FULL NAME: ____________________

Circle your teacher's name:

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TAYLORS COLLEGE SYDNEY

HSC MATHEMATICS

EXTENSION 2

30th November 2005

TIME ALLOWED: 1 hour

INSTRUCTIONS

1. All questions may be attempted.
2. All necessary working should be shown in every question. Marks may be deducted for careless or badly arranged work.
3. Approved non-programmable calculators and templates may be used.
4. **Start each question on a new page.**
5. **Each question is to be handed in separately.**

Question 1 **Marks**

(a) If $z = 5 + 12i$ find , in cartesian form,

(i) $z\bar{z}$ **1**

(ii) $\dfrac{13}{z}$ **2**

(iii) $\sqrt{z}$ **3**

(iv) the roots of $x^2 - 3x + 1 - 3i = 0.$ **3**

(b) Let $\beta = -1 + i$

(i) Express β in modulus-argument form. **2**

(ii) Express β^{10} in modulus-argument form. **2**

(iii) Find the least positive integer value of n for which β^n is real. **1**

(iv) Solve $z^4 = 8\sqrt{2}\,\beta$. **3**

Express your answers in modulus- argument form.

(c) Sketch, on an Argand diagram, the locus of z where **2**

$$|z + 3 - 4i| = 5$$

Question 2 **Start a new Answer Booklet** **Marks**

(a) Shade the region on an Argand diagram where the inequalities **4**

$$|z| \le |z - 4 + 2i| \text{ and } |z - \bar{z}| < 4$$

hold simultaneously.

(b) On the **same** Argand diagram, sketch the locus of z where

(i) $arg(z - 2) = arg\,(z - 3 + i)$ **2**

(ii) $arg\,(z + 1 + i) = \dfrac{\pi}{4}$ **2**

(iii) Find z where $arg(z - 2) = arg\,(z - 3 + i)$ and $arg\,(z + 1 + i) = \dfrac{\pi}{4}$ intersect. **1**

(c) (i) Solve $z^3 = -1$. Give your answers in modulus - argument form. **2**

(ii) If α is a complex root of $z^3 = -1$, find the value of $\alpha^2 - \alpha + 1$. **1**

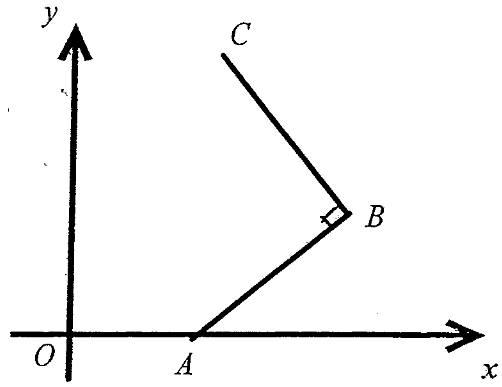
(iii) Hence evaluate $(1 - \alpha)^6$. **1**

Question 3 **Start a new Answer Booklet**

(a) Solve $z^2 + 2\bar{z} + 6 = 0$. **4**

Question 3 Continued **Marks**

(b) The Argand diagram shows the fixed points A, B and C where AB = BC and $\angle$ ABC = 90°. The points A and B represent the complex numbers a and 2a + bi, where a and b are positive constants.



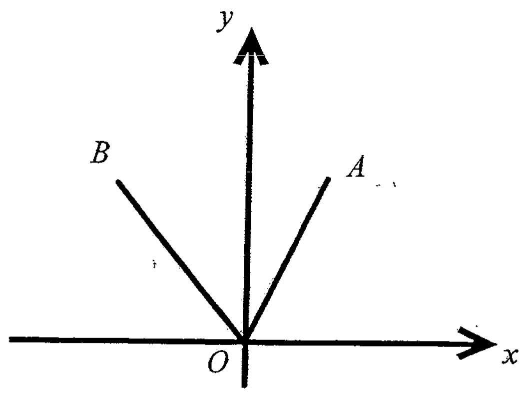
(i) Find the complex number represented by the vector $\overrightarrow{BA}$. **1**

(ii) Find the complex number represented by the vector $\overrightarrow{BC}$. **1**

(iii) Find the complex number represented by C. **1**

(iv) If ABCD is a square, find the complex number represented by D. **2**

(c)



The points A and B represent the complex numbers z_1 and z_2 respectively. ΔABO is equilateral where O is the origin.

