THE HEFFERNAN GROUP
P.O. Box 1180
Surrey Hills North VIC 3127
ABN 20 607 374 020
Phone 1800 100 521
Fax 1800 100 525

MATHEMATICS EXTENSION 2

2006

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

General Instructions

- Reading time – 5 minutes
- Writing time – 3 hours
- All eight questions should be attempted
- Total marks available - 120
- All questions are worth 15 marks
- An approved calculator may be used
- All relevant working should be shown for each question

The release date for this exam is Tuesday 15 August 2006. Teachers are asked not to release this trial exam to students until this date except under exam conditions where the trial exams are collected by teachers at the end of the exam.

This paper has been prepared independently of the Board of Studies NSW to provide additional exam preparation for students The publication is in no way connected with or endorsed by the Board of Studies NSW.

Question 1 (15 marks) **Marks**

(a) (i) Find $\int \frac{\sin^{-1} 2x}{\sqrt{1-4x^2}}\,dx$. **1**

(ii) Find $\int \frac{\cos^5 x}{\sin^2 x}\,dx$. **2**

(b) Evaluate $\int_0^2 x^3 e^{x^2}\,dx$. **2**

(c) By expressing $\frac{48}{x^3+64}$ as partial fractions, find $\int \frac{48}{x^3+64}\,dx$. **5**

(d) Let $I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n x\,dx$ where n is a positive integer.

(i) Find the value of I_1. **1**

(ii) Show $I_n + I_{n-2} = \frac{1}{n-1}$ for $n \geq 2$. **3**

(iii) Hence evaluate I_5. **1**

Question 2 (15 marks) **Marks**

(a) (i) If $z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$, find z^4, in the form $x + iy$. **2**

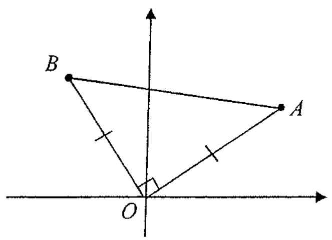
(ii) Hence or otherwise, find z^{13} in the form $x + iy$. **2**

(b) Find all the complex numbers $z = a + ib$, where a and b are real, such that $|z|^2 - 7 = 2i(z + 2)$. **4**

(c) (i) Express $z = 4 + 4\sqrt{3}i$ in modulus-argument form. **1**

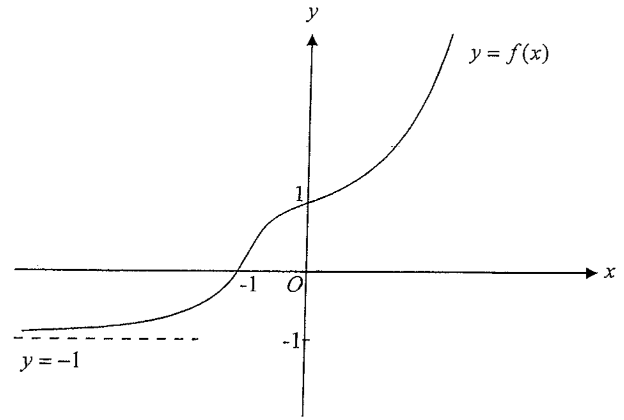
(ii) Hence find the three values of $z^{\frac{1}{3}}$ in modulus-argument form. **3**

(d) The Argand diagram shows the points A and B, which represent the complex numbers z_1 and z_2 respectively. Given that ΔBOA is a right-angled, isosceles triangle, show that $(z_1 + z_2)^2 = 2z_1 z_2$. **3**



Question 3 (15 marks) **Marks**

(a) The diagram below shows the graph of $y = f(x)$ which has a horizontal asymptote at $y = -1$



Draw large (half page), separate sketches of each of the following:

(i) $y = -f(|x|)$ **1**

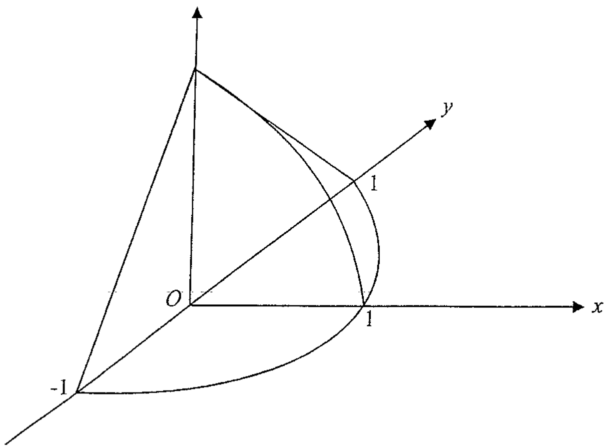
(ii) $y = |f(x)|$ **2**

(iii) $y = \frac{1}{f(x)}$ **2**

(iv) $y = e^{f(x)}$ **2**

Question 3 (cont'd) **Marks**

(b)



The base of a solid is the semi-circular region in the x - y plane with the straight edge running from the point $(0,-1)$ to the point $(0,1)$ and the point $(1,0)$ on the curved edge of the semicircle.
Each cross-section perpendicular to the x-axis is an isosceles triangle with each of the two equal sidelengths three quarters the length of the third side.

(i) Show that the area of the triangular cross-section at $x = a$ is $\frac{\sqrt{5}}{2}\left(1-a^2\right)$. **2**

(ii) Hence find the volume of the solid. **2**

(c) The ellipse with equation $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the hyperbola with equation $\frac{x^2}{A^2}-\frac{y^2}{B^2}=1$, have the same directrices. **4**
Show that $A^2 = \frac{9}{5}\sqrt{5\left(A^2+B^2\right)}$.

Question 4 (15 marks) **Marks**

(a) The equation $x^3-3x-2=0$ has roots α,β and γ.

(i) Find the equation with roots α^3,β^3 and γ^3. **2**

(ii) Hence or otherwise, find $\alpha^7\beta\gamma+\alpha\beta^7\gamma+\alpha\beta\gamma^7$. **2**

(b) (i) Find the equation of the normal to $x^2-xy-y^2=1$ at the point $(2,1)$. **2**

(ii) Find the coordinates of the other point of intersection where the normal intersects with the curve. **2**

(c) Using mathematical induction, prove that **3**

$$\sum_{r=1}^{n} r^3 < n^2(n+1)^2 \text{ for } n=1,2,3...$$

(d) In a tidal river, the top of an old anchorage post measured 0.8 metre below the water level at high tide and 0.2 metre above the river level at low tide. High tide occurred at 6.30am and low tide occurred at 12.35pm on the day that the measurements were taken. The motion of the tide can be assumed to be simple harmonic. Between high tide and low tide on this day, when was there at least 0.5 metre of water above the top of the old anchorage post? Express your times to the nearest minute. **4**

Question 5 (15 marks) **Marks**

(a) A group of 7 boys and 5 girls go to watch a movie together.

(i) In how many ways can they sit in a row if the boys sit together and the girls sit together? **1**

(ii) In how many ways can they sit together in a row if no two girls are to sit together? **2**

(iii) After the movie the group of twelve go to a café and sit at a round table so that two particular girls sit together, five particular boys sit together, the remaining three girls sit together and the remaining two boys sit together. The four groups around the table don't mind which other group they are seated next to and within each group no one minds who they sit next to.
In how many different ways can they be arranged? **2**

(b) The equation $ax^4+bx^3+cx^2+dx+e=0$ has a quadruple root α.

(i) Find α in terms of a and b. **2**

(ii) Hence, show $\left(1+\frac{b}{4a}\right)^4=\frac{a+b+c+d+e}{a}$. **2**

(c) The region S is enclosed by the line $x+y=a;\ a>0$, the curve $y=x^3-ax^2$ and the y-axis.

(i) Sketch the region S; clearly labelling its intercepts with the axes. **2**

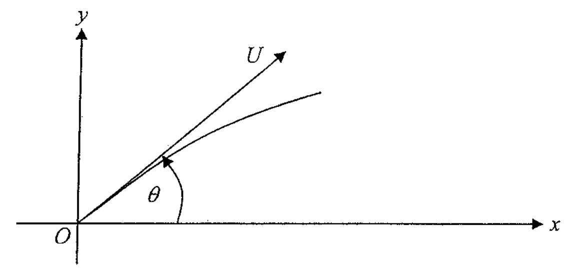
(ii) The region S is rotated around the line $x=a$ to form a solid. Use the method of cylindrical shells to find the volume of this solid. **4**

Question 6 (15 marks) **Marks**

(a) For what values of k does the equation $x^3-3x^2-24x+k=0$ have one real root? **4**

(b) A polynomial $P(x)$ gives remainders -2 and 1 when divided by $2x-1$ and $x-2$ respectively. What is the remainder when $P(x)$ is divided by $2x^2-5x+2$? **3**

(c) In an experiment, a particle of mass m is projected at an angle of θ to the horizontal and with an initial velocity of U. The forces acting on the particle are the gravitational force and a resistance force that is proportional to the square of its velocity in the horizontal and in the vertical directions.



