Full Name: ______________________________

Student Number: ___________________________

TAYLORS COLLEGE
SYDNEY CAMPUS

Term Four, 2009

HSC MATHEMATICS

Extension One (July Students)

Weighting: (25/50)

Time Allowed: 60 minutes

Instructions

- Attempt all three (3) questions. (Total marks: 36)
- Begin each question in a SEPARATE answer booklet.
- Show ALL necessary working, neatly and clearly.
- Hand-held, non-programmable scientific calculators may be used.

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

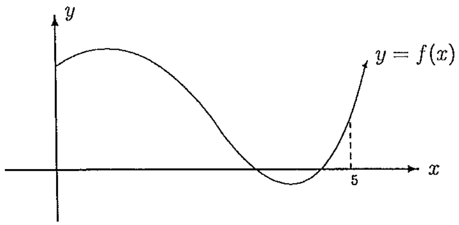
NOTE : $\ln x = \log_e x, \quad x > 0$

Question One (13 marks)

Use a SEPARATE writing booklet. **Marks**

(a) Using the substitution $u = x^4 + \overset{9}{\cancel{8}}$, find $\int_0^2 x^3\sqrt{x^4+9}\,dx$. **3**

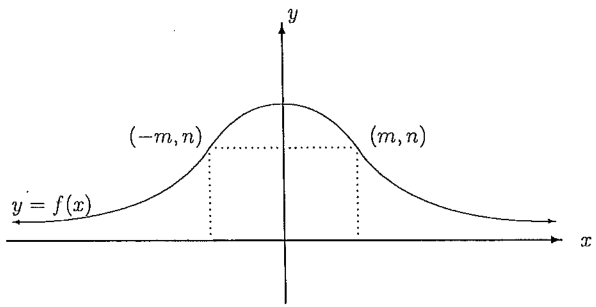
(b) The graph of the curve $y = f(x)$ is given below. **3**



If $\int_0^5 f(x)\,dx = 8$, find the value of $\int_0^5 3f(x) + 1\,dx$.

(c) Evaluate $\sum_{k=1}^{10}(2^n - 2n - 2)$. **3**

(d) The graph of the curve $y = \dfrac{1}{x^2+1}$ is shown below.



Let (m, n) be a point on the curve in the first quadrant. A rectangle is drawn between the points (m, n), $(-m, n)$, and the x axis as shown.

Question One continues next page

(i) Show that the area, $\mathcal{A}$, of the rectangle is given by **1**

$$\mathcal{A} = \frac{2m}{m^2+1}.$$

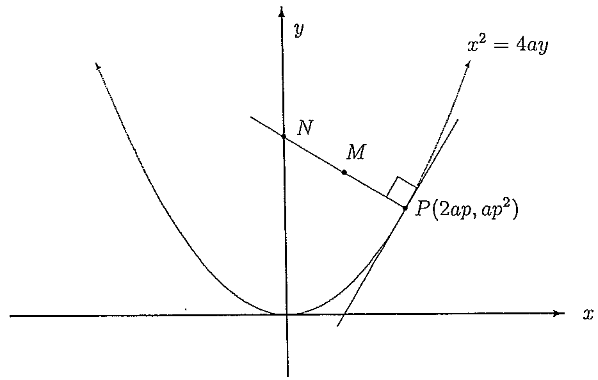
(ii) Find the value of m which maximises the area of the rectangle. Justify your answer. **3**

Question Two (12 marks)

Use a SEPARATE writing booklet. **Marks**

(a) Given that $\int_1^2 (ax+b)^4\,dx = \dfrac{P^5 - Q^5}{5a}$, find a and b in terms of P and Q. **3**

(b) The point $P(2ap, ap^2)$ lies on the parabola $x^2 = 4ay$.



(i) Show that the equation of the normal to the parabola at P is given by **3**

$$x + py = 2ap + ap^3.$$

(ii) The normal at P meets the parabola's axis of symmetry at N. Find the coordinates of N. **1**

(iii) If M is the midpoint of NP, find the coordinates of M. **1**

(iv) Show that the locus of M as P varies is a parabola. **2**

(v) State the coordinates of the vertex and focus of the parabola found in part (iv). **2**

Question Three (11 marks)

Marks

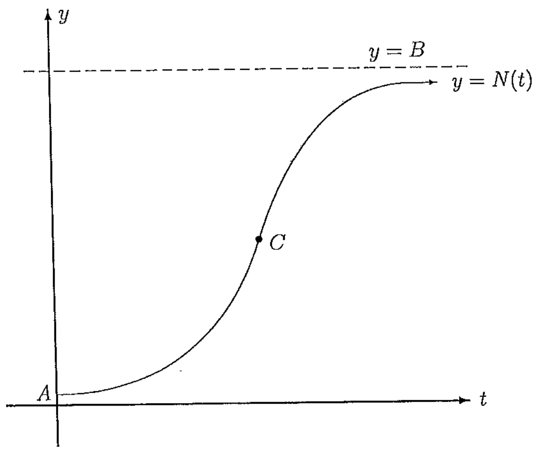
Use a SEPARATE writing booklet.

