FULL NAME:________________________ STUDENT NUMBER:____________

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TAYLORS COLLEGE

SYDNEY CAMPUS

HSC ASSESSMENT

MATHEMATICS

June 2010

Weighting: 30%

Time Allowed: 1.5 hours

Instructions

- Start each question in a NEW answer booklet.
- Write your FULL name and TEACHER'S name at the top of each booklet.
- Show ALL necessary working.
- Approved templates and calculators may be used.

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C, \quad n \neq -1; \; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x + C, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax} + C, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax + C, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax + C, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax + C, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax + C, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a} + C, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right) + C, \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right) + C$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Question 1

(a) Evaluate $\log_3 \sqrt{3}$ 1

(b) The first two terms of a geometric series are $\sqrt{5}-1$ and $\sqrt{5}+2$. 3
Find the common ratio, giving your answer with a rationalised denominator.

(c) Solve $|1-2x|< 7$, graphing your solution on a number line. 3

(d) Find the equation of the line passing through (2,7) and parallel to the line $2x-3y=5$. 2

(e) The graph of $y=f(x)$ passes through the point (0,3) and $f'(x)=e^{3x}$. 2
Find $f(x)$.

START A NEW ANSWER BOOKLET

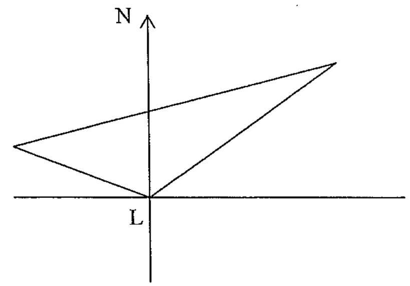
Question 2

(a) Differentiate $\log_e\left(\frac{3x}{x+1}\right)$ 2

(b) Find $\int \frac{1}{1+3x}dx$ 2

(c) Show that the line $3x+4y+18=0$ is a tangent to the circle $(x-2)^2+(y+1)^2=16$. 3

(d) A ship S sails 8 nautical miles on a bearing of 053°T from a lighthouse L. A boat B sails 6 nautical miles from the lighthouse on a bearing of 293°T.



i) Copy the diagram into your answer booklet showing all the given information. 1

ii) Find the distance between the ship S and the boat B, correct to the nearest nautical mile. 2

iii) Find the bearing of the ship S from the boat B. 3
Give your answer to the nearest degree.

START A NEW ANSWER BOOKLET

Question 3

(a) The population of pigeons P in a park is increasing at a rate proportional to the population such that:

$$\frac{dP}{dt}=kP,$$

where t is the time in weeks after the pigeons were first counted and k is a constant.

i) Show that $P=P_0 e^{kt}$ is a solution to the above differential equation, where P_0 is a constant. 1

ii) The population of pigeons increases from 84 to 252 in 5 weeks. Evaluate the constant P_0 and find k correct to four decimal places. 3

iii) Sketch the graph of $P=P_0 e^{kt}$. 1

(b) Byron borrows \$10 000 and agrees to repay the loan in equal monthly instalments over 5 years. Interest is charged at 12% per annum on money owing. If A_n represents the amount of money owing after n months and R is the monthly repayments.

i) Show that after 2 months $A_n = 10201 - R(1 + 1.01)$. 2

ii) Calculate the amount of the monthly instalments R to the nearest cent. 3

iii) What is the total amount repaid? 1

START A NEW ANSWER BOOKLET

Question 4

(a) Given $\log_a \frac{b}{c}>1$, which of the following statements are true. There may be more than one answer. 2

Write your answer(s) in your ANSWER BOOKLET.

For example, if you think the correct statements are B and C write "*statement B and statement C*".

Statement A: $b>c$

Statement B: $\log_e \frac{b}{c}>\log_e a$

Statement C: $\log_{10} \frac{b}{c}> 1$

Statement D: $\log_a \frac{c}{b}< 0$

(b) A particle moves along the x – axis. Its displacement x cm from the origin after t seconds is given by $x = 5 - 2\sin \pi t,\ 0\le t\le 2$.

i) Sketch the displacement – time graph. 3

ii) Find an expression for the velocity and acceleration as functions of t. 2

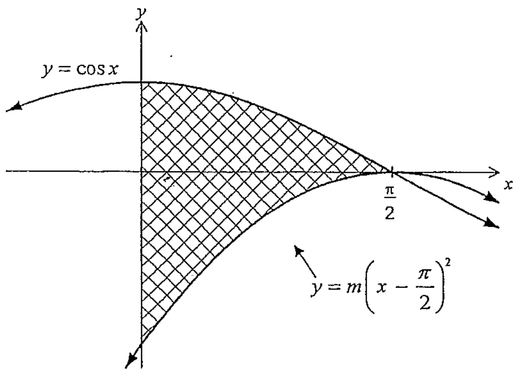
iii) Find when the particle first comes to rest. 2

iv) Find the maximum speed reached by the particle. 2

START A NEW ANSWER BOOKLET

Question 5

(a)



In the figure, the shaded area is enclosed by the curves $y = \cos x$, $y = m(x - \frac{\pi}{2})^2$ and the y – axis two square units. 4

Find the exact value of the constant m.

question 5 continues over the page.......