The equation of motion in the horizontal direction is $\ddot{x}=-k\dot{x}^2$ where k is a constant. The equation of motion in the vertical direction is $\ddot{y}=-k\dot{y}^2-g$ where g is the acceleration due to gravity.

(i) Verify that $\dot{x}=\frac{U\cos\theta}{kUt\cos\theta+1}$ satisfies the appropriate equation of motion and initial conditions. **2**

(ii) Show that $t=\frac{e^{kx}-1}{kU\cos\theta}$. **2**

(iii) Verify that $t=\frac{1}{\sqrt{kg}}\left\{\tan^{-1}\left(\sqrt{\frac{k}{g}}U\sin\theta\right)-\tan^{-1}\left(\sqrt{\frac{k}{g}}\dot{y}\right)\right\}$ satisfies the appropriate equation of motion and initial condition. **3**

(iv) Find the value of t when the particle reaches its maximum height. **1**

Question 7 (15 marks) **Marks**

(a) The real numbers $a > 0$, $b > 0$ and $c > 0$ are such that $\frac{1}{a}$, $\frac{1}{b}$ and $\frac{1}{c}$ are in geometric progression.

(i) Using the fact that $a^2 + b^2 \geq 2ab$, show that $a^2 + c^2 \geq ab + bc$. **2**

(ii) Show $\frac{1}{a^2} + \frac{1}{c^2} \geq \frac{2}{b^2}$. **1**

(b) Consider the function $f(x) = x - \ln(x^2 + 1)$ for $x \geq 0$.

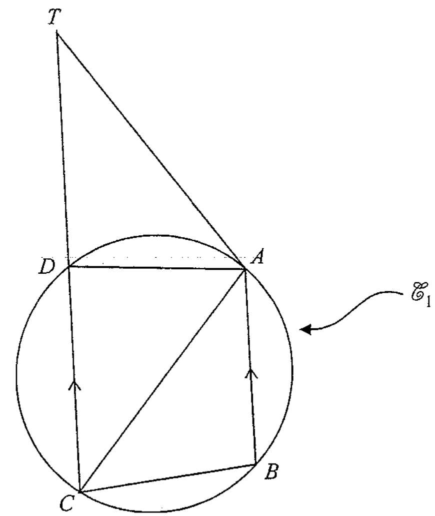
(i) Show that $x > \ln(x^2 + 1)$ for $x > 0$. **2**

(ii) By evaluating $\int_0^1 x\, dx$ and $\int_0^1 \ln(x^2 + 1)\, dx$, show that $5 > 2\ln 2 + \pi$. **3**

Question 7 continues on the next page

Question 7 (cont'd) **Marks**

(c)



The points A, B, C and D lie on the circle $\mathscr{C}_1$. From the exterior point T, a tangent is drawn to point A on $\mathscr{C}_1$. The line CT passes through D and TC is parallel to AB.

(i) Prove that ΔADT is similar to ΔABC. **3**

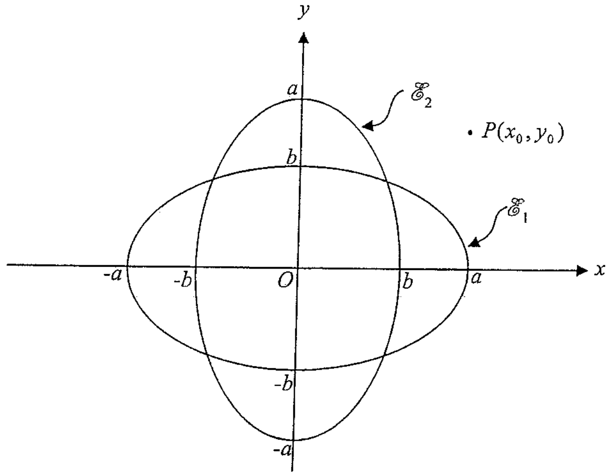
(ii) The line BA is produced through A to point M, which lies on a second circle $\mathscr{C}_2$. The points A, D, T also lie on $\mathscr{C}_2$ and the line DM crosses AT at O.

(1) Show that ΔOMA is isosceles. **1**

(2) Show that $TM = BC$ **3**

Question 8 (15 marks) **Marks**

(a)



The ellipse $\mathscr{E}_1$ has equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the ellipse $\mathscr{E}_2$ has equation $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$. The point $P(x_0, y_0)$ lies outside both $\mathscr{E}_1$ and $\mathscr{E}_2$.

(i) Find all the points of intersection of $\mathscr{E}_1$ with $\mathscr{E}_2$. **2**

(ii) The chord of contact to $\mathscr{E}_1$ from the point $P(x_0, y_0)$ has equation **3**

$$\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1.$$

This chord of contact intersects with the chord of contact to $\mathscr{E}_2$ from the point $P(x_0, y_0)$ at the point $Q(x_2, y_2)$.

Find the coordinates of the point Q.

(iii) Using your answer to part (i) or otherwise show that Q cannot lie outside both $\mathscr{E}_1$ and $\mathscr{E}_2$. **3**

Question 8 (cont'd) **Marks**

(b) (i) Show graphically or otherwise that **3**

$$\ln(n-1)! > \int_2^n \ln(x-1)\,dx \text{ for } n \geq 3$$

given that n is an integer.

(ii) Hence show that $n! > ne^{2-n}(n-1)^{n-1}$, $n \geq 3$. **4**

END OF EXAM

THE HEFFERNAN GROUP
P.O. Box 1180
Surrey Hills North VIC 3127
ABN 20 607 374 020
Phone 1800 100 521
Fax 1800 100 525

MATHEMATICS EXTENSION 2
HIGHER SCHOOL CERTIFICATE
TRIAL EXAMINATION
SOLUTIONS
2006

Question 1 (15 marks)

(a) (i)

$$\int \frac{\sin^{-1} 2x}{\sqrt{1-4x^2}}\,dx = \frac{1}{2}\int \frac{2\sin^{-1} 2x}{\sqrt{1-4x^2}}\,dx$$

$$= \frac{1}{4}\left(\sin^{-1} 2x\right)^2 + c$$

1 mark	Correct answer

(ii)

$$\int \frac{\cos^5 x}{\sin^2 x}\,dx = \int \frac{\cos x\left(1-\sin^2 x\right)^2}{\sin^2 x}\,dx$$

$$= \int \frac{\cos x\left(1-2\sin^2 x+\sin^4 x\right)}{\sin^2 x}\,dx$$

$$= \int \left(\cos x(\sin x)^{-2} - 2\cos x + \cos x(\sin x)^2\right)dx$$

$$= -(\sin x)^{-1} - 2\sin x + \frac{1}{3}(\sin x)^3$$

$$= -\operatorname{cosec} x - 2\sin x + \frac{1}{3}\sin^3 x + c$$

2 marks	Correct answer
1 mark	Obtaining $\int \left(\cos x(\sin x)^{-2} - 2\cos x + \cos x(\sin x)^2\right)dx$

(b)

$$\int_0^2 x^3 e^{x^2}\,dx = \int_0^2 x^2 x e^{x^2}\,dx$$

let $f(x) = xe^{x^2}$ so $F(x) = \frac{1}{2}e^{x^2}$

$g(x) = x^2$ so $g'(x) = 2x$

$$= \left[\frac{x^2}{2}e^{x^2}\right]_0^2 - \int_0^2 xe^{x^2}\,dx$$

$$= 2e^4 - \frac{1}{2}\left[e^{x^2}\right]_0^2$$

$$= 2e^4 - \frac{1}{2}\left(e^4 - 1\right)$$

$$= \frac{3}{2}e^4 + \frac{1}{2}$$

$$= \frac{1}{2}\left(3e^4 + 1\right)$$

2 marks	Correct answer
1 mark	Obtaining $\left[\frac{x^2}{2}e^{x^2}\right]_0^2 - \int_0^2 xe^{x^2}\,dx$