(i) Write z_2 in terms of z_1. **1**

(ii) Show $z_1^2 + z_2^2 = z_1 z_2$. **2**

(iii) Deduce that if z_1, z_2 and z_3 are the vertices of any equilateral triangle in the complex number plane then $z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$ **1**

Maths Extension 2 T4 2005

Q1. (a) (i) $z\bar{z} = (5+12i)(5-12i)$

$= 25 + 144$

$= 169$ ① ans

OR $z\bar{z} = |z|^2$

$= \left(\sqrt{5^2+12^2}\right)^2$

$= 169.$

(ii) $\dfrac{13}{z} = \dfrac{13}{5+12i} \times \dfrac{5-12i}{5-12i}$ ① "realising denom"

$= \dfrac{\cancel{13}(5-12i)}{\cancel{169}\,13}$

$= \dfrac{5}{13} - \dfrac{12}{13}i$ ① ans

(iii) Let $(a+ib)^2 = 5+12i$ where a & b are real

Equating real & Imaginary Parts

$a^2 - b^2 = 5$ --(1) ① simult. eq

$2ab = 12$ --(2)

from (2) $b = \dfrac{6}{a}$

sub in (1) $a^2 - \left(\dfrac{6}{a}\right)^2 = 5$

$a^4 - 5a^2 - 36 = 0$

$(a^2-9)(a^2+4) = 0$

$a^2 = 9$ ($a^2 \neq -4$ as a

$a = \pm 3$ is real) ① value of a

$\therefore \sqrt{z} = \pm(3+2i)$ ① ans.

(iv) $z = \dfrac{2i \pm \sqrt{(2i)^2 + 4.2(3+6i)}}{4}$ (1) sub in quad formula

$= \dfrac{2i \pm \sqrt{-4 + 24 + 48i}}{4}$

$= \dfrac{2i \pm \sqrt{20 + 48i}}{4}$

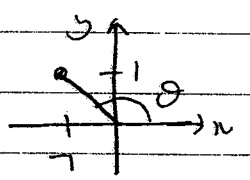
$= \dfrac{2i \pm 2\sqrt{5+12i}}{4}$

$= \dfrac{i \pm \sqrt{5+12i}}{2}$ (1) eval. $\sqrt{\ }$

$= \dfrac{i \pm (3+2i)}{2}$ from (iii)

$= \dfrac{3+3i}{2}$ or $\dfrac{-3-i}{2}$ (1) ans.

b) (i) $|-1+i| = \sqrt{(-1)^2 + 1^2} = \sqrt{2}$ (1) mod



$\tan\theta = -1$

$\theta = 3\pi/4$ (1) argument

$\therefore -1+i = \sqrt{2}\ \text{cis}\ \dfrac{3\pi}{4}$

(ii) $\beta^{10} = \left(2^{1/2}\ \text{cis}\ 3\pi/4\right)^{10}$

$= 2^5\ \text{cis}\ \dfrac{15\pi}{2}$ by de Moivre's thm (1)

$= 32\ \text{cis}\ \dfrac{3\pi}{2}$

$= 32\ \text{cis}\ (-\pi/2)$ (1) ans

(iii) $\beta^n = \left(2^{\frac{1}{2}} \text{cis} \frac{3\pi}{4}\right)^n$

$= 2^{n/2} \text{cis} \frac{3n\pi}{4}$ by de Moivre's

If β^n real then $\text{Im}(\beta^n) = 0$

$\therefore 2^{n/2} \sin \frac{3n\pi}{4} = 0$

$\therefore \sin \frac{3n\pi}{4} = 0$ ½ for statem

when $n = 4$, $\sin 3\pi = 0$

$\therefore$ least pos. integer is 4. ½ answer

(iv) $z^4 = 8\sqrt{2}.\sqrt{2} \text{cis}\left(\frac{3\pi}{4} + 2k\pi\right)$ 1 $2k\pi$