The spread of an influenza epidemic at Taylors College during winter is modelled by the equation

$$N(t) = 800\left(1 + \frac{t-3}{\sqrt{t^2-6t+10}}\right),$$

where $N(t)$ is the number of students who have been infected with influenza t weeks after the first infection is diagnosed.

The graph of $y = N(t)$ is shown below.



Note that

$$N'(t) = \frac{800}{(t^2-6t+10)^{3/2}},$$
$$N''(t) = -\frac{2400(t-3)}{(t^2-6t+10)^{5/2}}.$$

(You do NOT have to prove this.)

(a) Find the value of A to the nearest whole number. 1

(b) By completing the square, show that $t^2 - 6t + 10$ is positive for all values of t. 2

(c) Show that $N(t)$ is increasing for all real t. 1

Question Three continues next page

(d) (i) Show that $\dfrac{x}{\sqrt{x^2+1}} = \dfrac{1}{\sqrt{1+\frac{1}{x^2}}}$. 2

(ii) Hence evaluate $\displaystyle\lim_{t\to\infty} \frac{t-3}{\sqrt{t^2-6t+10}}$. 2

(iii) As $t \to \infty$, the curve approaches the line $y = B$. 1

Determine the value of B.

(e) The point C on the graph corresponds to the time when the increase in the number of new infections is the greatest. (That is to say, $N'(t)$ is at a maximum at the point C.) 2

Find the coordinates of C.

End of Paper

HSC MATHEMATICS - Extension One (July Students)

Term Four, 2009 - Solutions

Question One

(a) Since $u = x^4 + 9$, $du = 4x^3\,dx$. (1)

If $x = 0$, $u = 9$.
If $x = 2$, $u = 25$. (1)

$$\begin{aligned}
\int_0^2 x^3\sqrt{x^4+9}\,dx &= \frac{1}{4}\int_0^2 4x^3\sqrt{x^4+9}\,dx \\
&= \frac{1}{4}\int_9^{25}\sqrt{u}\,du \qquad (1) \\
&= \frac{1}{4}\left[\frac{2u^{3/2}}{3}\right]_9^{25} \\
&= \frac{1}{6}\left[u^{3/2}\right]_9^{25} \\
&= \frac{1}{6}\{(25)^{3/2} - (9)^{3/2}\} \\
&= \frac{49}{3}. \qquad (1)
\end{aligned}$$

(b) Using properties of the definite integral,

$$\begin{aligned}
\int_0^5 3f(x) + 1\,dx &= 3\int_0^5 f(x)\,dx + \int_0^5 1\,dx \qquad (1) \\
&= 3 \times 8 + [x]_0^5 \\
&= 24 + 5 \\
&= 29. \qquad (1)
\end{aligned}$$

(c) Splitting the sum into a geometric and an arithmetic series, we have

$$\begin{aligned}
\sum_{k=1}^{10}(2^n - 2n - 2) &= \sum_{k=1}^{10} 2^n - \sum_{k=1}^{10}(2n+2) \qquad (1) \\
&= \frac{2(2^{10}-1)}{2-1} \; (1) - \frac{10}{2}(4+22) \; (1) \\
&= 1916.
\end{aligned}$$

(d) (i) The area of the rectangle is $\mathcal{A} = 2m \times n$.

Since (m, n) lies on the curve, $n = \dfrac{1}{m^2+1}$. (1)

Hence $\mathcal{A} = \dfrac{2m}{m^2+1}$.

(ii) Differentiating $\mathcal{A}$ with respect to m, we have,

$$\begin{aligned}
\mathcal{A}' &= \frac{2(m^2+1) - 2m \times 2m}{(m^2+1)^2} \\
&= \frac{2 - 2m^2}{(m^2+1)^2}. \qquad (1)
\end{aligned}$$

If $\mathcal{A}' = 0$, then $2 - 2m^2 = 0$, so $m = 1$ (since (m, n) lies in the first quadrant). (1)

Testing points on either side of $m = 1$,

m	0	1	2
$\mathcal{A}'$	2	0	$-\frac{2}{25}$

(1)

Thus the stationary point at $m = 1$ is a maximum.

