

MATHEMATICS EXTENSION 1

2006

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

General Instructions

- Reading time – 5 minutes
- Writing time – 2 hours
- All seven questions should be attempted
- Total marks available - 84
- All questions are worth 12 marks
- An approved calculator may be used
- All relevant working should be shown for each question

The release date for this exam is Tuesday 15 August 2006. Teachers are asked not to release this trial exam to students until this date except under exam conditions where the trial exams are collected by teachers at the end of the exam.

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Question 1 (12 marks)

Marks

- (a) Evaluate $\lim_{x \rightarrow 0} \frac{2 \sin 2x}{x}$. 2
- (b) The polynomial $P(x) = 2x^3 + ax^2 + x + 2$ has a factor $(2x + 1)$. Find the value of a . 2
- (c) Solve the inequality $\frac{1}{x-2} \geq 2$. 2
- (d) Using the substitution $u = x + 2$, find $\int \frac{x}{3} \sqrt{x+2} \, dx$. 2
- (e) Divide the interval AB externally in the ratio $4:3$, where A is the point $(2, -1)$ and B is $(1, -3)$. 2
- (f) Find the obtuse angle between the lines $3x - y + 5 = 0$ and $2x + 3y - 1 = 0$. Give your answer correct to the nearest degree. 2

Question 2 (12 marks)

Marks

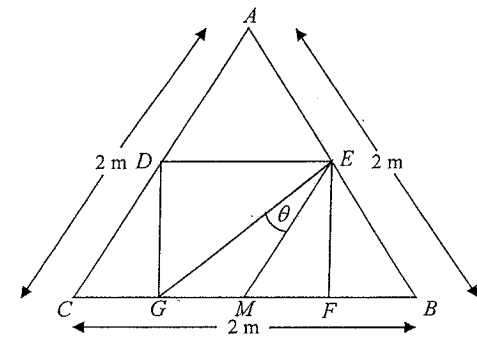
- (a) Differentiate $\cos^{-1}(\sin x)$. 2
- (b) Find:
- (i) $\int \frac{2}{\sqrt{1-9x^2}} dx$ 2
- (ii) $\int \sin^2 \frac{x}{2} dx$. 2
- (c) How many different arrangements can be made using the 8 letters in the word PARALLEL? 2
- (d) Consider the function $f(x) = \frac{1}{2} \cos^{-1}(1-3x)$.
- (i) State the domain and range of $f(x)$. 2
- (ii) Hence sketch the graph of $y = f(x)$. 2

Question 3 (12 marks)

Marks

- (a) Use the definition $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$ to find $f'(x)$ where $f(x) = x^2 - 2x$. 2

(b)



NOT TO SCALE

The equilateral triangle ABC has sidelengths of 2 metres. The points D and E are the mid-points of AC and AB respectively and G, F lies on CB . Also, $DEFG$ is a rectangle. Point M is the midpoint of GF . $\angle GEM = \theta$.

- (i) Show that $GE = \frac{\sqrt{7}}{2}$ metres. 1
- (ii) Show that $\cos \theta = \frac{5\sqrt{7}}{14}$. 2
- (iii) Hence show that the area of $\triangle GEM$ is $\frac{\sqrt{3}}{8}$ square metres. 1

Question 3 continues on the next page.

Question 3 (cont'd)	Marks
(c) (i) Show that $f(x) = e^x - x^3 + 1$ has a zero between 4.4 and 4.6.	1
(ii) Find an approximation; correct to 1 decimal place, for this zero using the method of halving the interval.	2
(d) (i) Express $\sqrt{3} \cos x - \sin x$ in the form $R \cos(x + \alpha)$ where $0 < \alpha < \frac{\pi}{2}$ and $R > 0$.	2
(ii) Hence, solve $\sqrt{3} \cos x - \sin x = 1$ for $0 \leq x \leq \frac{\pi}{2}$.	1

Question 4 (12 marks)	Marks
(a) Prove $\tan^{-1} \frac{2}{3} + \cos^{-1} \frac{2}{\sqrt{5}} = \tan^{-1} \frac{7}{4}$.	3
(b) Find $\int \frac{\cos x \sin x}{\sqrt{2 - \sin^2 x}} dx$ using the substitution $u = \sin x$.	3
(c) Sketch the curve $y = x + \frac{4}{x}$ showing clearly all the stationary points and asymptotes. Hence find the values of k such that $x + \frac{4}{x} = k$ where there are no real roots.	3
(d) Use the method of mathematical induction to prove that $(1+1) + (2+3) + (3+5) + \dots + (n + (2n-1)) = \frac{1}{2}n(3n+1)$ where n is a positive integer.	3

Question 5 (12 marks)

Marks

- (a) A particle with displacement x and velocity v , is moving in simple harmonic motion such that $\ddot{x} = -12x$. Initially, it is stationary at the point where $x = -4$.

- (i) Show that

$$v^2 = 12(16 - x^2).$$

- (ii) By assuming the general form for x or otherwise, find x as a function of t .

- (b) The probability of George being late to class is 0.2. He is supposed to attend 20 classes in a week. The Year Level Coordinator has said that if George is late to three or more classes in a week he will receive a detention.

- (i) What is the probability that George receives a detention for being late? Express your answer correct to two decimal places.

- (ii) The principal said that if George received n or more detentions for being late in a ten week term, he would be suspended. George's mates worked out that the probability of George being suspended was 0.65 (correct to two decimal places). Using your answer to part (i), find the value of n .

