Consider $\sqrt{a+ib}$.

Let $\sqrt{a+ib} = x + iy$

ie $a+ib = (x+iy)^2$

$= (x^2-y^2) + 2xy\,i$

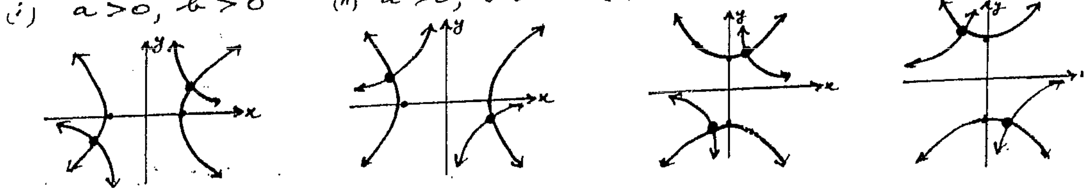
ie $x^2 - y^2 = a$, $2xy = b$ } equating real and imaginary parts

ie $x^2 - y^2 = a$ ——— (1)

$xy = \frac{b}{2}$ ——— (2)

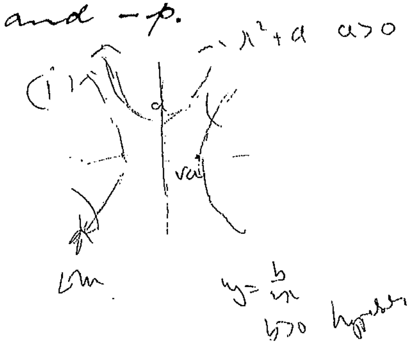
Consider the graphs of (1) and (2) above under the following conditions

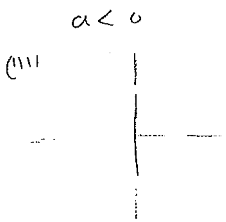
(i) $a>0, b>0$ (ii) $a>0, b<0$ (iii) $a<0, b>0$ (iv) $a<0, b<0$



In each case we see that there are two solutions and, because of the symmetrical properties of the curves involved, we see that if $x+iy$ is one solution then $-x-iy$ is another solution.

Hence, any complex number has two square roots and they are of the form p and $-p$.





1. If $z = 2+3i$, $w = 3+i$, find each of the following in the form $a+ib$

(a) $z+w$ (b) $z-w$ (c) zw (d) $\frac{z}{w}$ (e) $z\bar{z}$ (f) $z-\bar{z}$

[ANSWERS: (a) $5+4i$ (b) $-1+2i$ (c) $3+11i$ (d) $\frac{9}{10}+\frac{7}{10}i$ (e) 13 (f) $6i$]

2. If a is real, prove that $\overline{z+ai} = \bar{z}-ai$

3. If $z = 1+i$, find (i) z^2 (ii) z^4 (iii) $z^2 + \frac{1}{z^2}$

[ANSWERS: (i) $2i$ (ii) -4 (iii) $\frac{3}{2}i$]

4. Find (i) $\sqrt{-5-12i}$ (ii) $\sqrt{-8-6i}$

[ANSWERS: (i) $2-3i, -2+3i$ (ii) $-1+3i, 1-3i$]

5. Prove that for any two complex numbers z, w

(i) $\overline{zw} = \bar{z}\,\bar{w}$ (ii) $\overline{\left(\frac{z}{w}\right)} = \frac{\bar{z}}{\bar{w}}$

Hence prove that $\overline{\left(\frac{z_1}{z_2 z_3}\right)} = \frac{\bar{z}_1}{\bar{z}_2\,\bar{z}_3}$

Can also show

a) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

b) $\overline{i z_1} = -i\,\bar{z}_1$

6. Solve for z, w

$(2+i)z + (2-i)w = 1$ ——— (1)

$(2-i)z + (2+i)w = 2$ ——— (2)

[ANSWERS: $z = \frac{3}{8}+\frac{1}{4}i$, $w = \frac{3}{8}-\frac{1}{4}i$]

7. Solve for z: $z^2 = i\bar{z}$ [ANSWERS: $z = 0, -i, \frac{\sqrt{3}}{2}+\frac{1}{2}i, -\frac{\sqrt{3}}{2}+\frac{1}{2}i$]

8. If $z = \cos\theta + i\sin\theta$, show that $\frac{1}{z} = \bar{z} = \cos\theta - i\sin\theta$

Hence prove that if $z_1 = \cos\alpha + i\sin\alpha$, $z_2 = \cos\beta + i\sin\beta$, $z_3 = \cos\gamma + i\sin\gamma$, and $z_1+z_2+z_3 = 0$, then

$$\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = 0$$

Worksheet #1 contd.

7. $z^2 = i\bar{z}$

$(x+iy)^2 = i(x-iy)$.

$x^2-y^2+2xyi = ix+y$.

$(x^2-y-y^2) + i(2xy-x) = 0+0i$

$x^2-y-y^2 = 0$ $\quad$ $2xy - x = 0$.

$x(2y-1) = 0$

when $x=0$,

$-(y+y^2) = 0$

$y(y+1) = 0$

$y = -1$.

$\therefore z = 0$ or $-i$

when $y = \frac{1}{2}$

$x^2 = \frac{3}{4}$

$x = \pm\frac{\sqrt{3}}{2}$.

