely handled 2

QUESTION 4 (START A NEW ANSWER BOOKLET)

(a) Find $\int \cos^2 3x\, dx$ 2

(b) A square PQRS of side 1 cm is being "pushed over" to form a rhombus. The angle at P, θ, decreases at a constant rate of 0.1 radians/s



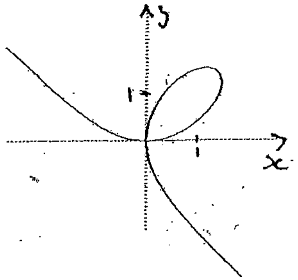
(i) Show that the area of the rhombus is given by $A=\sin\theta$ 1

(ii) At what rate is A decreasing when $\theta=\frac{\pi}{6}$ 2

(iii) At what rate is the "shorter" diagonal, QS, decreasing when $\theta=\frac{\pi}{6}$ 3

Marks

(c) 1



Which of the following sets of parametric equations could be used to describe the curve above. Write the ANSWER ONLY in your answer booklet. FOR EXAMPLE if you think the answer is A, write A in your answer booklet.

A) $x = \cos^3 t, y = \sin^3 t$ B) $x = 1 + e^{t}, y = t - e^{-t}$

C) $x = t^2 + t^3, y = 1 + t + t^3$ D) $x = \dfrac{3t}{1+t^3}, y = \dfrac{-6t}{1+t^3}$

E) NONE OF THE ABOVE

QUESTION 5 (START A NEW ANSWER BOOKLET)

(a) Evaluate exactly $\displaystyle\int_0^1 \frac{1}{\sqrt{2-x^2}}dx$ 2

(b) For the function $f(x) = \dfrac{1}{x^2 - 4x + 5} \neq 0$.

(i) Show that $y = f(x)$ has a local maximum at $(2,1)$ 3

(ii) Sketch the curve $y = f(x)$ 2

(iii) State the largest positive domain for which the function has an inverse 1

(iv) Find the inverse function and state its domain and range 2

(v) Sketch $y = f^{-1}(x)$ 2

(vi) If a is a real number NOT in the domain found in part (iii), find $f^{-1}(f(a))$ 1

Questn 1

EXCELLENT

(a) $y = \sin^{-1} 3x$

$\frac{dy}{dx} = \frac{3}{\sqrt{1-9x^2}}$ ✓

(b) $2\cos x = -\sqrt{3}$

$\cos x = \frac{-\sqrt{3}}{2}$ ✓

$x = 2n\pi \pm \frac{5\pi}{6}$ ✓

(c) $\int_0^3 \frac{1}{\sqrt{x^2+4}}\,dx = \left[\ln\left(x + \sqrt{x^2+4}\right)\right]_0^3$ ✓

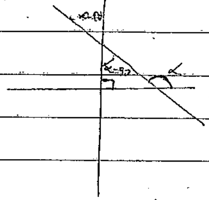
$= \left[\ln(3+\sqrt{13}) - \ln 2\right]$ ✓

~~$= \ln\left(\frac{3+\sqrt{13}}{2}\right)$~~

d) $(2-x)^5 = 2^5 x^0 - {}^5C_1 2^4 x^1 + {}^5C_2 2^3 x^2 + \ldots$ ✓

$= 32x^0 - 80x^1 + 80x^2$ ✓

e)



$\tan\alpha = m$

$= -3$

$\therefore \alpha = \tan^{-1} -3$

acute agl $\Rightarrow$ $\alpha - 90 = \tan^{-1}(-3) - 90$

$\approx 18°26'$ ✓

Question 2

a) $LHS = \cot A - \tan A$

$= \frac{1}{\tan A} - \tan A$

$= \frac{1 - \tan^2 A}{\tan A}$ ✓

Qustn 2

(a) $LHS = \frac{1 - \tan^2 A}{\tan A}$

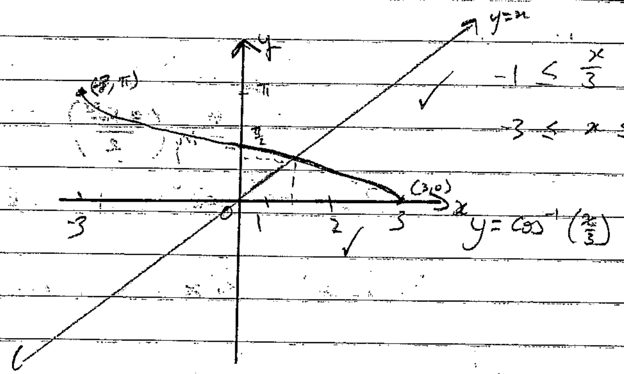
$= \frac{2(1 - \tan^2 A)}{2\tan A}$ ✓

$= 2\left(\frac{1}{\tan 2A}\right)$ ✓

$= 2\cot 2A$

$= RHS$ GOOD

b) (i)