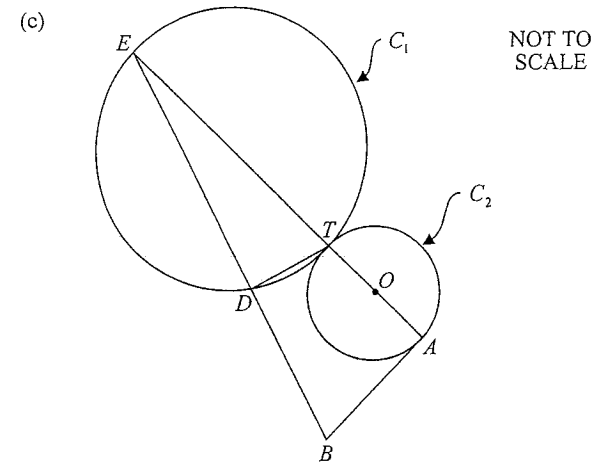
2

2

2

Question 5 (cont'd)

Marks



Two circles C_1 and C_2 touch at T . The line AE passes through O , the centre of C_2 , and through T . The point A lies on C_2 and E lies on C_1 . The line AB is a tangent to C_2 at A , D lies on C_1 and BE passes through D . The radius of C_1 is R and the radius of C_2 is r .

- (i) Find the size of $\angle EDT$, giving reasons for your answer.

1

- (ii) If $DE = 2r$ find an expression for the length of EB in terms of r and R .

3

Question 6 (12 marks)

Marks

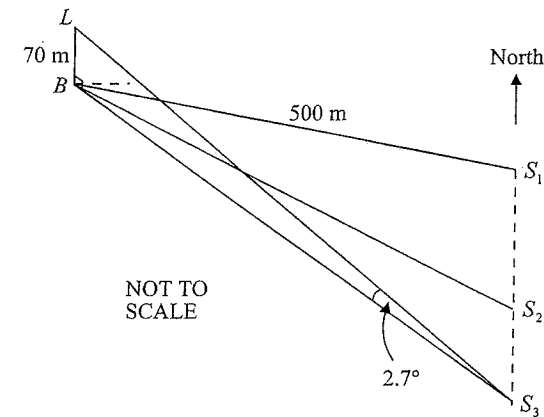
- (a) Find the coefficient of x^6 in the expansion $(x+2)^2(2x-1)^7$. 2
- (b) A particle moves in a straight line so that its velocity v at a position x is given by $v=9+4x^2$. It starts from a fixed point where $x=0$ at a velocity of 9 ms^{-1} .
- (i) Find its acceleration, \ddot{x} as a function of the displacement, x . 1
- (ii) Express its displacement, x , as a function of time, t . 3
- (iii) Find the displacement after $\frac{\pi}{24}$ seconds from its starting point. 1
- (c) (i) Find the equation of the normal at $P(at^2, 2at)$ on the parabola $y^2 = 4ax$. 2
- (ii) The normal intersects with the x -axis at point Q . Find the coordinates of Q and hence find the coordinates of R where R is the midpoint of PQ . 2
- (iii) Hence find the Cartesian equation of the locus of R . 1

Question 7 (12 marks)

Marks

- (a) A life saver sits at point L , 70 m above sea level, in a lookout on a cliff. He spots a surfer lying on his board at point S_1 ; on a bearing of 120° in relation to the life savers position, and a distance of 500 m from point B which is located at sea level directly below the life saver.

Twenty minutes later the life saver notes that the surfer is still lying on his surfboard but is now located at point S_2 on a bearing of 160° in relation to the life savers position and has moved in a line due south.



The life saver raises the alarm and twenty-three minutes after his initial sighting a rescue boat is launched from point B . From point S_3 , which is the point at which the rescue boat reaches the surfer, the angle of elevation of the life saver in the tower is 2.7° .

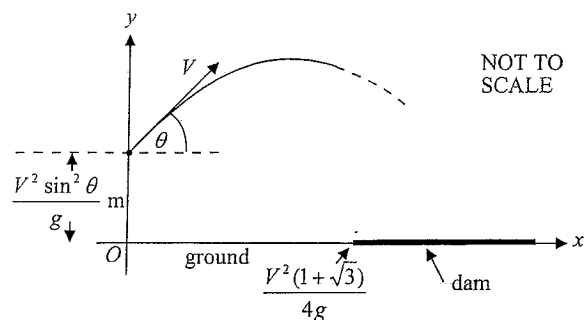
Assume that the rescue boat has travelled at a constant speed and in a straight line to reach the surfer and that the surfer has drifted at a constant speed in a direction due south since the initial sighting.

- (i) Show that the speed at which the surfer was drifting south was 2.82 km/h (correct to 2 decimal places). 2
- (ii) Find the speed at which the rescue boat was moving. Express your answer in km/h correct to 2 decimal places. 3

Question 7 (cont'd)

Marks

(b)



A man in an ascending hot air balloon throws a set of car keys to his wife who is on the ground. The keys are projected at a constant velocity of $V \text{ ms}^{-1}$ at an angle of θ to the horizontal, $0^\circ < \theta < 90^\circ$, and from a point $\frac{V^2 \sin^2 \theta}{g}$ m vertically above the ground.

Also, the near edge of a dam located near where the balloon took off, lies $\frac{V^2(1+\sqrt{3})}{4g}$ m horizontally from the point of projection. The dam is $\frac{V^2}{2g}$ m wide.

The position of the keys at time t seconds after they are projected is given by

$$x = Vt \cos \theta$$

$$\text{and } y = -\frac{gt^2}{2} + Vt \sin \theta + \frac{V^2 \sin^2 \theta}{g}$$

- (i) Show that the Cartesian equation of the path of the keys is given by $y = \frac{-gx^2 \sec^2 \theta}{2V^2} + x \tan \theta + \frac{V^2 \sin^2 \theta}{g}$. 1
- (ii) Show that the horizontal range of the keys on the ground is given by $x = \frac{V^2(1+\sqrt{3}) \sin 2\theta}{2g}$. 2
- (iii) Find the values of θ for which the keys will **not** land in the dam or on the edge of the dam. 4

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**MATHEMATICS EXTENSION 1
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION
SOLUTIONS
2006**

Question 1 (12 marks)

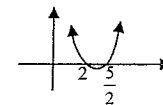
$$\begin{aligned} \text{(a)} \quad \lim_{x \rightarrow 0} \frac{2 \sin 2x}{x} &= 4 \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \\ &= 4 \times 1 \text{ since } \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1 \\ &= 4 \end{aligned}$$