(b) A student is doing a question in an examination. 5

THE STUDENT'S SOLUTION IS:

(*A*) $V = 200t^{\frac{3}{4}} + 10t + c$

When $t = 16$, $V = 3000$

$\therefore c = 1240$

$\therefore V = 200t^{\frac{3}{4}} + 10t + 1240$

(*B*) *When* $t = 196$, $V = 200(196)^{\frac{3}{4}} + 10(196) + 1240$

$\therefore$ *10 700 litres is added*

THE QUESTION IS:

The rate at which a reservoir is being filled is given by _______(i)___________ litres/sec.

(*A*) Find *V(t)* the volume of water in the reservoir at time *t* seconds given that the volume was ______(ii)______ after __________(iii)_______ seconds.

(*B*) How much will be added to the reservoir, to the nearest ___________(iv)________ litres in the next ________(v)_______ minutes.

Use the student's solution to complete the missing parts of the question.

Write your answer(s) in your ANSWER BOOKLET.

For example, if you think (v) should be 10000 write *"(v) = 10000"*.

END OF EXAMINATION

HSC Mathematics

Term Two Exam - Solutions

Question One

(a) $\log_3 \sqrt{3} = \frac{1}{2}$.

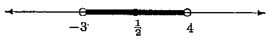
(b)

$$\begin{aligned}\frac{\sqrt{5}+2}{\sqrt{5}-1} &= \frac{\sqrt{5}+2}{\sqrt{5}-1} \times \frac{\sqrt{5}+1}{\sqrt{5}+1} \\ &= \frac{5+\sqrt{5}+2\sqrt{5}+2}{5-1} \\ &= \frac{7+3\sqrt{5}}{4}.\end{aligned}$$

(c)

$$|1-2x| < 7$$

$$\left|x - \frac{1}{2}\right| < \frac{7}{2}$$



$$-3 < x < 4$$

(d) Since the line is parallel to $2x - 3y = 5$,

$$\begin{aligned}2x - 3y &= k \\ 2(2) - 3(7) &= k \\ \therefore k &= -17 \\ \therefore 2x - 3y + 17 &= 0.\end{aligned}$$

(e)

$$\begin{aligned}f'(x) &= e^{3x} \\ f(x) &= \frac{1}{3}e^{3x} + C \\ f(0) = 3 &\Rightarrow C = \frac{8}{3} \\ \therefore f(x) &= \frac{1}{3}e^{3x} + \frac{8}{3}.\end{aligned}$$

Question Two

(a)

$$\begin{aligned}\frac{d}{dx}\ln\left(\frac{3x}{x+1}\right) &= \frac{d}{dx}(\ln 3x - \ln(x+1)) \\ &= \frac{1}{x} - \frac{1}{x+1}.\end{aligned}$$

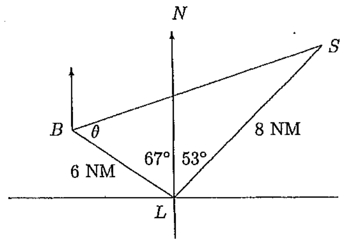
(b)

$$\int \frac{1}{1+3x}\,dx = \frac{1}{3}\ln(1+3x) + C.$$

(c) The circle has radius 4 and centre $(2, -1)$. We shall show that the perpendicular distance from $(2, -1)$ to the line $3x + 4y + 18 = 0$ is also 4 units.

$$\begin{aligned}p &= \frac{|3(2) + 4(-1) + 18|}{\sqrt{3^2 + 4^2}} \\ &= \frac{|20|}{5} \\ &= 4.\end{aligned}$$

(d) (i) The picture should be as follows.



(ii)

$$\begin{aligned}SB^2 &= 6^2 + 8^2 - 2(6)(8)\cos 120° \\ &= 148 \\ \therefore SB &= \sqrt{148} \\ &= 12.16552506 \\ &= 12NM \text{ (nearest nautical mile).}\end{aligned}$$

(iii) Let $\angle SBL = \theta$. By the sine rule,

$$\frac{\sin\theta}{8} = \frac{\sin 120^\circ}{12.16552506}$$
$$\sin\theta = \frac{8\sin 120^\circ}{12.16552506}$$
$$= 0.5694947973$$
$$\theta = 34.71500\ldots$$
$$= 35^\circ \text{ (to nearest degree).}$$

The bearing is $113^\circ - 35^\circ = 078^\circ T$.