$-1 \le \frac{x}{3} \le 1$

$-3 \le x \le 3$ ✓

(ii) let $f(x) = \cos^{-1}\left(\frac{x}{3}\right) - x$

$f(1) \approx 0.23096$ ✓ (5.d.p)

$f(2) \approx -0.15893$ (5.d.p) ✓

$\therefore$ Since $f(1)$ & $f(2)$ are opposite in sign, $f(x)$ has a root between $x = 1$ & $x = 2$ ✓

b) (iii) $f(x) = \cos^{-1}\left(\frac{x}{3}\right) - x$

$f'(x) = \frac{-1}{\sqrt{9-x^2}} - 1$ ✓

if $x = 1.5$ is a good approxml

$x_1 = 1.5 - \frac{f(1.5)}{f'(1.5)}$ ✓

≈ 1.65208 ✓ (5 d.p)

c) (i) $\sum_{r=1}^{n} r \times r! = (n+1)! - 1$

rtp for $n=1$

$LHS = \sum_{r=1}^{1} r \times r! = 1 \times 1!$

$= 1$ ✓

$RHS = (1+1)! - 1 = 2 - 1$

$= 1$ ✓

∴ LHS = RHS

True for $n=1$

assume true for $n=k$ ie. $\sum_{r=1}^{k} r \times r! = (k+1)! - 1$

rtp for $n=k+1$

$LHS = \sum_{r=1}^{k+1} r \times r! = (k+1)! - 1 + (k+1)(k+1)!$

$= (k+1)!\,(1+k+1) - 1$

$= (k+1)!\,(k+2) - 1$

$= (k+2)! - 1$ ✓

$= RHS$.

∴ true for $n=k+1$

∴ Blah Blah Blah ...

It's true.

c) (ii) $1 \times 1! + 2 \times 2! + 3 \times 3! + \dots + n \times n! > 10^{21}$

$(n+1)! - 1 > 10^{21}$ ✓

$(n+1)! > 10^{21} - 1$

By Trial & Error $n = 21$ since $22! > 10^{21}$

$\Downarrow$

1.124×10^{21}

$\cancel{\log_{10}(n+1)! > 21 - 0}$

$\log_{10}\cancel{(n+1)!} > 21$

Quest 3

a) (α) (i) Always true. ✓

(ii) Sometimes true ✓

(iii) Sometimes true ✓ (because it is true for $n=1$ but not $n=2$)

(iv) Sometimes true. ✓

(β) Statement 2

Let the polynomial be of the 3^{rd} degree.

e.g. $P(x) = x^3$ this is an odd function

$P(x) = -P(-x) = x^3$

but, polynomial of the 5^{th} degree

e.g. $P(x) = x^3(x-1)(x-2)$ this is not an odd function

$P(x) \neq -P(-x)$

Hence Statement 2 is only sometimes true.

Quest 3

b) (i) $T = M + Ae^{-kt}$ but $Ae^{kt} = T - M$

$\frac{dT}{dt} = kAe^{-kt}$ ✓

$= k(T - M)$ ✓

Hence it is proven QED

(ii) $t = 0$, $T = 90$

$\therefore 90 = 18 + Ae^{0}$

$A = 72$ ✓

$t = 3$ $T = 60$

$60 = 18 + 72e^{3k}$

$\frac{7}{12} = e^{3k}$ ✓

$\ln\left(\frac{7}{12}\right) = 3k$

$k = \frac{\ln\left(\frac{7}{12}\right)}{3}$ ✓

≈ -0.18 ✓

(iii) $50 = 18 + 72e^{kt}$

$\frac{4}{9} = e^{kt}$ /

$\ln\left(\frac{4}{9}\right) = kt$

$t \approx 4.51356$ (5 d.p) ✓

$\therefore$ 5 mins after its been taken out.

Quest 4

(a) $\int \cos^2 3x \, dx = \frac{1}{2}\int \cos 6x + 1 \, dx$

$= \frac{1}{2}\left[\frac{1}{6}\sin 6x + x\right] + c$ ✓

(b) (i) $A = A_{\Delta OPS} + A_{\Delta QRS}$

$= \frac{1}{2}\sin\theta + \frac{1}{2}\sin\theta$ ✓

$= \sin\theta$ ✓

QED.