2 marks	Correct answer
1 mark	Obtaining $4 \times \lim_{x \rightarrow 0} \frac{\sin 2x}{2x}$

$$\begin{aligned} \text{(b)} \quad P(x) &= 2x^3 + ax^2 + x + 2 \\ \text{Now, } P\left(-\frac{1}{2}\right) &= 0 \\ \text{so } P\left(-\frac{1}{2}\right) &= 2\left(-\frac{1}{2}\right)^3 + a\left(-\frac{1}{2}\right)^2 + \left(-\frac{1}{2}\right) + 2 = 0 \\ -\frac{1}{4} + \frac{1}{4}a - \frac{1}{2} + 2 &= 0 \\ \frac{1}{4}a &= -\frac{5}{4} \\ a &= -5 \end{aligned}$$

2 marks	Correct answer
1 mark	Obtaining correct expression for $P\left(-\frac{1}{2}\right)$ and then equating it to zero

$$\begin{aligned} \text{(c)} \quad \frac{1}{x-2} &\geq 2, \quad x \neq 2 \\ x-2 &\geq 2(x-2)^2 \\ 2(x-2)^2 - (x-2) &\leq 0 \\ (x-2)[2(x-2)-1] &\leq 0 \\ (x-2)(2x-5) &\leq 0 \end{aligned}$$



$$\text{so } 2 < x \leq \frac{5}{2} \text{ since } x \neq 2$$

2 marks	Correct answer
1 mark	Obtaining $(x-2)[2(x-2)-1] \leq 0$

$$\begin{aligned} \text{(d)} \quad \int_{\frac{x}{3}}^x \sqrt{x+2} \, dx &= \frac{1}{3} \int (u-2)u^{\frac{1}{2}} \, du \quad \text{where } u = x+2, \frac{du}{dx} = 1 \text{ and } x = u-2 \\ &= \frac{1}{3} \int (u^{\frac{3}{2}} - 2u^{\frac{1}{2}}) \, du \\ &= \frac{1}{3} \left(\frac{2}{5}u^{\frac{5}{2}} - \frac{4}{3}u^{\frac{3}{2}} \right) \\ &= \frac{2}{3} \left\{ \frac{1}{5}(x+2)^{\frac{5}{2}} - \frac{2}{3}(x+2)^{\frac{3}{2}} \right\} \end{aligned}$$

2 marks	Correct answer
1 mark	Obtaining $\frac{1}{3} \int (u^{\frac{3}{2}} - 2u^{\frac{1}{2}}) \, du$

$$\begin{aligned} \text{(e)} \quad A(2, -1), B(1, -3) \\ 4: -3 \\ \left(\frac{4(1) - 3(2)}{4-3}, \frac{4(-3) - 3(-1)}{4-3} \right) \\ = (-2, -9) \end{aligned}$$

2 marks	Correct answer
1 mark	One correct coordinate

(f) $3x - y + 5 = 0, 2x + 3y - 1 = 0$

$$m_1 = 3, \quad m_2 = -\frac{2}{3}$$

$$\tan \theta = \frac{3 + \frac{2}{3}}{1 + 3\left(-\frac{2}{3}\right)} \quad \text{OR} \quad \tan \theta = \left| \frac{3 + \frac{2}{3}}{1 + 3\left(-\frac{2}{3}\right)} \right|$$

$$= -3\frac{2}{3} \qquad \theta = 75^\circ$$

$\theta = 105^\circ$ so the obtuse angle is 105°

2 marks	Correct answer
1 mark	Obtaining $m_1 = 3$ and $m_2 = -\frac{2}{3}$

Question 2 (12 marks)

(a) $\frac{d}{dx}(\cos^{-1}(\sin x)) = \frac{-\cos x}{\sqrt{1 - \sin^2 x}}$

$$= \frac{-\cos x}{|\cos x|}$$

$$= \begin{cases} -1, & \text{if } \cos x > 0 \\ 1, & \text{if } \cos x < 0 \end{cases}$$

2 marks	Correct answer
1 mark	Obtaining $\frac{d}{dx}(\cos^{-1}(\sin x)) = \frac{-\cos x}{\sqrt{1 - \sin^2 x}}$

(b) (i) $\int \frac{2dx}{\sqrt{1 - 9x^2}} = \frac{2}{3} \int \frac{3dx}{\sqrt{1 - (3x)^2}}$

$$= \frac{2}{3} \sin^{-1}(3x) + c$$

2 marks	Correct answer
1 mark	Obtaining $\frac{2}{3} \int \frac{3dx}{\sqrt{1 - (3x)^2}}$

(ii) $\int \sin^2 \frac{x}{2} dx = \int \left(\frac{1}{2} - \frac{1}{2} \cos x\right) dx$

$$= \frac{1}{2}x - \frac{1}{2} \sin x + c$$

2 marks	Correct answer
1 mark	Obtaining $\int \left(\frac{1}{2} - \frac{1}{2} \cos x\right) dx$

(c) PARALLEL
P, R, E, A, A, L, L

$$\text{Number of ways} = \frac{8!}{2!3!}$$

$$= 3\,360$$

2 marks	Correct answer
1 mark	Giving $8!$ as answer

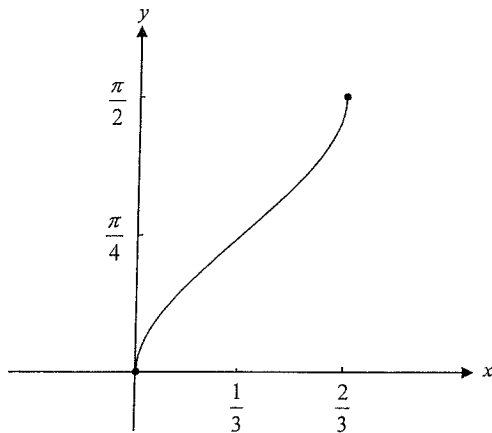
(d) $f(x) = \frac{1}{2} \cos^{-1}(1-3x)$

(i) Domain:
 $-1 \leq 1-3x \leq 1$
 $-2 \leq -3x \leq 0$
 $\frac{2}{3} \geq x \geq 0$
 so $0 \leq x \leq \frac{2}{3}$