$\therefore z = \frac{\sqrt{3}}{2} + \frac{1}{2}i$ or $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$

8. $\frac{1}{\cos\theta + i\sin\theta} = \frac{\cos\theta - i\sin\theta}{1} = \bar{z} = \frac{1}{z}$ $(\cos\theta)$

$\cos\alpha + i\sin\alpha + \cos\beta + i\sin\beta + \cos\gamma + i\sin\gamma = 0$.

$(\cos\alpha + \cos\beta + \cos\gamma) + i(\sin\alpha + \sin\beta + \sin\gamma) = 0$

Equating Re and Im $(\cos\alpha + \cos\beta + \cos\gamma) = 0$ and $(\sin\alpha + \sin\beta + \sin\gamma) = 0$.

Now, $\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} = (\cos\alpha + \cos\beta + \cos\gamma) - i(\sin\alpha + \sin\beta + \sin\gamma)$

$= 0 - 0$

$= 0$

Worksheet #1

1. a) $z + w = 2+3i+3+i$
$= 5+4i$ ✓

b) $z - w = 2+3i-3-i$
$= -1+2i$ ✓

c) $(2+3i)(3+i) = 6-3+2i+9i$
$= 3+11i$ ✓

d) $\frac{(2+3i)(3-i)}{10} = \frac{6+3-2i+9i}{10} = \frac{9}{10} + \frac{7}{10}i$ ✓

e) $(2+3i)(2-3i)$
$= 2^2+3^2$
$= 4+9$
$= 13$. ✓

f) $2+3i-2+3i$
$= 6i$ ✓

2. $\overline{\bar{z} - ai} = \bar{\bar{z}} - \bar{a}.\bar{i} = z - ai$ ✓

3. (i) $(1+i)^2$
$= 2i$ ✓
(ii) $= -4$ ✓
(iii) $2i + \frac{1}{2i} = 2i - \frac{i}{2}$
$= \frac{3}{2}i$ ✓

4. (i) $-5-12i = x^2-y^2+2xyi$
$x^2-y^2 = -5$ $\quad$ $2xy = -12$.
$y = -\frac{6}{x}$.
$x^4+5x^2-36 = 0$
$(x^2+9)(x^2-4) = 0$.
$x = \pm 2$ only $(x \in \mathbb{R})$
$y = \mp 3$.
$\sqrt{-5-12i} = \pm(2-3i)$ ✓

(ii) $-8-6i = x^2-y^2+2xyi$
$x^2-y^2 = -8$ $\quad$ $2xy = -6$
$y = -\frac{3}{x}$
$x^4+8x^2-9 = 0$.
$(x^2+9)(x^2-1) = 0$.
$x = \pm 1$ only $(x \in \mathbb{R})$
$y = \mp 3$
$\sqrt{-8-6i} = \pm(1-3i)$ ✓

5. see later for (i) and (ii)

a) $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$
$(x_1 \pm x_2) - (y_1 \pm y_2)i$
$= x_1 - y_1 i \pm (x_2 - y_2 i)$
$= \bar{z}_1 \pm \bar{z}_2$ ✓

b) $\overline{iz_1} = -i\bar{z}_1$
$\overline{iz_1} = \overline{xi+yi^2}$
$= yi^2 - xi$
$= -i(x-yi)$
$= -i\bar{z}$ ✓

6. ① is $(2-i)\cancel{(2+i)}z + (2-i)^2 w = 2-i$

② is $(2+i)(2-i)z + (2+i)^2 w = 4+2i$

$w[(2-i)^2 - (2+i)^2] = 2-i-4-2i$

$w(\cancel{2}-i-\cancel{2}-i)(2-\cancel{i}+2+\cancel{i}) = -2-3i$

$w = \frac{3}{8} - \frac{1}{4}i$

→ z

contd next page

6) → Eliminating w,

$$(2+i)\cancel{(2-i)}z + (2-i)^2 w = 2-i$$
$$(2-i)\cancel{(2+i)} + (2+i)^2 w = 4+2i$$

$$w\left[(2-i)^2 - (2+i)^2\right] = 2-i-4-2i$$
$$w(\cancel{4-1-4i-4+1}-4i) = 2-i-4-2i$$
$$w(-8i) = -2-3i$$
$$w = \frac{2}{8i} + \frac{3\cancel{i}}{8\cancel{i}}$$
$$= \frac{3}{8} - \frac{1}{4}i$$

5) (i) Let $z = a+ib$ and $w = x+iy$.

$$\overline{zw} = \overline{(a+ib)(x+iy)} = \overline{(ax-by)+i(ay+bx)}$$
$$= ax - by - ayi - bxi$$
$$= ax - ayi - by - bxi$$
$$= a(x-yi) - bi(x-yi)$$
$$= (a-bi)(x-yi)$$
$$= \bar{z}\cdot\bar{w}$$

(ii) $\overline{\left(\frac{z}{w}\right)} = \overline{z \times w^{-1}} = \bar{z} \times \overline{w^{-1}}$
$$= \frac{\bar{z}}{\bar{w}}$$

Hence $\overline{\left(\frac{z_1}{z_2 z_3}\right)} = \frac{\bar{z_1}}{\overline{z_2 z_3}} = \frac{\bar{z}}{\bar{z_2} \times \bar{z_3}}$