(c)

$$\frac{48}{x^3+64} = \frac{48}{(x+4)\left(x^2-4x+16\right)}$$

$$\frac{48}{(x+4)\left(x^2-4x+16\right)} \equiv \frac{A}{x+4} + \frac{Bx+C}{x^2-4x+16}$$

$$48 \equiv A\left(x^2-4x+16\right) + (Bx+C)(x+4)$$

Sub $x = -4$, $48 = 48A$

$A = 1$

$x = 0$, $48 = 16 + 4C$

$C = 8$

$x = 1$, $48 = 13 + 5(B+8)$

$B + 8 = 7$

$B = -1$

$$\int \frac{48}{x^3+64}\,dx = \int\left(\frac{1}{x+4} + \frac{8-x}{x^2-4x+16}\right)dx$$

$$= \int\left(\frac{1}{x+4} - \frac{2x-4}{2(x^2-4x+16)} + \frac{6}{x^2-4x+16}\right)dx$$

$$= \ln|x+4| - \frac{1}{2}\ln\left|x^2-4x+16\right| + \int \frac{6}{(x-2)^2+12}\,dx$$

$$= \ln\left|\frac{x+4}{\sqrt{x^2-4x+16}}\right| + \frac{6}{2\sqrt{3}}\tan^{-1}\left(\frac{x-2}{2\sqrt{3}}\right)$$

$$= \ln\left|\frac{x+4}{\sqrt{x^2-4x+16}}\right| + \sqrt{3}\tan^{-1}\left(\frac{x-2}{2\sqrt{3}}\right) + c$$

5 marks	Correct answer
4 marks	Obtaining $\ln\|x+4\| - \frac{1}{2}\ln\left\|x^2-4x+16\right\| + \int \frac{6}{(x-2)^2+12}\,dx$
3 marks	Obtaining $\int\left(\frac{1}{x+4} - \frac{2x-4}{2(x^2-4x+16)} + \frac{6}{x^2-4x+16}\right)dx$
2 marks	Finding A, B and C correctly
1 mark	Finding A or B or C correctly

(d) (i) $I_1 = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot x\,dx$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x}\,dx$$

$$= \left[\ln|\sin x|\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$= \ln\left|\sin\frac{\pi}{2}\right| - \ln\left|\sin\frac{\pi}{4}\right|$$

$$= \ln 1 - \ln\frac{1}{\sqrt{2}}$$

$$= \frac{1}{2}\ln 2$$

1 mark	Correct answer

(ii) $I_n = \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^n x\,dx$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \cot^2 x \cot^{n-2} x\,dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\operatorname{cosec}^2 x - 1\right)\cot^{n-2} x\,dx$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\operatorname{cosec}^2 x \cot^{n-2} x - \cot^{n-2} x\right)dx$$

$$= \left[-\frac{\cot^{n-1} x}{n-1}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - I_{n-2} \quad \left(\text{let } u = \cot x,\ \frac{du}{dx} = -\operatorname{cosec}^2 x\right)$$

$$I_n + I_{n-2} = -\frac{\cot^{n-1}\frac{\pi}{2}}{n-1} + \frac{\cot^{n-1}\frac{\pi}{4}}{n-1}$$

$$I_n + I_{n-2} = \frac{1}{n-1} \text{ for } n \ge 2$$

3 marks	Correct answer
2 marks	Obtaining $\left[-\frac{\cot^{n-1} x}{n-1}\right]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - I_{n-2}$
1 mark	Obtaining $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \left(\operatorname{cosec}^2 x - 1\right)\cot^{n-2} x\,dx$

(d) (iii) Find I_5.

Sub $n = 5$ into $I_n + I_{n-2} = \frac{1}{n-1}$

$$I_5 + I_3 = \frac{1}{4}$$

Sub $n = 3$ into $I_n + I_{n-2} = \frac{1}{n-1}$

$$I_3 + I_1 = \frac{1}{2}$$

$$I_5 = \frac{1}{4} - \left(\frac{1}{2} - I_1\right)$$

$$= -\frac{1}{4} + \frac{1}{2}\ln 2$$

1 mark	Correct answer

Question 2 (15 marks)

(a) (i)

$z=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i$

$z^4=\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)^4$

$=\frac{1}{4}\left(1+4i+6i^2+4i^3+i^4\right)$

$=\frac{1}{4}(1+4i-6-4+1)$

$=-1$

OR $z=\text{cis}\frac{\pi}{4}$

$z^4=\left(\text{cis}\frac{\pi}{4}\right)^4$

$=\text{cis}\,\pi$ (De Moivre)

$=\cos\pi+i\sin\pi$

$=-1$

2 marks	Correct answer
1 mark	Obtaining $\frac{1}{4}\left(1+4i+6i^2+4i^3+i^4\right)$ or $z^4=\text{cis}\,\pi$

(ii)

$z^{13}=z^{12}\cdot z$

$=\left(z^4\right)^3\cdot z$

$=(-1)^3\cdot z$

$=-\left(\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}i\right)$

OR $z^{13}=\left(\text{cis}\frac{\pi}{4}\right)^{13}$

$=\text{cis}\frac{13\pi}{4}$ (De Moivre)

$=\text{cis}\left(\frac{-3\pi}{4}\right)$

$=\cos\frac{3\pi}{4}-i\sin\frac{3\pi}{4}$

$=-\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i$

2 marks	Correct answer
1 mark	Obtaining $(-1)^3\cdot z$ or $\text{cis}\frac{13\pi}{4}$

(b)

$|z|^2-7=2i(z+2)$

Let $z=a+ib$

$|z|=\sqrt{a^2+b^2}$

$|z|^2=a^2+b^2$

$a^2+b^2-7=2i(a+ib+2)$

$=-2b+2ai+4i$

$a^2+b^2+2b-2ai=7+4i$

so $-2a=4$

$a=-2$

so $a^2+b^2+2b=7$

becomes $b^2+2b-3=0$

$(b+3)(b-1)=0$

$b=-3$ or $b=1$

$z=-2+i,\quad z=-2-3i$

4 marks	Correct answer
3 marks	Obtaining $b=-3$ and $b=1$
2 marks	Obtaining $a=-2$
1 mark	Obtaining $a^2+b^2+2b-2ai=7+4i$

(c) (i)

$z=4+4\sqrt{3}i$

$|z|=\sqrt{4^2+\left(4\sqrt{3}\right)^2}$

$=8$

$\arg z=\tan^{-1}\sqrt{3}$

$=\frac{\pi}{3}$

so $z=8\text{cis}\frac{\pi}{3}$

1 mark	Correct answer

(ii) $z = 8\text{cis}\left(2k\pi + \frac{\pi}{3}\right), \quad k \in J$

$z^{\frac{1}{3}} = 2\text{cis}\left(\frac{2k\pi}{3} + \frac{\pi}{9}\right)$ (De Moivre)

$k = 0, \quad z^{\frac{1}{3}} = 2\text{cis}\frac{\pi}{9}$

$k = 1, \quad z^{\frac{1}{3}} = 2\text{cis}\frac{7\pi}{9}$

$k = -1, \quad z^{\frac{1}{3}} = 2\text{cis}\left(\frac{-5\pi}{9}\right)$

Therefore the three values of $z^{\frac{1}{3}}$ are

$2\left(\cos\left(\frac{-5\pi}{9}\right) + i\sin\left(\frac{-5\pi}{9}\right)\right)$, $2\left(\cos\frac{\pi}{9} + i\sin\frac{\pi}{9}\right)$ and $2\left(\cos\frac{7\pi}{9} + i\sin\frac{7\pi}{9}\right)$

3 marks	Correct answers in modulus-argument form
2 marks	Correctly finding the 3 different arguments
1 mark	Finding $z^{\frac{1}{3}} = 2\text{cis}\left(\frac{2k\pi}{3} + \frac{\pi}{9}\right)$

(d) Show $(z_1 + z_2)^2 = 2z_1z_2$

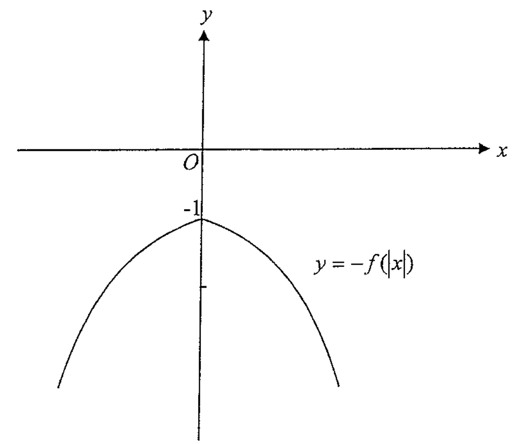
$z_2 = iz_1$, since $\angle BOA = \frac{\pi}{2}$

$$\begin{aligned} z_1 + z_2 &= z_1 + iz_1 \\ &= z_1(1+i) \\ (z_1 + z_2)^2 &= z_1^2(1+i)^2 \\ &= 2z_1^2 i \\ &= 2z_1(iz_1) \\ &= 2z_1z_2 \end{aligned}$$