Range:
 $0 \leq 2y \leq \pi$
 $0 \leq y \leq \frac{\pi}{2}$

2 marks	Correct answer
1 mark	Correct domain or correct range

(ii)



Test $x = 0, f(0) = \frac{1}{2} \cos^{-1} 1$
 $= 0$

$x = \frac{2}{3}, f\left(\frac{2}{3}\right) = \frac{1}{2} \cos^{-1}(-1)$
 $= \frac{\pi}{2}$

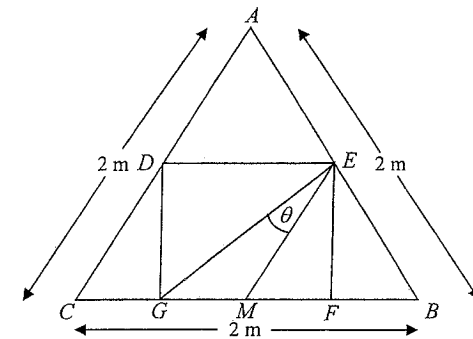
2 marks	Correct graph
1 mark	Correct shape without endpoints marked or point of inflection labelled

Question 3 (12 marks)

(a) $f(x) = x^2 - 2x$
 $f(x+h) = (x+h)^2 - 2(x+h)$
 $= x^2 + 2hx + h^2 - 2x - 2h$
 $f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$
 $= \lim_{h \rightarrow 0} \frac{x^2 + 2hx + h^2 - 2x - 2h - x^2 + 2x}{h}$
 $= \lim_{h \rightarrow 0} \frac{2hx + h^2 - 2h}{h}$
 $= \lim_{h \rightarrow 0} (2x + h - 2)$
 $= 2x - 2$

2 marks	Correctly derived derivative
1 mark	Correctly substituting into $f'(x)$

(b) (i)



NOT TO SCALE

In $\triangle CDG$

$$DG = \sqrt{1 - \left(\frac{1}{2}\right)^2}$$

$$= \frac{\sqrt{3}}{2} \text{ m}$$

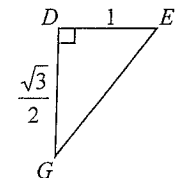
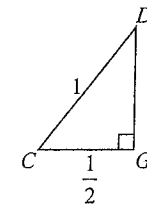
In $\triangle DEG$

$$GE = \sqrt{1^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{\sqrt{7}}{2} \text{ metres}$$

as required

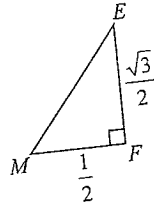
1 mark	Correct answer
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(ii) In $\triangle EFM$

$$EM = \sqrt{\frac{1}{4} + \frac{3}{4}}$$

$$= 1$$

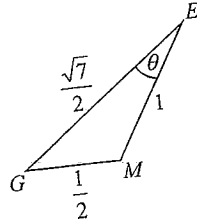


In $\triangle GEM$,

$$\left(\frac{1}{2}\right)^2 = \frac{7}{4} + 1 - 2 \times \frac{\sqrt{7}}{2} \times 1 \cos \theta$$

$$-\frac{5}{2} = -\sqrt{7} \cos \theta$$

$$\cos \theta = \frac{5\sqrt{7}}{14} \text{ as required}$$



2 marks	Correct answer
1 mark	Finding EM and attempting to use the cosine rule

(iii) Area = $\frac{1}{2} \times GE \times EM \sin \theta$

$$= \frac{1}{2} \times \frac{\sqrt{7}}{2} \times 1 \sin \theta$$

$$= \frac{\sqrt{7}\sqrt{21}}{56}$$

$$= \frac{7\sqrt{3}}{56}$$

$$= \frac{\sqrt{3}}{8} \text{ m}^2 \text{ as required.}$$

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\sin \theta = \sqrt{1 - \frac{175}{196}}$$

$$= \frac{\sqrt{21}}{14}$$

$$\text{or Area} = \frac{1}{2} \times \frac{1}{2} \times \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{8}$$

1 mark	Correct answer
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(c) (i) $f(x) = e^x - x^3 + 1$

$$f(4.4) = -2.7331\dots$$

$$f(4.6) = 3.1483\dots$$

The sign of the function changes between $x = 4.4$ and $x = 4.6$ and hence the zero must lie between $x = 4.4$ and 4.6 .

1 mark	Correct answer
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x	4.4	4.6	4.5	4.55	4.525
$f(x)$	-2.7331...	3.1483...	-0.1078..	1.4360...	0.6437...

The zero is between $x = 4.50$ and 4.525 . Therefore, correct to 1 decimal place, the zero is 4.5.

2 marks	Correct answer
1 mark	Using the correct method but with an arithmetic mistake

(d) (i) $R \cos(x + \alpha) = R \cos x \cos \alpha - R \sin x \sin \alpha$ for all x

Now, $\sqrt{3} \cos x - \sin x = R \cos x \cos \alpha - R \sin x \sin \alpha$

so, $\sqrt{3} = R \cos \alpha$

and $1 = R \sin \alpha$

So, $\tan \alpha = \frac{1}{\sqrt{3}}$

$$\alpha = \frac{\pi}{6}$$

Also $R^2 \cos^2 \alpha + R^2 \sin^2 \alpha = 4$

$$R = 2, R > 0$$

So $\sqrt{3} \cos x - \sin x = 2 \cos\left(x + \frac{\pi}{6}\right)$

2 marks	Correct answer
1 mark	Correctly finding R or α

(ii) $\sqrt{3} \cos x - \sin x = 1$

$$2 \cos\left(x + \frac{\pi}{6}\right) = 1 \text{ from part (i)}$$

$$\cos\left(x + \frac{\pi}{6}\right) = \frac{1}{2}$$

$$x + \frac{\pi}{6} = \frac{\pi}{3}$$

$$x = \frac{\pi}{6}$$

1 mark	Correct answer
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Question 4 (12 marks)