Question Three

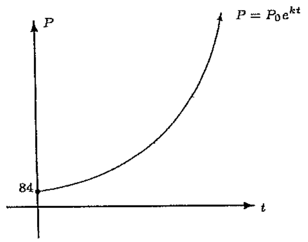
(a) (i) If $P(t) = P_0 e^{kt}$, then

$$\frac{dP}{dt} = kP_0 e^{kt}$$
$$= kP.$$

(ii) $P(0) = 84 \Rightarrow P_0 = 84$.

$$P(t) = 84e^{kt}$$
$$252 = 84e^{5k}$$
$$3 = e^{5k}$$
$$k = \frac{1}{5}\ln 3$$
$$\approx 0.2197224578$$
$$= 0.2197 \text{ (to 4 dp.)}$$

(iii) The graph should be as follows.



(b) (i) $P = 10\,000$, $r = 12\%$ p.a. $= 0.01$ per month.

$$A_0 = 10\,000$$
$$A_1 = A_0 \times 1.01 - R$$
$$= 10,000 \times 1.01 - R$$
$$A_2 = A_1 \times 1.01 - R$$
$$= (10,000 \times 1.01 - R) \times 1.01 - R$$
$$= 10\,000 \times (1.01)^2 - R(1 + 1.01)$$
$$= 10201 - R(1 + 1.01).$$

(ii) We can see that

$$A_n = 10\,000 \times (1.01)^n - R(1 + 1.01 + \ldots + 1.01^{n-1})$$
$$= 10\,000 \times (1.01)^n - R\left(\frac{(1.01)^n - 1}{1.01 - 1}\right)$$
$$= 10\,000 \times (1.01)^n - R\left(\frac{(1.01)^n - 1}{0.01}\right).$$

Since $A_{60} = 0$,

$$0 = 10\,000 \times (1.01)^{60} - R\left(\frac{(1.01)^{60} - 1}{0.01}\right)$$
$$R = \left(\frac{0.01}{(1.01)^{60} - 1}\right) \times 10\,000 \times (1.01)^{60}$$
$$R = 222.4444768$$
$$= \$222.44 \text{ (to nearest cent.)}$$

(iii) $60 \times 222.44 = \$13\,346.40$.

Question Four

(a) Statement D.

Statement A is false.
Counter example: $\log_{\frac{1}{2}} \frac{1}{4} = 2$.

Statement B is also false.
Since $\log_a \frac{b}{c} > 1$, by the change of base formula,

$$\frac{\ln \frac{b}{c}}{\ln a} > 1.$$

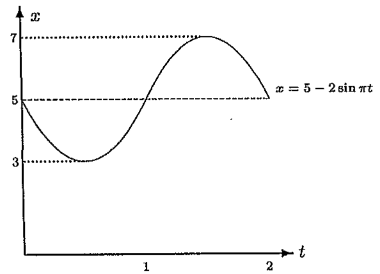
So the desired inequality is only true if $\ln a > 0$, that is, if $a > 1$.

Statement C is only true if $\frac{b}{c} > 10$, thus false in general.

Statement D is true.

$$\log_a \frac{c}{b} = -\log_a \frac{b}{c}$$
$$\log_a \frac{b}{c} > 1$$
$$\therefore \log_a \frac{c}{b} < -1 < 0.$$

(b) (i) The graph should be as follows.



(ii) Differentiating with respect to time,

$$v(t) = 2\pi \cos \pi t,$$
$$a(t) = -2\pi^2 \sin \pi t.$$

(iii) The particle is at rest if $v(t) = 0$, which first occurs at $t = \frac{1}{2}$.

(iv) Since $|2\pi \cos \pi t| \leq 2\pi$, the maximum speed is 2π m/s.

Question Five

(a) The shaded area, A, is given by

$$A = \int_0^{\pi/2} \cos x - m\left(x - \frac{\pi}{2}\right)^2 \, dx$$
$$\therefore 2 = \left[\sin x - \frac{m}{3}\left(x - \frac{\pi}{2}\right)^3\right]_0^{\pi/2}$$
$$= \left(\sin\frac{\pi}{2} - \frac{m}{3}\left(\frac{\pi}{2} - \frac{\pi}{2}\right)^3\right) - \left(\sin 0 - \frac{m}{3}\left(0 - \frac{\pi}{2}\right)^3\right)$$
$$= 1 - \frac{m}{3}\left(\frac{\pi}{2}\right)^3$$
$$\therefore m = -\frac{24}{\pi^3}.$$

(b) (i) $\frac{dV}{dt} = 150t^{-1/4} + 10$

(ii) 3000 litres

(iii) 16

(iv) 100

(v) 3

End of Solutions