3 marks	Correctly showing $(z_1 + z_2)^2 = 2z_1z_2$
2 marks	Showing $(z_1 + z_2)^2 = 2{z_1}^2 i$
1 mark	Showing $z_1 + z_2 = z_1(1+i)$

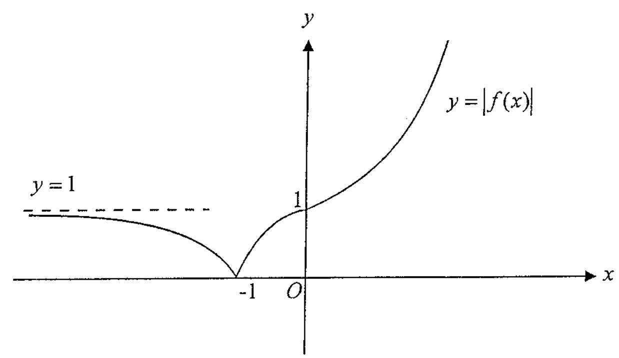
Question 3 (15 marks)

(a) (i) $y = -f(|x|)$.



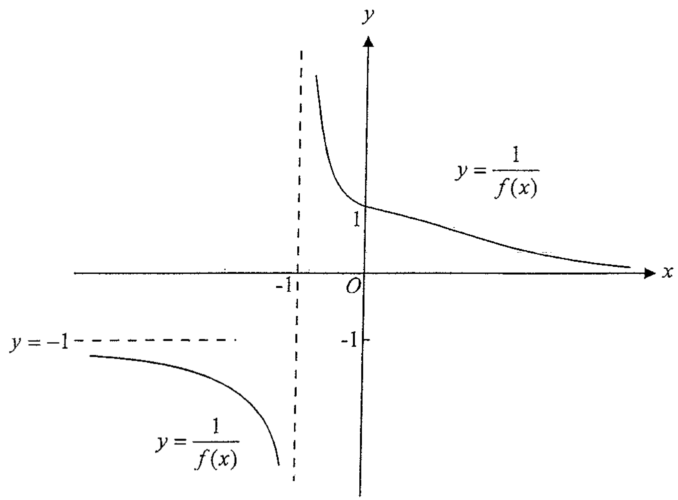
1 mark	Correct graph

(ii) $y = |f(x)|$



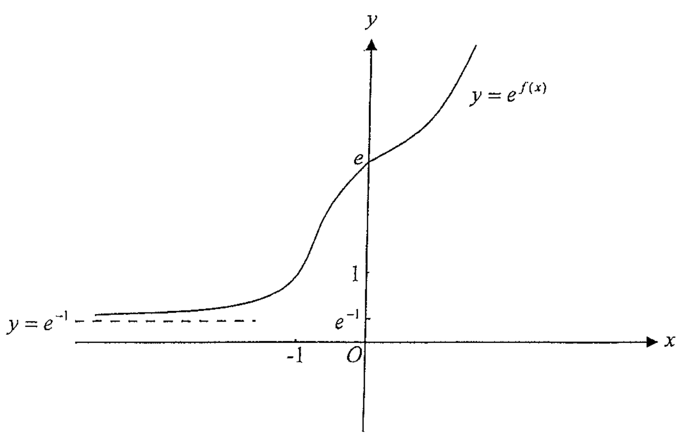
2 marks	Correct graph
1 mark	Correct graph with cusp and asymptote not labelled or incorrectly labelled

(iii) $y=\frac{1}{f(x)}$



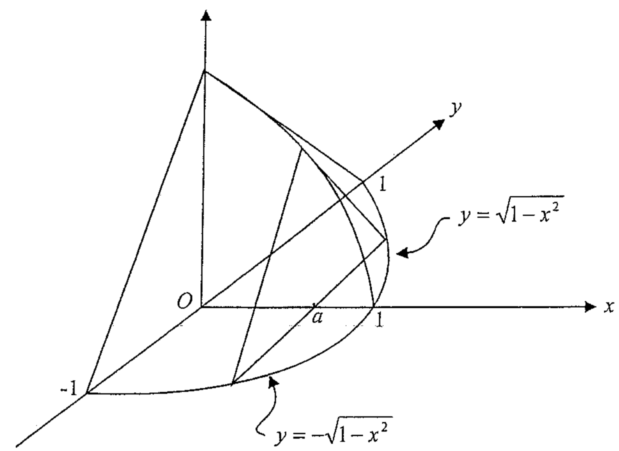
2 marks	Correct graph
1 mark	Correct graph with asymptotes and intercepts not labelled or not labelled correctly

(iv) $y=e^{f(x)}$



2 marks	Correct graph
1 mark	Correct graph with asymptotes and intercepts not labelled or not labelled correctly

(b) (i)

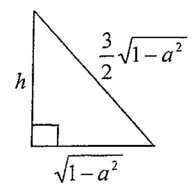


The base of the shaded isosceles triangle has length $2\sqrt{1-a^2}$.
The two equal sidelengths are therefore of length

$\frac{3}{4}\times 2\sqrt{1-a^2}$

$=\frac{3}{2}\sqrt{1-a^2}$

The height of the isosceles triangle is given by



$h=\sqrt{\frac{9}{4}\left(1-a^2\right)-\left(1-a^2\right)}$

$=\sqrt{\frac{5}{4}-\frac{5}{4}a^2}$

$=\sqrt{\frac{5}{4}\left(1-a^2\right)}$

So the area of the isosceles triangle is

$\frac{1}{2}\times \text{base}\times \text{height}$

$=\frac{1}{2}\times 2\sqrt{1-a^2}\times\sqrt{\frac{5}{4}\left(1-a^2\right)}$

$=\sqrt{\frac{5}{4}\left(1-a^2\right)^2}$

$=\frac{\sqrt{5}}{2}\left(1-a^2\right)$ as required.

2 marks	Correct derivation
1 mark	Correct method with one mistake OR identifying the base length of the isosceles triangle as $2\sqrt{1-a^2}$

(ii) Now $\delta V = A\delta x$ where A = area of isosceles triangle

$$= \frac{\sqrt{5}}{2}\left(1 - x^2\right)\delta x$$

$$V = \lim_{\delta x \to 0} \sum_{x=0}^{1} \frac{\sqrt{5}}{2}\left(1 - x^2\right)\delta x$$

$$= \frac{\sqrt{5}}{2}\int_0^1 \left(1 - x^2\right)dx$$

$$= \frac{\sqrt{5}}{2}\left[x - \frac{x^3}{3}\right]_0^1$$

$$= \frac{\sqrt{5}}{2}\left\{\left(1 - \frac{1}{3}\right) - 0\right\}$$

$$= \frac{\sqrt{5}}{2} \times \frac{2}{3}$$

$$= \frac{\sqrt{5}}{3} \text{ cubic units}$$

2 marks	Correct answer
1 mark	Correct method with one mistake OR obtaining $V = \frac{\sqrt{5}}{2}\int_0^1 \left(1 - x^2\right)dx$