(a) Prove $\tan^{-1} \frac{2}{3} + \cos^{-1} \frac{2}{\sqrt{5}} = \tan^{-1} \frac{7}{4}$

$$\tan\left(\tan^{-1} \frac{2}{3} + \cos^{-1} \frac{2}{\sqrt{5}}\right) = \frac{7}{4}$$

let $A = \tan^{-1} \frac{2}{3}$ and $B = \cos^{-1} \frac{2}{\sqrt{5}}$

so $\tan A = \frac{2}{3}$ and $\cos B = \frac{2}{\sqrt{5}}$

so $\tan B = \frac{1}{2}$

$$LHS = \tan(A+B)$$

$$= \frac{\tan A + \tan B}{1 - \tan A \tan B}$$

$$= \frac{\frac{2}{3} + \frac{1}{2}}{1 - \frac{2}{3} \times \frac{1}{2}}$$

$$= \frac{\frac{7}{6}}{\frac{1}{3}}$$

$$= \frac{7}{2} \times \frac{3}{3} = \frac{7}{2}$$

$$= \frac{7}{2}$$

$$= RHS$$

3 marks	Correct answer
2 marks	Obtaining $\tan A = \frac{2}{3}$ and $\tan B = \frac{1}{2}$
1 mark	Obtaining $\tan\left(\tan^{-1} \frac{2}{3} + \cos^{-1} \frac{2}{\sqrt{5}}\right) = \frac{7}{4}$

(b) $\int \frac{\cos x \sin x}{\sqrt{2 - \sin^2 x}} dx$ let $u = \sin x$ so $\frac{du}{dx} = \cos x$
 $= \int \frac{\cos x \times u}{\sqrt{2 - u^2}} \cos x dx$
 $= \int u(2 - u^2)^{-\frac{1}{2}} du$
 $= -\frac{1}{2} \int -2u(2 - u^2)^{-\frac{1}{2}} du$
 $= -\frac{1}{2} \frac{(2 - u^2)^{\frac{1}{2}}}{\frac{1}{2}}$
 $= -\sqrt{2 - u^2}$
 $= -\sqrt{2 - \sin^2 x} + c$

3 marks	Correct answer
2 marks	Obtaining $\int u(2 - u^2)^{-\frac{1}{2}} du$
1 mark	Obtaining $dx = \frac{du}{\cos x}$

(c) $y = x + \frac{4}{x}$

$$\frac{dy}{dx} = 1 - \frac{4}{x^2}$$

$$\frac{d^2y}{dx^2} = \frac{8}{x^3}$$

Stationary points occur when $\frac{dy}{dx} = 0$.

When $1 - \frac{4}{x^2} = 0$

$$x^2 = 4$$

$$x = \pm 2$$

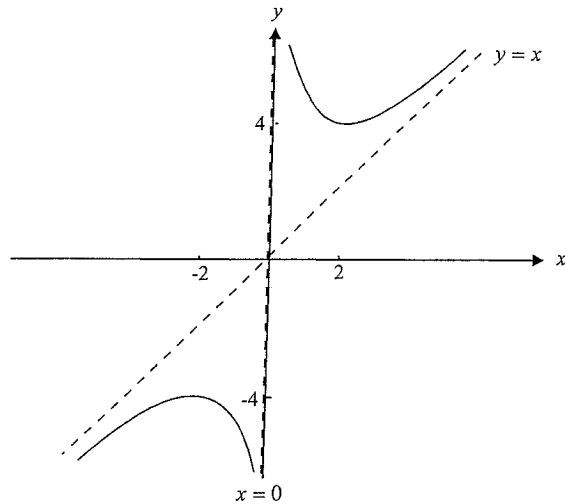
$$x = 2, y = 2 + \frac{4}{2} = 4$$

$$x = -2, y = -2 + \frac{4}{-2} = -4$$

At the point $(-2, -4)$, $\frac{d^2y}{dx^2} < 0$ so we have a max. at this point.

At the point $(2, 4)$, $\frac{d^2y}{dx^2} > 0$ so we have a min. at this point.

Asymptotes $x = 0$, $y = x$



From the graph we see that there are no real roots for $x + \frac{4}{x} = k$ when $-4 < k < 4$.

3 marks	Correct answer and graph
2 marks	Correct graph
1 mark	Finding correct local minimum and local maximum OR correct asymptotes

(d) $(1+1) + (2+3) + (3+5) + \dots + (n + (2n-1)) = \frac{1}{2}n(3n+1)$, n is a positive integer

Step 1

For $n = 1$,

$$\begin{aligned} LHS &= 1+1 \\ &= 2 \end{aligned}$$

$$\begin{aligned} RHS &= \frac{1}{2}(1)(3+1) \\ &= 2 \end{aligned}$$

$$\therefore LHS = RHS$$

Hence true for $n = 1$

Step 2: Assume true for $n = k$,

$$(1+1) + (2+3) + (3+5) + \dots + (k + (2k-1)) = \frac{1}{2}k(3k+1) \quad (*)$$

Prove true for $n = k+1$,

$$LHS = (1+1) + (2+3) + \dots + (k + (2k-1)) + (k+1 + (2(k+1)-1))$$

$$= \frac{1}{2}k(3k+1) + (k+1 + 2k+1) \quad (\text{from } *)$$

$$= \frac{3}{2}k^2 + \frac{1}{2}k + 3k + 2$$

$$= \frac{3}{2}k^2 + \frac{7}{2}k + 2$$

$$= \frac{1}{2}(3k^2 + 7k + 4)$$

$$= \frac{1}{2}(k+1)(3k+4)$$

$$= \frac{1}{2}(k+1)(3(k+1)+1)$$

So it is true for $n = (k+1)$

Step 3 Since it is true for $n = 1, n = 2$ and so on and it is true for $n = k+1$ by assuming true for $n = k$, therefore by mathematical induction, it is true for all n where n is a positive integer.