$= 16 \text{ cis } \frac{\pi}{4}(3 + 8k)$

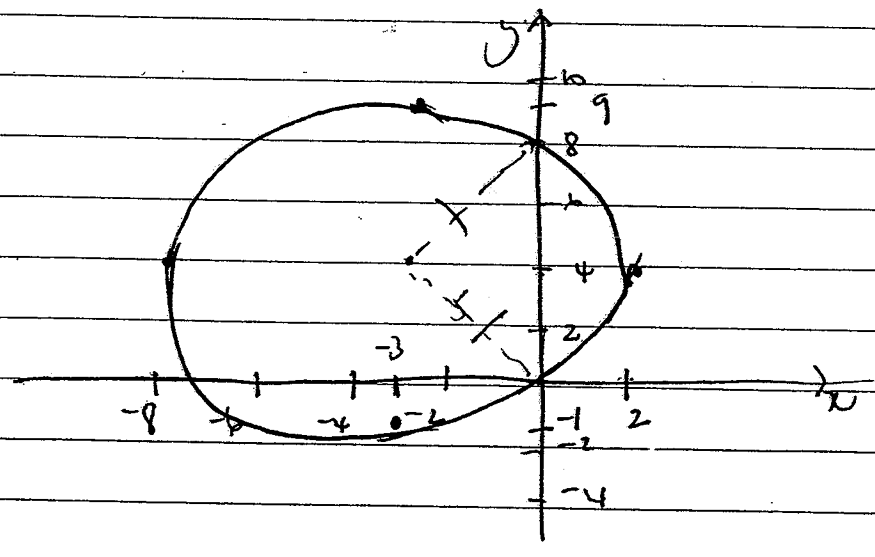
$z = 16^{1/4} \text{ cis } \frac{\pi}{4}(8k + 3.)^{1/4}$

1 de Moivre

$= 2 \text{ cis } \frac{\pi}{16}(8k + 3)$, $k = 0, \pm 1, -2$ 1 4 solns

$z = 2 \text{ cis } \frac{3\pi}{16}, \ 2 \text{ cis } \frac{11\pi}{16}, \ 2 \text{ cis } -\frac{5\pi}{16}, \ 2 \text{ cis } -\frac{13\pi}{16}$

c) Circle centre $(-3 + 4i)$ radius $= 5$. 1



1 passes through O or domain + range correct

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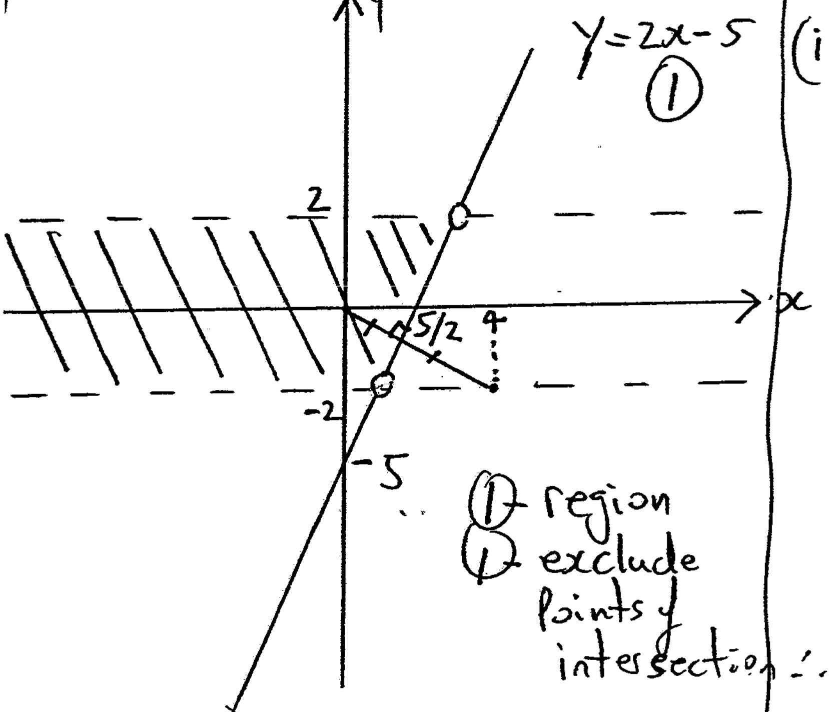
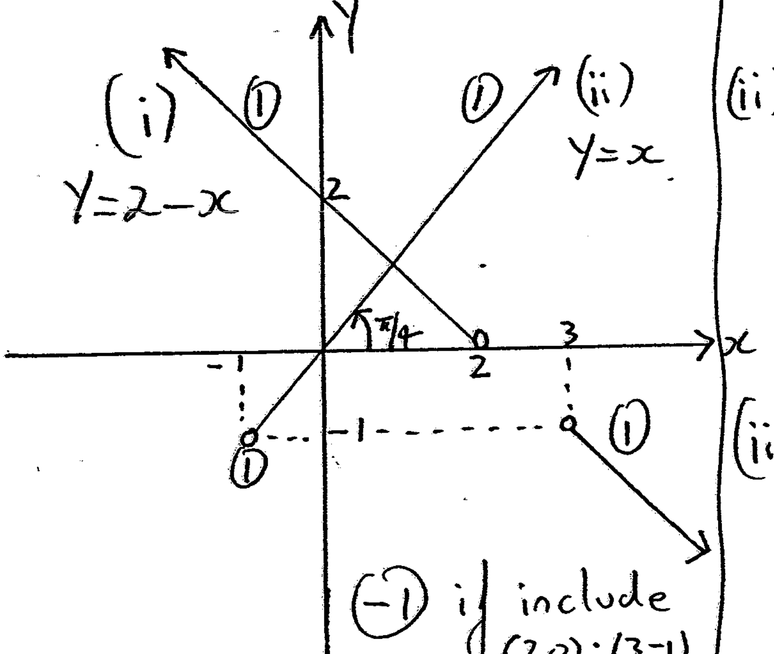