(c) For $\frac{x^2}{9} + \frac{y^2}{4} = 1$,

$$b^2 = a^2\left(1 - e^2\right)$$

$$4 = 9\left(1 - e^2\right)$$

$$e^2 = \frac{5}{9}$$

$$e = \frac{\sqrt{5}}{3}, \quad e > 0$$

The directrices of the ellipse are given by $x = \pm 3 \div \frac{\sqrt{5}}{3}$

$$= \pm \frac{9\sqrt{5}}{5}$$

For $\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$

$$b^2 = a^2\left(e^2 - 1\right)$$

$$B^2 = A^2\left(e^2 - 1\right)$$

$$e^2 = 1 + \frac{B^2}{A^2}$$

$$e = \sqrt{\frac{A^2 + B^2}{A^2}}, \quad e > 0$$

$$= \frac{1}{A}\sqrt{A^2 + B^2}$$

The directrices of the hyperbola are given by $x = \pm \frac{A}{\frac{1}{A}\sqrt{A^2 + B^2}}$

$$= \pm \frac{A^2}{\sqrt{A^2 + B^2}}$$

Since the directrices are the same, $\frac{A^2}{\sqrt{A^2 + B^2}} = \frac{9\sqrt{5}}{5}$

$$5A^2 = 9\sqrt{5}\sqrt{A^2 + B^2}$$

$$A^2 = \frac{9}{5}\sqrt{5\left(A^2 + B^2\right)}$$

4 marks	Showing correctly the expression for A^2
3 marks	Obtaining $\frac{A^2}{\sqrt{A^2 + B^2}} = \frac{9\sqrt{5}}{5}$
2 marks	Finding the correct directrices of the ellipse and the hyperbola
1 mark	Finding the correct directrices of the ellipse or the hyperbola OR finding the correct value of e for the ellipse and the hyperbola

Question 4 (15 marks)

(a) (i) $x^3 - 3x - 2 = 0$ let $x = \alpha^3$

$\left(x^{\frac{1}{3}}\right)^3 - 3x^{\frac{1}{3}} - 2 = 0$ so $\alpha = x^{\frac{1}{3}}$

$$3x^{\frac{1}{3}} = x - 2$$
$$27x = (x-2)^3$$
$$= x^3 - 6x^2 + 12x - 8$$
$$x^3 - 6x^2 - 15x - 8 = 0$$

2 marks	Correct answer
1 mark	Showing $3x^{\frac{1}{3}} = x - 2$

(ii) $\alpha^7\beta\gamma + \alpha\beta^7\gamma + \alpha\beta\gamma^7$

$$= \alpha\beta\gamma\left(\alpha^6 + \beta^6 + \gamma^6\right)$$
$$\alpha^6 + \beta^6 + \gamma^6 = \left(\alpha^3 + \beta^3 + \gamma^3\right)^2 - 2\left(\alpha^3\beta^3 + \alpha^3\gamma^3 + \beta^3\gamma^3\right)$$
$$= 6^2 - 2(-15) \quad \text{(from i)}$$
$$= 66$$
$$\alpha\beta\gamma = 2$$
$$\alpha^7\beta\gamma + \alpha\beta^7\gamma + \alpha\beta\gamma^7 = 2(66)$$
$$= 132$$

2 marks	Correct answer
1 mark	Showing $\alpha^6 + \beta^6 + \gamma^6 = 66$

(b) (i) $\frac{d}{dx}(x^2 - xy - y^2 = 1)$

$$2x - y - x\frac{dy}{dx} - 2y\frac{dy}{dx} = 0$$
$$\frac{dy}{dx} = \frac{2x-y}{x+2y}$$

At $(2,1)$, $\frac{dy}{dx} = \frac{3}{4}$

The equation of the normal at $(2,1)$ is given by

$$y - 1 = -\frac{4}{3}(x-2)$$
$$3y - 3 = -4x + 8$$
$$4x + 3y - 11 = 0$$
$$y = \frac{11-4x}{3}$$

2 marks	Correct answer
1 mark	Finding gradient of tangent

(ii) Substitute $y = \frac{11-4x}{3}$ into $x^2 - xy - y^2 = 1$.

$$x^2 - x\frac{(11-4x)}{3} - \left(\frac{11-4x}{3}\right)^2 = 1$$
$$9x^2 - 3x(11-4x) - (11-4x)^2 = 9$$
$$9x^2 - 33x + 12x^2 - 121 + 88x - 16x^2 - 9 = 0$$
$$5x^2 + 55x - 130 = 0$$
$$x^2 + 11x - 26 = 0$$
$$(x-2)(x+13) = 0$$
$$x = 2 \text{ or } x = -13$$

When $x = 2$, $y = 1$

When $x = -13$, $y = 21$

$(-13, 21)$ is the other point of intersection.

2 marks	Correct answer
1 mark	Obtaining $5x^2 + 55x - 130 = 0$

(c) <u>To prove</u>: $\sum_{r=1}^{n} r^3 < \{n(n+1)\}^2$

that is, $1^3 + 2^3 + ... + n^3 < n^2(n+1)^2$

<u>Step 1</u>

Let $n = 1$

$LS = 1$

$RS = 1 \times 4$

$= 4$

$LS < RS$

So it is true for $n = 1$.

<u>Step 2</u>

Let us assume that it is true for $n = k$. That is, let us assume that

$1^3 + 2^3 + ... + k^3 < k^2(k+1)^2$ $\quad -(A)$

Let $n = k+1$

We have

$1^3 + 2^3 + ... + k^3 + (k+1)^3 < (k+1)^2(k+2)^2$ $\quad -(B)$

$LS = 1^3 + 2^3 + ... + k^3 + (k+1)^3$

$< k^2(k+1)^2 + (k+1)^3$ $\quad$ given (A)

$= (k+1)^2(k^2 + k + 1)$

$RS = (k+1)^2(k+2)^2$

$= (k+1)^2(k^2 + 4k + 4)$

So $LS < RS$

<u>Step 3</u>

So it is true for $n = 1$ and it is true for $n = k+1$ so it is true for $n = 2$ and so on using the principle of mathematical induction.

3 marks	Correct proof
2 marks	Obtaining the expression (B)
1 mark	Showing that it is true for $n = 1$

(d) amplitude $= \dfrac{0 \cdot 8 + 0 \cdot 2}{2}$

$= 0 \cdot 5$

period $= 2 \times 365$

$= 730$

and $n = \dfrac{2\pi}{730} = \dfrac{\pi}{365}$

We have *SHM* so $\ddot{x} = -\left(\dfrac{\pi}{365}\right)^2 x$ and $x = 0 \cdot 5\cos\left(\dfrac{\pi t}{365} + \theta\right) \quad 0 \le \theta < 2\pi$

When $t = 0$, $\quad x = 0 \cdot 5$

$0 \cdot 5 = 0 \cdot 5\cos(0 + \theta)$

$\cos\theta = 1$

$\theta = 0$

So, $\quad x = 0 \cdot 5\cos\left(\dfrac{\pi t}{365}\right)$

When there is 0.5 metre above the top of the post, $x = 0 \cdot 2$.

So, $\quad 0 \cdot 5\cos\left(\dfrac{\pi t}{365}\right) = 0 \cdot 2$

$t = 134 \cdot 6886...$

$= 135$ (to the nearest minute)

Between high tide and low tide there is at least 0.5m of water above the top of the old anchorage post between 6.30am and 8.45am and again between 6.40 pm and 8.55 pm. Note that high tide occurs at 6.30 am and again at 6.40 pm. We are asked for the times between "high tide and low tide" for which the water is at least 0.5 above the post. This condition limits the possible times to immediately after 6.30 am and immediately after 6.40 pm and not immediately before those times.

4 marks	Correct answer
3 marks	Obtaining $t = 134 \cdot 6886...$
2 marks	Obtaining $x = 0 \cdot 5\cos\left(\dfrac{\pi t}{365}\right)$
1 mark	Obtaining an expression for amplitude and period

Question 5 (15 marks)

(a) (i) Consider the boys as one "unit" and the girls as one "unit". The two "units" can be arranged in 2! ways.
Number of ways the group can be arranged $= 7!5!2!$
$= 1\,209\,600$

1 mark	Correct answer

(ii) _B_B_B_B_B_B_B_
The 5 girls can be seated in 8 possible seats available as shown between the boys or at either end of the boys. For every arrangement of the girls, the boys can be arranged in7! ways.
Number of ways$= {}^8P_5 \times 7!$ OR $8\times 7\times 6\times 5\times 4\times 7!$
$= 33\ 868\ 800$

2 marks	Correct answer
1 mark	Obtaining $7!$ OR ${}^8P_5$

(iii) There are 2! ways that the two girls can be arranged and 5!, 3! and 2! ways respectively for the other 3 groups. There are 4 groups to be arranged but because they are to be arranged in a circle the groups can be arranged in 3! ways.
Total number of ways $= 2!\times 5!\times 3!\times 2!\times 3!$
$= 17\,280$ ways

2 marks	Correct answer
1 mark	Gives some of the terms 3!, 2!, 5!