3 marks	Correct proof
2 marks	Verifies the case for $n = 1$ and attempts to prove that it is true for $n = k+1$
1 mark	Verifies the case for $n = 1$

Question 5 (12 marks)

- (a) (i) $\ddot{x} = -12x$
- $$\frac{d}{dx} \left(\frac{1}{2} v^2 \right) = -12x$$
- $$\frac{1}{2} v^2 = -12 \int x dx$$
- $$\frac{1}{2} v^2 = \frac{-12x^2}{2} + c$$
- When $v = 0$, $x = -4$
- $$0 = -96 + c$$
- $$c = 96$$
- So, $\frac{1}{2} v^2 = -6x^2 + 96$
- $$v^2 = -12x^2 + 192$$
- $$v^2 = 12(16 - x^2)$$
- as required.

2 marks	Correct answer
1 mark	Obtaining $\frac{1}{2} v^2 = -6x^2 + c$ OR Correct substitution of initial conditions into incorrect function

- (ii) Since the motion starts at the 'negative' end of the path of motion we use the general form $x = -a \cos nt$.
- Now, $\ddot{x} = -12x$ so $n^2 = 12$ and so $n = 2\sqrt{3}$, $n > 0$.
- Also, the particle is stationary at $x = -4$ and since $\ddot{x} = -12x$, the centre of motion is at $x = 0$ so $a = 4$.
- So $x = -4 \cos(2\sqrt{3}t)$.

2 marks	Correct answer
1 mark	Finding a and n correctly OR giving correct general form for x

- (b) (i) $P(\text{receiving a detention})$
 $= P(\text{George being late to 3 or more classes in a week})$
 $= 1 - \{P(X=0) + P(X=1) + P(X=2)\}$
 $= 1 - \{ {}^{20}C_0 (0.2)^0 (0.8)^{20} + {}^{20}C_1 (0.2)^1 (0.8)^{19} + {}^{20}C_2 (0.2)^2 (0.8)^{18} \}$ - *
 $= 1 - \{0.0115... + 0.0576... + 0.1369...\}$
 $= 1 - 0.2060...$
 $= 0.79$ (correct to 2 decimal places)

2 marks	Correct answer
1 mark	Obtaining line marked *

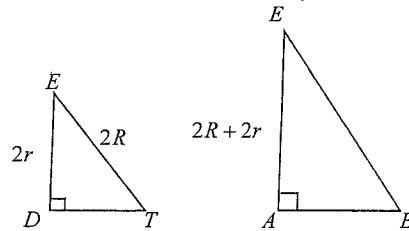
- (ii) Using trial and error, if $n = 10$,
 $P(\text{George is suspended})$
 $= P(X=10)$
 $= {}^{10}C_{10} (0.79)^{10} (0.21)^0$
 $= 0.094...$
 If $n = 9$, $P(\text{George is suspended})$
 $= P(X=9) + P(X=10)$
 $= {}^{10}C_9 (0.79)^9 (0.21)^1 + 0.094...$
 $= 0.3456...$
 If $n = 8$, $P(\text{George is suspended})$
 $= P(X=8) + P(X=9) + P(X=10)$
 $= {}^{10}C_8 (0.79)^8 (0.21)^2 + 0.3456...$
 $= 0.3010... + 0.3456...$
 $= 0.65$ (correct to two decimal places)
 The value of n is 8.

2 marks	Correct answer
1 mark	Showing an example of $P(X=n)$ with the correct values of n , p and q .

- (c) (i) Since C_1 and C_2 touch at T and AE passes through O , the centre of C_2 , and through T , then AE also passes through the centre of C_1 . So TE is a diameter of C_1 . (When circles touch, the line through the centres passes through the point of contact.)
So $\angle EDT = 90^\circ$ (Angles in a semi-circle are right angles.)

1 mark	Correct answer
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- (ii) $\angle EAB = 90^\circ$ (The tangent to a circle is perpendicular to the radius drawn to the point of contact.)
In Δ 's ABE and DTE ,
 $\angle EDT = \angle EAB = 90^\circ$ (from part (i) and from above)
 $\angle DET$ is common
 $\angle ETD = \angle EBA$ (Angles in a triangle add to give 180° .)
So ΔABE is similar to ΔDTE . (Three corresponding angles are equal.)



Now, $DE = 2r$, $ET = 2R$, and $EA = 2R + 2r$

Since the Δ 's are similar,

$$\frac{EB}{ET} = \frac{EA}{ED}$$

$$EB = 2R \left(\frac{2R + 2r}{2r} \right)$$

$$= \frac{2R(R + r)}{r}$$

3 marks	Correct answer including showing that ΔABE is similar to ΔDTE
2 marks	Showing that ΔABE is similar to ΔDTE
1 mark	Stating that $\angle EAB = 90^\circ$

Question 6 (12 marks)

(a)

$$(x+2)^2(2x-1)^7 = (x^2 + 4x + 4) \left((2x)^7 - \binom{7}{1}(2x)^6 + \binom{7}{2}(2x)^5 - \binom{7}{3}(2x)^4 + \dots \right)$$

$$\text{coefficient of } x^6 = -\binom{7}{3}2^4 + 4\binom{7}{2}2^5 - 4\binom{7}{1}2^6$$

$$= 336$$

2 marks	Correct answer
1 mark	Correct expansion of $(x+2)^2(2x-1)^7$

(b) (i) $v = 9 + 4x^2$

$$\ddot{x} = \frac{d}{dx} \left(\frac{1}{2} v^2 \right)$$

$$= \frac{d}{dx} \left(\frac{1}{2} (9 + 4x^2)^2 \right)$$

$$= (9 + 4x^2)(8x)$$

$$= 8x(9 + 4x^2)$$

1 mark	Correct answer
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(ii) $v = 9 + 4x^2$

$$\frac{dx}{dt} = 9 + 4x^2$$

$$\frac{dt}{dx} = \frac{1}{9 + 4x^2}$$

$$t = \int \frac{1}{9 + 4x^2} dx$$

$$= \frac{1}{4} \int \frac{1}{\frac{9}{4} + x^2} dx$$

$$= \frac{1}{4} \times \frac{2}{3} \tan^{-1}\left(\frac{2x}{3}\right) + c$$

$$= \frac{1}{6} \tan^{-1}\left(\frac{2x}{3}\right) + c$$

$t = 0$ and $x = 0$, so $c = 0$

$$t = \frac{1}{6} \tan^{-1}\left(\frac{2x}{3}\right)$$

$$6t = \tan^{-1}\left(\frac{2x}{3}\right)$$

$$\frac{2x}{3} = \tan 6t$$

$$x = \frac{3}{2} \tan 6t$$

3 marks	Correct answer
2 marks	Obtaining $t = \frac{1}{6} \tan^{-1}\left(\frac{2x}{3}\right)$
1 mark	Obtaining $t = \int \frac{1}{9 + 4x^2} dx$

(iii) $t = \frac{\pi}{24}$, $x = \frac{3}{2} \tan 6\left(\frac{\pi}{24}\right)$

$$= \frac{3}{2} \tan \frac{\pi}{4}$$

$$= 1\frac{1}{2} \text{ m}$$

The displacement is 1.5m after $\frac{\pi}{24}$ seconds from its starting point.