Question Two

(a) Evaluating the integral using the reverse chain rule,

$$\begin{aligned}
\int_1^2 (ax+b)^4\,dx &= \left[\frac{(ax+b)^5}{5a}\right]_1^2 \\
&= \frac{(2a+b)^5}{5a} - \frac{(a+b)^5}{5a}. \qquad (1)
\end{aligned}$$

$\therefore P = 2a + b$
$Q = a + b$ (1)

$\therefore a = P - Q$
$b = 2Q - P$. (1)

(b) (i) Making y the subject of the equation,

$$y = \frac{x^2}{4a}. \qquad (1)$$

$$\therefore y' = \frac{x}{2a}.$$

If $x = 2ap$, $y' = p$, so the normal will have gradient $-\frac{1}{p}$. (1)

$$\begin{aligned}
y - ap^2 &= -\frac{1}{p}(x - 2ap) \qquad (1) \\
py - ap^3 &= -x + 2ap \\
x + py &= 2ap + ap^3.
\end{aligned}$$

(ii) If $x = 0$, then $y = 2a + ap^2$, so the coordinates of N are $(0, 2a + ap^2)$. (1)

(iii) Using the midpoint formula,

$$\begin{aligned}
x &= \frac{0 + 2ap}{2} \\
&= ap \\
y &= \frac{2a + ap^2 + ap^2}{2} \\
&= a + ap^2.
\end{aligned}$$

So the coordinates of M are $(ap, a + ap^2)$. (1)

(iv) Since $x = ap$,

$$p = \frac{x}{a}. \qquad (1)$$

$$\begin{aligned}
\therefore y &= a + a\left(\frac{x}{a}\right)^2 \\
&= a + \frac{x^2}{a}. \qquad (1)
\end{aligned}$$

Thus the equation of the locus is $x^2 = a(y - a)$.

(v) The vertex of the parabola is $(0, a)$ and the focus is $(0, \frac{5a}{4})$.
(1) (1)

Question Three

(a) $N(0) = 800\left(1 - \frac{3}{\sqrt{10}}\right) \approx 41$ (to the nearest whole number). So $A \approx 41$. (1)

(b) Completing the square on the quadratic expression,

$$\begin{aligned} t^2 - 6t + 10 &= t^2 - 6t + 9 + 1 \\ &= (t-3)^2 + 1. \quad (1) \end{aligned}$$

For all real t, $(t-3)^2 \geq 0$, thus $(t-3)^2 + 1 > 0$ for all real t. (1)

(c) $N'(t) = \dfrac{800}{(t^2 - 6t + 10)^{3/2}}$.

From part (b), $t^2 - 6t + 10$ is positive for all real t, so both the numerator and denominator of $N'(t)$ are positive and hence the function is always increasing. (1)

(d) (i)

$$\begin{aligned} R.H.S &= \frac{1}{\sqrt{1 + \frac{1}{x^2}}} \\ &= \frac{x}{x} \cdot \frac{1}{\sqrt{1 + \frac{1}{x^2}}} \quad (1) \\ &= \frac{x}{\sqrt{x^2\left(1 + \frac{1}{x^2}\right)}} \\ &= \frac{x}{\sqrt{x^2 + 1}} \quad (1) \\ &= L.H.S. \end{aligned}$$

(ii) From part(b),

$$\begin{aligned} \lim_{t \to \infty} \frac{t-3}{\sqrt{t^2 - 6t + 10}} &= \lim_{t \to \infty} \frac{t-3}{\sqrt{(t-3)^2 + 1}} \quad (1) \\ &= \lim_{t \to \infty} \frac{1}{\sqrt{1 + \frac{1}{(t-3)^2}}} \quad \text{(from (d)(i))} \quad (1) \\ &= 1. \end{aligned}$$

(iii) $\lim_{t \to \infty} 800\left(1 + \dfrac{t-3}{\sqrt{t^2 - 6t + 10}}\right) = 800(1+1) = 1600.$

So $B = 1600$. (1)

(iv) When $N'(t)$ reaches a maximum, $N''(t) = 0$.

If $N''(t) = 0$,

$$\begin{aligned} -\frac{2400(t-3)}{(t^2 - 6t + 10)^{5/2}} &= 0 \\ 2400(t-3) &= 0 \\ t &= 3. \quad (1) \end{aligned}$$

Furthermore, if $t < 3$, $N''(t) > 0$, and if $t > 3$, $N''(t) < 0$, so $N'(t)$ has a local maximum at $t = 3$.

$$\begin{aligned} N(3) &= 800\left(1 + \frac{3-3}{\sqrt{9 - 18 + 10}}\right) \\ &= 800. \quad (1) \end{aligned}$$

So the coordinates of C are $(3, 800)$.

End of Solutions