(b) (i) Let $P(x) = ax^4 + bx^3 + cx^2 + dx + e = 0$

$$P'(x) = 4ax^3 + 3bx^2 + 2cx + d$$
$$P''(x) = 12ax^2 + 6bx + 2c$$
$$P'''(x) = 24ax + 6b = 0 \text{ (since quadruple root)}$$
$$x = -\frac{6b}{24a}$$
$$\alpha = -\frac{b}{4a}$$

2 marks	Correctly showing $\alpha = -\dfrac{b}{4a}$
1 mark	Showing $P'''(x) = 24ax + 6b = 0$

(ii) Since $\alpha = -\dfrac{b}{4a}$ is a quadruple root,

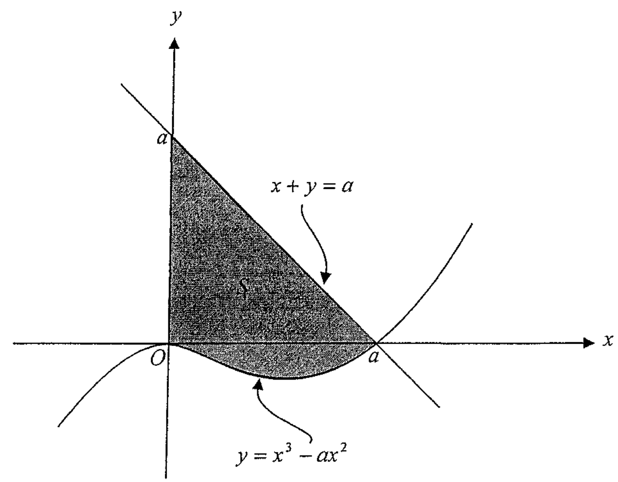
$$P(x) = a(x-\alpha)^4 = ax^4 + bx^3 + cx^2 + dx + e = 0$$
$$(x-\alpha)^4 = \frac{ax^4 + bx^3 + cx^2 + dx + e}{a}$$

Substitute $x = 1$, $\alpha = -\dfrac{b}{4a}$

$$\text{so, } \left(1 + \frac{b}{4a}\right)^4 = \frac{a+b+c+d+e}{a}$$

2 marks	Correct answer
1 mark	Showing $(x-\alpha)^4 = \dfrac{ax^4 + bx^3 + cx^2 + dx + e}{a}$

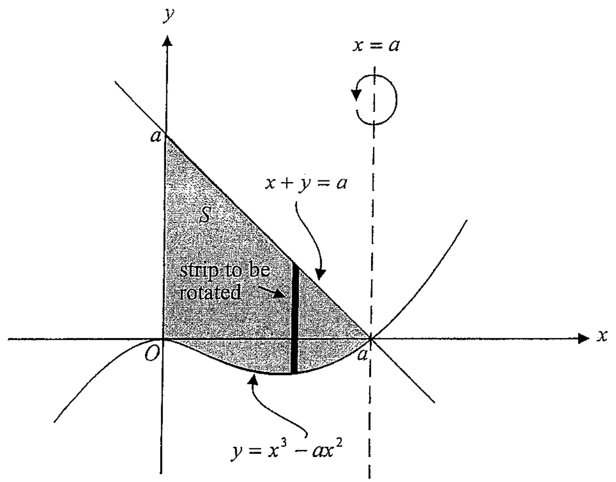
(c) (i)



$$y = x^3 - ax^2$$
$$= x^2(x-a)$$

2 marks	Correctly labelled and shaded diagram
1 mark	Reasonable attempt at sketching but leaving off intercepts or shading

(ii)



A typical strip that is to be rotated around the line $x = a$ is shown. The cylindrical shell that is formed has height $(a-x)-(x^3-ax^2)=-x^3+ax^2-x+a$, circumference $2\pi(a-x)$ and thickness δx. If cut vertically, it forms a rectangular solid with the dimensions $-x^3+ax^2-x+a$ by $2\pi(a-x)$ by δx.

So, $\delta V = 2\pi(-x^3+ax^2-x+a)(a-x)\delta x$

$$V = 2\pi\int_0^a (-x^3+ax^2-x+a)(a-x)dx, \qquad a>0$$

$$= 2\pi\int_0^a \left(x^4-2ax^3+(a^2+1)x^2-2ax+a^2\right)dx$$

$$= 2\pi\left[\frac{x^5}{5}-\frac{ax^4}{2}+(a^2+1)\frac{x^3}{3}-ax^2+a^2x\right]_0^a$$

$$= 2\pi\left(\frac{a^5}{5}-\frac{a^5}{2}+\frac{a^5}{3}+\frac{a^3}{3}\right)$$

$$= 2\pi\left(\frac{a^5+10a^3}{30}\right) \text{ cubic units}$$

4 marks	Correct solution
3 marks	Shows $V = 2\pi\int_0^a \left(x^4-2ax^3+(a^2+1)x^2-2ax+a^2\right)dx$
2 marks	Obtains an integral with correct expression for height and circumference of cylindrical shell
1 mark	Obtains the correct expression for the height or circumference of the shell

Question 6 (15 marks)

(a) Let $P(x) = x^3-3x^2-24x+k=0$

$P'(x) = 3x^2-6x-24$

For stationary points $(P'(x)=0)$:

$3x^2-6x-24=0$

$x^2-2x-8=0$

$(x-4)(x+2)=0$

$x=4,\ x=-2$

$x=4, \quad P(4)=4^3-3(4)^2-24(4)+k$

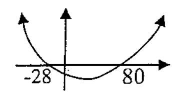
$= k-80$

$x=-2, \quad P(-2)=(-2)^3-3(-2)^2-24(-2)+k$

$= k+28$

$P(-2)\times P(4)>0$ (since one real root)

$(k-80)(k+28)>0$



$k<-28,\ k>80$

4 marks	Correct answer
3 marks	Showing $(k-80)(k+28)>0$
2 marks	Showing $P(4)=k-80$ and $P(-2)=k+28$
1 mark	Correctly finding x-values of stationary points

(b) Let $P(x)=(2x^2-5x+2)Q(x)+R(x)$

since $\deg D(x) > \deg R(x)$

then $\deg R(x) < 2$

let $R(x) = ax+b$

$P(x) = (2x-1)(x-2)Q(x)+ax+b$

$P\left(\frac{1}{2}\right) = \frac{1}{2}a+b=-2 \qquad -(1)$

$P(2) = 2a+b=1 \qquad -(2)$

$(2)-(1) \Rightarrow \frac{3a}{2}=3$

$a=2,\ b=-3$

so $R(x)=2x-3$

3 marks	Correct answer
2 marks	Correctly finding $P\left(\frac{1}{2}\right)$ and $P(2)$
1 mark	Finding $P(x)=(2x-1)(x-2)Q(x)+ax+b$

(c) (i) <u>To verify</u>: $\ddot{x}=-k\dot{x}^2$

$$\begin{aligned}\text{LS} &= \ddot{x}\\ &= \frac{d}{dt}\left(\frac{U\cos\theta}{kUt\cos\theta+1}\right)\\ &= -U\cos\theta(kUt\cos\theta+1)^{-2}\times kU\cos\theta\\ &= \frac{-kU^2\cos^2\theta}{(kUt\cos\theta+1)^2}\\ &= -k\dot{x}^2 \text{ given } \dot{x}=\frac{U\cos\theta}{kUt\cos\theta+1}\\ &= \text{RS}\end{aligned}$$

Have verified.

Also, when $t=0$, $\dot{x}=U\cos\theta$.

For $\dot{x}=\dfrac{U\cos\theta}{kUt\cos\theta+1}$,

when $t=0$, $\dot{x}=\dfrac{U\cos\theta}{1}$
$=U\cos\theta$

so the expression for $\dot{x}$ satisfies the initial conditions.