1 mark	Correct answer
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(c) (i) $x = at^2$, $y = 2at$

$$\frac{dx}{dt} = 2at$$

$$\frac{dy}{dt} = 2a$$

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= 2a \times \frac{1}{2at}$$

$$= \frac{1}{t}$$

$$m_T = \frac{1}{t} \text{ so } m_N = -t \text{ (since } m_1 m_2 = -1)$$

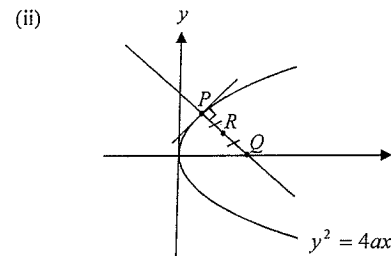
Equation of normal at $P(at^2, 2at)$ is given by

$$y - 2at = -t(x - at^2)$$

$$= -tx + at^3$$

$$tx + y = 2at + at^3$$

2 marks	Correct answer
1 mark	Obtaining the correct gradient of the normal



At Q , $y = 0$.

sub $y = 0$ into $tx + y = 2at + at^3$

$$tx = 2at + at^3$$

$$x = a(2 + t^2)$$

So we have $Q(a(2 + t^2), 0)$

and $R\left(\frac{a(2 + t^2) + at^2}{2}, \frac{0 + 2at}{2}\right)$

$$R(a(1 + t^2), at)$$

2 marks	Correct answer
1 mark	Obtaining one correct coordinate of R OR finding the correct coordinates of Q

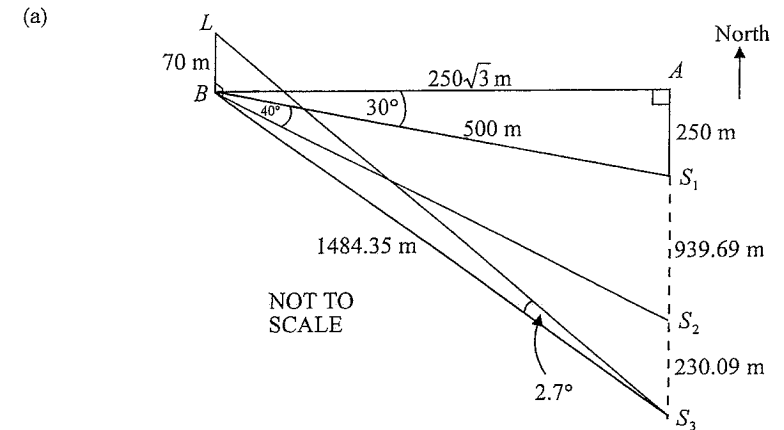
(iii) At R , $x = a(1+t^2)$ and $y = at$
 so $x = a\left(1 + \frac{y^2}{a^2}\right)$ and $t = \frac{y}{a}$

$$\frac{x}{a} = 1 + \frac{y^2}{a^2}$$

$$\frac{y^2}{a^2} = \frac{x}{a} - 1$$

$$y^2 = ax - a^2$$
 so $y^2 = a(x - a)$

1 mark	Correct answer
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Question 7 (12 marks)


(i)

$$\text{In } \triangle ABS_1, AB = 500 \cos 30^\circ$$

$$= 250\sqrt{3}$$

$$\text{Also, } AS_1 = 500 \sin 30^\circ$$

$$= 250$$

$$\text{In } \triangle ABS_2, AS_2 = 250\sqrt{3} \tan 70^\circ$$

$$= 1189.692\dots$$

$$\text{so, } S_1S_2 = AS_2 - AS_1$$

$$= 939.692\dots$$

$$\text{Speed of surfer} = \frac{S_1S_2}{20 \text{ mins}}$$

$$= \frac{0.939692\dots}{0.3} \text{ km/h}$$

$$= 2.82 \text{ km/h (correct to 2 decimal places)}$$

as required

2 marks	Correct answer
1 mark	Making a reasonable attempt to find the distance S_1S_2

$$(ii) \text{ In } \triangle BLS_3, BS_3 = \frac{70}{\tan 2.7^\circ} \\ = 1484.346\dots$$

$$\text{In } \triangle ABS_3, AS_3 = \sqrt{(BS_3)^2 - (250\sqrt{3})^2} \\ = 1419.783\dots$$

$$\text{So } S_2S_3 = 1419.783 - 250 - 939.692 = 230.091$$

The time taken for the surfer to drift from S_2 to S_3 is $\frac{0.230091}{2.82} = 0.08159$ h (to 5 decimal places).

The rescue boat leaves 3 minutes or $\frac{3}{60} = 0.05$ hours after the lifesaver spotted the surfer at point S_2 . Therefore the rescue boat took $0.08159 - 0.05 = 0.03159$ hours to cover the distance of 1.48435 km.

The speed of the rescue boat was therefore $\frac{1.48435}{0.03159} = 46.99$ km/h (correct to 2 decimal places).