$z - \bar{z}| < 4 \rightarrow |y| < 2$ (1)

)



iii) $(1,1) \rightarrow z = 1 + i$ (1)

c) $z^3 = -1$

(i) $|z^3| = |z|^3 = |-1| = 1$

$\therefore |z| = 1$

Let $\arg z = \theta$

$\therefore \arg z^3 = 3\arg z = \arg(-1)$

$\therefore 3\theta = \pi, 3\pi, 5\pi$

so $\theta = \frac{\pi}{3}, \pi, \frac{5\pi}{3}$ (1)

$\therefore z = \text{cis}\,\frac{k\pi}{3}$, $k = 1, 3, 5$ (1)

or

$= \text{cis} \pm \pi/3, \text{cis}\,\pi$

(ii) $(z+1)(z^2 - z + 1) = 0$

If α is a complex root then $\alpha^2 - \alpha + 1 = 0$ (1)

(iii) From (ii) $1 - \alpha = -\alpha^2$

$$\begin{aligned} \therefore (1-\alpha)^6 &= (-\alpha^2)^6 \\ &= (-1)^6(\alpha^2)^6 \\ &= (\alpha^{12}) \\ &= (\alpha^3)^4 \\ &= (-1)^4 \\ &= 1 \end{aligned}$$ (1)

3a. $(x+iy)^2 + 2(x-iy) + 6 = 0$, $x, y \in \mathbb{R}$, $z = x+iy$

$x^2 - y^2 + 2x + 6 = 0$

$2xy - 2y = 0$ (1)

$y(x-1) = 0 \Rightarrow y = 0$ or $x = 1$.

If $y = 0$, $x = \frac{-2 \pm \sqrt{4-24}}{2} = -1 \pm i\sqrt{5}$, but $x \in \mathbb{R}$ $\therefore y \neq 0$ (1)

$\therefore x = 1$ & $-y^2 + 1 + 2 + 6 = -y^2 + 9 = 0$ $\therefore y = \pm 3$ (1) $\therefore z = 1 \pm 3i$ (1)

b.(i) $\vec{OB} = \vec{OA} + \vec{AB} = a + \vec{AB} = 2a + bi$ $\therefore \vec{BA} = -\vec{AB} = -(2a - a + bi) = -a - bi$ (1)

(ii) $\vec{BA} = \vec{BC}\, i$ $\therefore \vec{BC} = \vec{BA}/i = -\vec{BA}\, i = -(-a-bi)i = -b + ai$ (1)

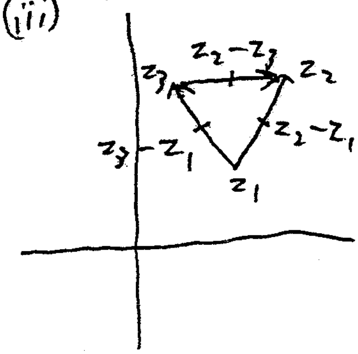
(iii) $\vec{OC} = \vec{OB} + \vec{BC} = 2a + bi + -b + ai = 2a - b + (a+b)i$ $\therefore C$ represents $2a - b + (a+b)i$ (1)

(iv) $\vec{OD} = \vec{OA} + \vec{AD} = \vec{OA} + \vec{BC} = a + (-b + ai) = (a-b) + ai$ (1) $\therefore D$ represents $(a-b) + ai$. (1)

c. (i) $z_2 = z_1 \operatorname{cis}\frac{\pi}{3}$ (1)