2 marks	Verifying that the equation of motion AND the initial condition have been satisfied
1 mark	Verifying that one of the above have been satisfied

(ii) Now $\dot{x}=\dfrac{U\cos\theta}{kUt\cos\theta+1}$

so $x=\dfrac{1}{k}\displaystyle\int\frac{kU\cos\theta}{kUt\cos\theta+1}\,dt$

$x=\dfrac{1}{k}\ln(kUt\cos\theta+1)+c$

When $t=0$, $x=0$

$0=\dfrac{1}{k}\ln 1+c$

$c=0$

So, $x=\dfrac{1}{k}\ln(kUt\cos\theta+1)$

$kx=\ln(kUt\cos\theta+1)$

$e^{kx}=kUt\cos\theta+1$

$t=\dfrac{e^{kx}-1}{kU\cos\theta}$ as required

2 marks	Correctly showing $t=\dfrac{e^{kx}-1}{kU\cos\theta}$
1 mark	Showing $\dfrac{1}{k}\ln(kUt\cos\theta+1)$

(iii) To verify: $\ddot{y} = -k\dot{y}^2 - g$

$$\text{LS} = \ddot{y}$$
$$= \frac{d\dot{y}}{dt}$$
$$= \frac{1}{\frac{dt}{d\dot{y}}}$$

Now, $t = \frac{1}{\sqrt{kg}}\left\{\tan^{-1}\left(\sqrt{\frac{k}{g}}U\sin\theta\right) - \tan^{-1}\left(\sqrt{\frac{k}{g}}\dot{y}\right)\right\}$

$$\frac{dt}{d\dot{y}} = -\frac{1}{\sqrt{kg}}\left(\frac{\sqrt{g}}{\sqrt{k}} \div \left(\frac{g}{k} + \dot{y}^2\right)\right)$$ note that $\tan^{-1}\left(\sqrt{\frac{k}{g}}U\sin\theta\right)$ is a constant

$$= -\frac{1}{\sqrt{kg}}\left(\sqrt{\frac{g}{k}} \times \frac{k}{g + k\dot{y}^2}\right)$$
$$= \frac{-1}{g + k\dot{y}^2}$$

So $\text{LS} = \frac{1}{\frac{-1}{g + k\dot{y}^2}}$

$$= -k\dot{y}^2 - g$$
$$= \text{RS}$$

Have verified.

Also when $t = 0$, $\dot{y} = U\sin\theta$.

For $t = \frac{1}{\sqrt{kg}}\left\{\tan^{-1}\left(\sqrt{\frac{k}{g}}U\sin\theta\right) - \tan^{-1}\left(\sqrt{\frac{k}{g}}\dot{y}\right)\right\}$

$\text{LS} = 0$

$$\text{RS} = \frac{1}{\sqrt{kg}}\left(\tan^{-1}\left(\sqrt{\frac{k}{g}}U\sin\theta\right) - \tan^{-1}\sqrt{\frac{k}{g}}U\sin\theta\right)$$
$$= 0$$
$$= \text{LS}$$

Have verified.

3 marks	Correctly verifying the motion of equation and the initial condition
2 marks	Correctly verifying the motion of equation OR attempting to verify the motion of equation and correctly verifying the initial condition
1 mark	Attempting to verify the motion of equation OR correctly verifying the initial condition

(iv) The particle reaches its maximum when height when $\dot{y} = 0$.
So, when $\dot{y} = 0$,

$$t = \frac{1}{\sqrt{kg}}\left\{\tan^{-1}\left(\sqrt{\frac{k}{g}}U\sin\theta - \tan^{-1}(0)\right)\right\}$$
$$= \frac{1}{\sqrt{kg}}\tan^{-1}\left(\sqrt{\frac{k}{g}}U\sin\theta\right)$$

1 mark	Correct answer

Question 7 (15 marks)

(a) (i)

$a^2 + b^2 \geq 2ab$ (1)

$a^2 + c^2 \geq 2ac$ (2)

$b^2 + c^2 \geq 2bc$ (3)

$(1)+(2)+(3)$ gives

$a^2 + b^2 + c^2 \geq ab + ac + bc$ (4)

Since $\frac{1}{a}, \frac{1}{b}$ and $\frac{1}{c}$ form a GP,

$$\frac{\frac{1}{b}}{\frac{1}{a}} = \frac{\frac{1}{c}}{\frac{1}{b}}$$

$b^2 = ac$

Sub $b^2 = ac$ into (4)

$a^2 + ac + c^2 \geq ab + ac + bc$

so $a^2 + c^2 \geq ab + bc$

2 marks	Correct answer
1 mark	Showing $b^2 = ac$

(ii) Show $\frac{1}{a^2} + \frac{1}{c^2} \geq \frac{2}{b^2}$

$\left(\frac{1}{a} - \frac{1}{c}\right)^2 \geq 0$

$\frac{1}{a^2} + \frac{1}{c^2} \geq \frac{2}{ac}$

since $b^2 = ac$

$\frac{1}{a^2} + \frac{1}{c^2} \geq \frac{2}{b^2}$

1 mark	Correctly showing $\frac{1}{a^2} + \frac{1}{c^2} \geq \frac{2}{b^2}$

(b) (i) $f(x) = x - \ln(x^2+1)$ for $x \geq 0$

$f'(x) = 1 - \frac{2x}{x^2+1}$

$= \frac{x^2+1-2x}{x^2+1}$

$= \frac{(x-1)^2}{x^2+1}$ for $x > 0$

For $x \neq 1$, $(x-1)^2 > 0$

so $f'(x) > 0$ for $x \neq 1$

at $x = 1$, $f(1) = 1 - \ln 2 > 0$

$x = 0$, $f(0) = 0$

since $f(0) = 0$, $f(1) > 0$ and $f'(x) > 0$ for $x \neq 1$,

$f(x) > 0$ for $x > 0$

so $x > \ln(x^2+1)$ for $x > 0$

2 marks	Correct answer
1 mark	Obtaining $f'(x) > 0$ for $x \neq 1$

(ii) $\int_0^1 x\,dx = \left[\frac{x^2}{2}\right]_0^1$

$= \frac{1}{2}$

$\int_0^1 \ln(x^2+1)dx$

let $f(x)=1 \quad F(x)=x$

$g(x)=\ln(x^2+1) \quad g'(x)=\frac{2x}{x^2+1}$

$= \left[x\ln(x^2+1)\right]_0^1 - \int_0^1 \frac{2x^2}{x^2+1}dx$

$= \ln 2 - 2\int_0^1 \frac{x^2+1-1}{x^2+1}dx$

$= \ln 2 - 2\int_0^1 (1-\frac{1}{x^2+1})dx$

$= \ln 2 - 2\left[x - \tan^{-1}x\right]_0^1$

$= \ln 2 - 2((1-\tan^{-1}1)-0)$

$= \ln 2 - 2 + \frac{\pi}{2}$

Since $x > \ln(x^2+1)$ for $x > 0$

$\int_0^1 x\,dx > \int_0^1 \ln(x^2+1)dx$

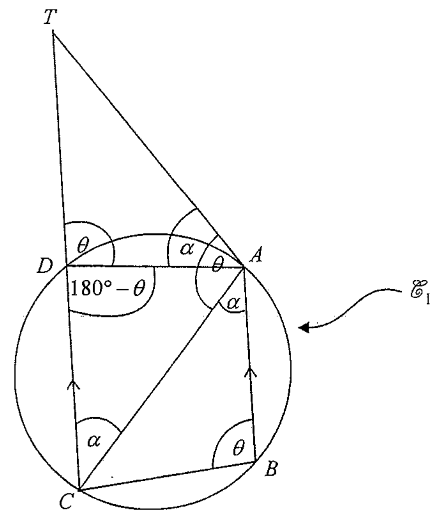
$\frac{1}{2} > \ln 2 - 2 + \frac{\pi}{2}$

$\frac{5}{2} > \ln 2 + \frac{\pi}{2}$

Hence $5 > 2\ln 2 + \pi$

3 marks	Correct answer
2 marks	Obtaining $\int_0^1 \ln(x^2+1)dx = \ln 2 - 2 + \frac{\pi}{2}$
1 mark	Obtaining $\int_0^1 \ln(x^2+1)dx = \left[x\ln(x^2+1)\right]_0^1 - \int_0^1 \frac{2x^2}{x^2+1}dx$

(c) (i)



$\angle TAC = \angle ABC$ (The angle between a tangent and a chord through the point of contact is equal to the angle in the alternate segment.)

$\angle TAD = \angle ACD$ (same reason)

$\angle BAC = \angle DCA$ (Alternate angles in parallel lines are equal.)

$\angle ADC = 180° - \angle ABC$ (Opposite angles in a cyclic quadrilateral are supplementary.)

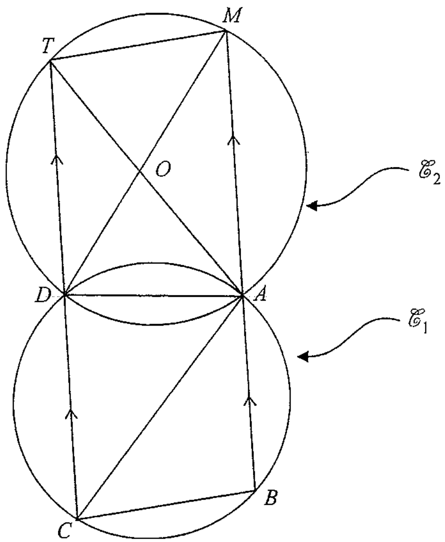
$\angle ADT = \angle ABC$ (Angles along a straight line are supplementary.)