3 marks	Correct answer
2 marks	Finding correctly the time taken for the surfer to drift from S_2 to S_3
1 mark	Finding correctly the distance from S_2 to S_3

$$(b) \quad (i) \quad x = Vt \cos \theta$$

$$\text{so } t = \frac{x}{V \cos \theta}$$

$$\text{in } y = \frac{-gt^2}{2} + Vt \sin \theta + \frac{V^2 \sin^2 \theta}{g}$$

$$\text{becomes } y = \frac{-gx^2}{2V^2 \cos^2 \theta} + \frac{Vx \sin \theta}{V \cos \theta} + \frac{V^2 \sin^2 \theta}{g} \\ = \frac{-gx^2 \sec^2 \theta}{2V^2} + x \tan \theta + \frac{V^2 \sin^2 \theta}{g}$$

1 mark	Correctly showing the formula
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(ii) When $y = 0$,

$$\frac{-gx^2 \sec^2 \theta}{2V^2} + x \tan \theta + \frac{V^2 \sin^2 \theta}{g} = 0$$

$$x = \frac{-\tan \theta \pm \sqrt{\tan^2 \theta + 4 \times \frac{g \sec^2 \theta}{2V^2} \times \frac{V^2 \sin^2 \theta}{g}}}{\frac{-g \sec^2 \theta}{V^2}}$$

$$= \frac{-\tan \theta \pm \sqrt{\tan^2 \theta + 2 \tan^2 \theta}}{\frac{-g \sec^2 \theta}{V^2}}$$

$$= \frac{-\tan \theta \pm \sqrt{3 \tan^2 \theta}}{\frac{-g \sec^2 \theta}{V^2}}$$

$$= \frac{(\tan \theta + \sqrt{3} \tan \theta) V^2}{g \sec^2 \theta} \text{ or } \frac{(\tan \theta - \sqrt{3} \tan \theta) V^2}{g \sec^2 \theta}$$

$$= \frac{V^2 \tan \theta (1 + \sqrt{3})}{g \sec^2 \theta} \text{ or } \frac{V^2 \tan \theta (1 - \sqrt{3})}{g \sec^2 \theta}$$

$$= \frac{V^2 (1 + \sqrt{3}) \sin \theta \cos \theta}{g}$$

since $1 - \sqrt{3} < 0$ this value of x occurs to the left of the y -axis

$$\text{So } x = \frac{V^2 (1 + \sqrt{3}) \sin 2\theta}{2g} \text{ as required.}$$

2 marks	Correctly showing formula including a reason for rejecting the second solution
1 mark	Substituting correctly into quadratic formula

(iii) From part (ii) the range of the projectile is given by $x = \frac{V^2(1+\sqrt{3})\sin 2\theta}{2g}$

The near edge of the dam is located at $\left(\frac{V^2(1+\sqrt{3})}{4g}, 0\right)$.

At this point,

$$\frac{V^2(1+\sqrt{3})}{4g} = \frac{V^2(1+\sqrt{3})\sin 2\theta}{2g}$$

$$\sin 2\theta = \frac{1}{2}$$

$$2\theta = 30^\circ, 150^\circ, \dots$$

$$\theta = 15^\circ, 75^\circ \text{ since } 0^\circ < \theta < 90^\circ$$

The far edge of the dam is located at $\left(\frac{V^2(1+\sqrt{3})}{4g} + \frac{V^2}{2g}, 0\right)$; that is, at

$$\left(\frac{V^2(3+\sqrt{3})}{4g}, 0\right).$$

At this point,

$$\frac{V^2(3+\sqrt{3})}{4g} = \frac{V^2(1+\sqrt{3})\sin 2\theta}{2g}$$

$$3+\sqrt{3} = 2(1+\sqrt{3})\sin 2\theta$$

$$\sin 2\theta = \frac{3+\sqrt{3}}{2+2\sqrt{3}}$$

$$= \frac{3+\sqrt{3}}{2+2\sqrt{3}} \times \frac{2-2\sqrt{3}}{2-2\sqrt{3}}$$

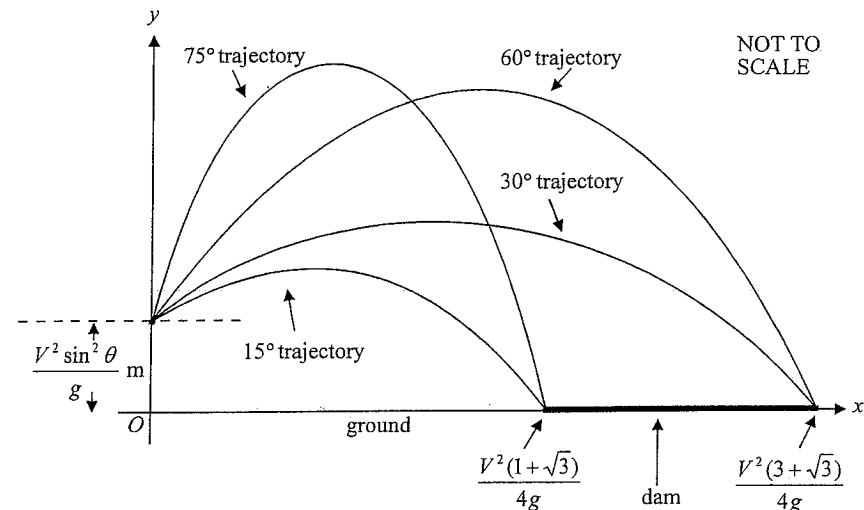
$$= \frac{6-6\sqrt{3}+2\sqrt{3}-6}{4-12}$$

$$= \frac{-4\sqrt{3}}{-8}$$

$$\sin 2\theta = \frac{\sqrt{3}}{2}$$

$$2\theta = 60^\circ, 120^\circ, \dots$$

$$\theta = 30^\circ, 60^\circ \quad 0^\circ < \theta < 90^\circ$$



The keys will **not** land in the dam or on the edge of the dam for $0^\circ < \theta < 15^\circ$, $30^\circ < \theta < 60^\circ$ and $75^\circ < \theta < 90^\circ$.

4 marks	Correct answer
3 marks	Obtaining two of the three correct ranges of values of θ
2 marks	Obtaining the values of 15° , 75° , 30° and 60° for θ
1 mark	Obtaining two correct values of θ for the near edge of the dam OR at the far edge of the dam OR obtaining one correct value of θ at both the near edge and the far edge of the dam