(ii) $z_1^2 + z_2^2 = z_1^2 + z_1^2 \operatorname{cis}\frac{2\pi}{3} = z_1^2\left(1 - \frac{1}{2} + \frac{\sqrt{3}i}{2}\right)$ (1) $= z_1^2\left(\frac{1}{2} + \frac{\sqrt{3}i}{2}\right) = z_1 z_1 \operatorname{cis}\frac{\pi}{3} = z_1 z_2$ (1)

(iii)



w.l.o.g., $z_3 - z_1$ & $z_2 - z_1$ satisfy the conditions in (i) & (ii) $\therefore z_3 - z_1 = (z_2 - z_1)\operatorname{cis}\frac{\pi}{3}$ and

$(z_3 - z_1)^2 + (z_2 - z_1)^2 = (z_3 - z_1)(z_2 - z_1)$ (1)

$\therefore z_3^2 - 2z_3z_1 + z_1^2 + z_2^2 - 2z_2z_1 + z_1^2 = z_3z_2 - z_1z_2 - z_3z_1 + z_1^2$ (1)

$\therefore z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$.

<u>Note</u>: Alternative Solution to (c)(ii) provided by LIU Long Hua (Henry):

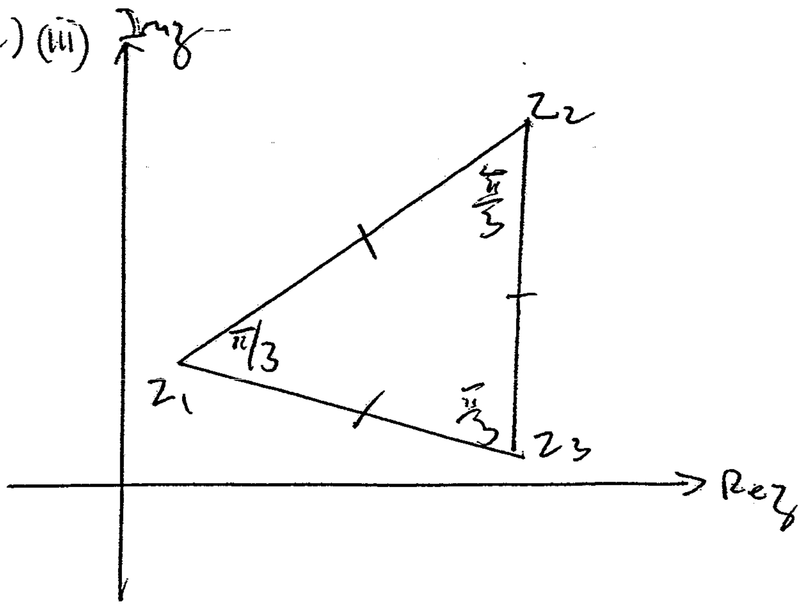
$z_2^3 = z_1^3 \operatorname{cis}\pi = -z_1^3$ $\therefore z_1^3 + z_2^3 = 0$

$\therefore (z_1 + z_2)(z_1^2 - z_1z_2 + z_2^2) = 0$

$z_1 \neq -z_2$

$\therefore z_1^2 + z_2^2 = z_1 z_2$.

Q3 c) (iii)



$$z_2 - z_1 = (z_3 - z_1)\,\text{cis}\,\pi/3$$
$$z_1 - z_3 = (z_2 - z_3)\,\text{cis}\,\pi/3$$

$$\therefore \frac{z_2 - z_1}{z_1 - z_3} = \frac{z_3 - z_1}{z_2 - z_3}$$

$$z_2^2 - z_2 z_3 - z_1 z_2 + \cancel{z_1 z_3} = \cancel{z_1 z_3} - z_1^2 - z_3^2 + z_3 z_1$$

$$\therefore z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$$

or

$$z_2 - z_3 = (z_1 - z_3)\,\text{cis}\left(-\frac{\pi}{3}\right)$$
$$z_2 - z_1 = (z_2 - z_3)\,\text{cis}\left(-\frac{\pi}{3}\right)$$

$$\frac{z_2 - z_3}{z_2 - z_1} = \frac{z_1 - z_3}{z_2 - z_3}$$

$$z_2^2 - z_3 z_2 - z_2 z_3 + \underset{z_3^2}{\cancel{z_2 z_3}} = z_1 z_2 - \cancel{z_2 z_3} - z_1^2 + \overset{z_1 z_3}{\cancel{z_2 z_3}}$$

$$\therefore z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_1 z_3$$