(ii) $\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt}$

$= \cos\theta \times 0.1$ ✓

$= 0.1\cos\theta$

for $\theta = \frac{\pi}{6}$

$\frac{dA}{dt} = \frac{\sqrt{3}}{20}$ ✓ cm^2/sec

(iii) $\frac{dQS}{dt} = \frac{dQS}{d\theta} \times \frac{d\theta}{dt}$

$QS = \sqrt{2 + 2\cos\theta} = (2 + 2\cos\theta)^{\frac{1}{2}}$

$\frac{dQS}{d\theta} = \frac{1}{\cancel{2}}(2 + 2\cos\theta)^{-\frac{1}{2}}(-\cancel{2}\sin\theta)$ ✓

$= \frac{-\sin\theta}{\sqrt{2 + 2\cos\theta}}$

$\therefore \frac{dQS}{dt} = \frac{-\sin\theta}{10\sqrt{2 + 2\cos\theta}}$

at $\theta = \frac{\pi}{6}$

$$\frac{ds}{dt} = \frac{-\sin(\frac{\pi}{6})}{10\sqrt{2+2\cos(\frac{\pi}{6})}}$$

$$= \frac{-0.5}{10\sqrt{2+2(\frac{\sqrt{3}}{2})}} = \frac{-1}{20\sqrt{2+\sqrt{3}}} \text{ cm/s}$$

$$\cancel{= \frac{-20}{\sqrt{2+\sqrt{3}}} \text{ cm/sec}}$$

c) E ✓

Question 5

(a) $\int_0^1 \frac{1}{\sqrt{2-x^2}} = \left[\sin^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^1$ ✓

$= \frac{\pi}{4}$ ✓

b) (i) $f(x) = \frac{1}{x^2-4x+5} = (x^2-4x+5)^{-1}$

$f'(x) = -(x^2-4x+5)^{-2}(2x-4)$

$= \frac{-2x+4}{(x^2-4x+5)^2}$ ✓

at $x = 2$

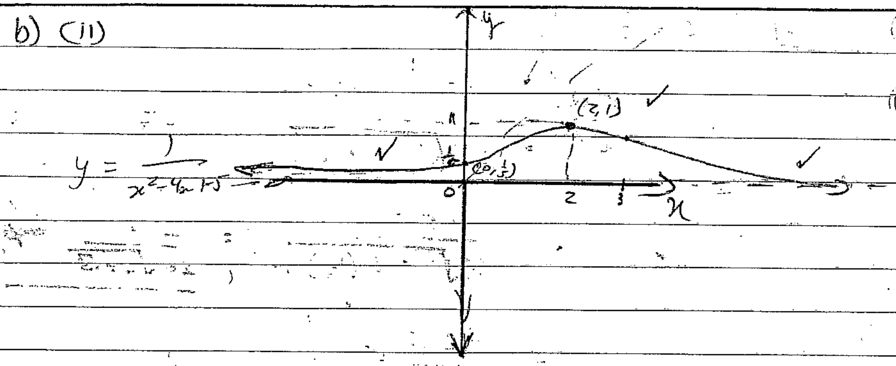
$f'(x) = 0$ ✓

∴ cond only: $-2x+4$ (since $(x^2-4x+5)^2 > 0$)

1	2	3
2	0	-2

∴ local max at (2,1)

b) (ii)



(iii) $x \geq 2$

(iv) $x = \frac{1}{y^2-4y+5}$

$\frac{1}{x} = y^2-4y+5$ ✓

$y^2 - 4y + (5 - \frac{1}{x}) = 0$

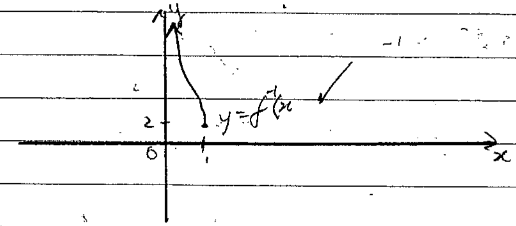
$y = \frac{4 \pm \sqrt{16-4(5-\frac{1}{x})}}{2}$ ✓

$= 2 + \sqrt{4-(5-\frac{1}{x})}$

D: $0 < x \leq 1$ ✓

R: $y \geq 2$ ✓

(v)



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Question 5

b) (vi) $f(a) = \frac{1}{a^2-4a+5}$

$f^{-1}(f(x)) = 2+\sqrt{4-\left(5-\frac{1}{x}\right)}$ ✓

$= 2+\sqrt{4-(5-(a^2-4a+5))}$ ✓

$= 2+\sqrt{4-(-a^2+4a)}$

$= 2+\sqrt{4+a^2-4a}$

$= 2+(a-2)$

$= a$ ✓

QED