$\angle ATD = \angle BCA$ (Angles in a triangle add to give $180°$.)

So ΔADT is similar to ΔABC (Three corresponding pairs of angles are equal.)

3 marks	Correctly reasoned proof
2 marks	Derives correctly that $\angle TAD = \angle BAC$ and attempts to derive that $\angle ADT = \angle ABC$
1 mark	States that $\angle TCA = \angle BAC$ OR attempts to derive that $\angle ADT = \angle ABC$ OR attempts to derive that $\angle TAD = \angle BAC$

(ii) (1) Draw a diagram

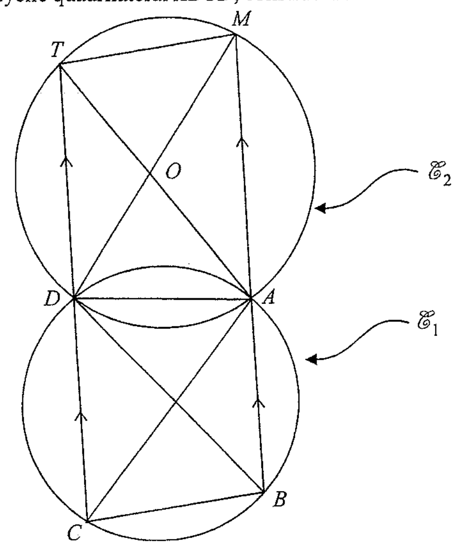


Now $\angle OMA = \angle OTD$ (Angles in the same segment are equal.)
and $\angle OTD = \angle OAM$ (Alternate angles in parallel lines are equal.)
So, $\angle OMA = \angle OAM$ and so ΔOMA is isosceles.

1 mark	Correctly explaining ΔOMA is isosceles

(2) From part (1) $\angle OMA = \angle OAM$
Also $\angle OTD = \angle OMA$ (Angles on the same segment are equal.)
and $\angle ODT = \angle OAM$ (Angles on the same segment are equal.)
so $\angle OTD = \angle OMA = \angle OAM = \angle ODT$
So since $\angle OTD$ and $\angle OMA$ stand on the same segment DA, and are both equal to $\angle ODT$ and $\angle OAM$ then the arc that these latter two angles stand on; namely TM, must be equal to DA.

In the cyclic quadrilateral $ABCD$, construct BD.



Similarly,
$\angle BAC = \angle ACD$ (Alternate angles in parallel lines are equal.)
Also $\angle BAC = \angle BDC$ (Angles on the same segment are equal.)
so $\angle ACD = \angle BDC$.
Also $\angle ABD = \angle ACD$ (Angles on the same segment are equal.)
and $\angle ACD = \angle BAC$ (Alternate angles in parallel lines are equal.)
so $\angle BAC = \angle ACD = \angle BDC = \angle ABD$

So since $\angle BDC$ and $\angle CAB$ stand on the same segment BC and are both equal to $\angle ACD$ and $\angle DBA$ then the arc that these latter two angles stand on; namely DA, must be equal to BC.
Since $TM = DA$ then $TM = BC$.

3 marks	Correct explanation
2 marks	Correctly explaining that $TM = DA$ AND showing a diagram
1 mark	Correctly explaining that $TM = DA$ OR showing a diagram

Question 8 (15 marks)

(a) (i) For $\mathscr{E}_1$, $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$x^2=a^2\left(1-\frac{y^2}{b^2}\right)$$

In $\frac{x^2}{b^2}+\frac{y^2}{a^2}=1$

gives $\frac{a^2}{b^2}\left(1-\frac{y^2}{b^2}\right)+\frac{y^2}{a^2}=1$

$$\frac{a^2}{b^2}-\frac{a^2y^2}{b^4}+\frac{y^2}{a^2}=1$$

$$y^2\left(\frac{1}{a^2}-\frac{a^2}{b^4}\right)=1-\frac{a^2}{b^2}$$

$$y^2\left(\frac{b^4-a^4}{a^2b^4}\right)=\frac{b^2-a^2}{b^2}$$

$$y^2=\frac{a^2b^2(b^2-a^2)}{(b^2-a^2)(b^2+a^2)}$$

$$=\frac{a^2b^2}{a^2+b^2}$$

$$y=\pm\frac{ab}{\sqrt{a^2+b^2}}$$

In $x^2=a^2\left(1-\frac{y^2}{b^2}\right)$

gives $x^2=a^2\left(1-\frac{a^2b^2}{b^2(a^2+b^2)}\right)$

$$=a^2-\frac{a^4}{a^2+b^2}$$

$$=\frac{a^2(a^2+b^2)-a^4}{a^2+b^2}$$

$$x^2=\frac{a^2b^2}{a^2+b^2}$$

$$x=\pm\frac{ab}{\sqrt{a^2+b^2}}$$

The four points of intersection are $\left(\frac{ab}{\sqrt{a^2+b^2}},\pm\frac{ab}{\sqrt{a^2+b^2}}\right)$ and $\left(\frac{-ab}{\sqrt{a^2+b^2}},\pm\frac{ab}{\sqrt{a^2+b^2}}\right)$.

2 marks	Four correct points
1 mark	Finding one correct point or finding 4 points with a mistake included.

(ii) The chord of contact to $\mathscr{E}_1$ from $P(x_0,y_0)$ is given by

$$\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1 \qquad -(A)$$

The chord of contact to $\mathscr{E}_2$ from $P(x_0,y_0)$ is given by

$$\frac{xx_0}{b^2}+\frac{yy_0}{a^2}=1 \qquad -(B)$$

Now, $\frac{xx_0}{a^2}+\frac{yy_0}{b^2}=1$

gives $\frac{xx_0}{a^2}=1-\frac{yy_0}{b^2}$

$$x=\frac{a^2}{x_0}\left(1-\frac{yy_0}{b^2}\right)$$

In (B) $\frac{a^2x_0}{x_0b^2}\left(1-\frac{yy_0}{b^2}\right)+\frac{yy_0}{a^2}=1$

$$\frac{a^2}{b^2}-\frac{a^2yy_0}{b^4}+\frac{yy_0}{a^2}=1$$

$$y\left(\frac{y_0}{a^2}-\frac{a^2y_0}{b^4}\right)=1-\frac{a^2}{b^2}$$

$$y\left(\frac{b^4y_0-a^4y_0}{a^2b^4}\right)=\frac{b^2-a^2}{b^2}$$

$$y=\frac{(b^2-a^2)}{b^2}\times\frac{a^2b^4}{y_0(b^2-a^2)(b^2+a^2)}$$

$$y=\frac{a^2b^2}{y_0(a^2+b^2)}$$

in $x=\frac{a^2}{x_0}\left(1-\frac{yy_0}{b^2}\right)$

$$=\frac{a^2}{x_0}\left(1-\frac{a^2b^2y_0}{y_0(a^2+b^2)b^2}\right)$$

$$=\frac{a^2}{x_0}\left(\frac{a^2+b^2-a^2}{a^2+b^2}\right)$$

$$=\frac{a^2b^2}{x_0(a^2+b^2)}$$

Q is the point $\left(\dfrac{a^2b^2}{x_0(a^2+b^2)}, \dfrac{a^2b^2}{y_0(a^2+b^2)}\right)$.

3 marks	Correctly finding Q
2 marks	Stating the equation of the chord of contact to $\mathscr{E}_2$ from P and finding one of the coordinates of Q
1 mark	Stating the equation of the chord of contact to $\mathscr{E}_2$ from P and making an attempt to find the coordinates of Q

(iii) $P(x_0, y_0)$ is a point that lies outside both $\mathscr{E}_1$ and $\mathscr{E}_2$.

$Q\left(\dfrac{a^2b^2}{x_0(a^2+b^2)}, \dfrac{a^2b^2}{y_0(a^2+b^2)}\right)$ lies outside both $\mathscr{E}_1$ and $\mathscr{E}_2$

if $x_0 = \dfrac{a^2b^2}{x_0(a^2+b^2)}$

$x_0{}^2 = \dfrac{a^2b^2}{(a^2+b^2)}$

so $x_0 = \pm\dfrac{ab}{\sqrt{a^2+b^2}}$

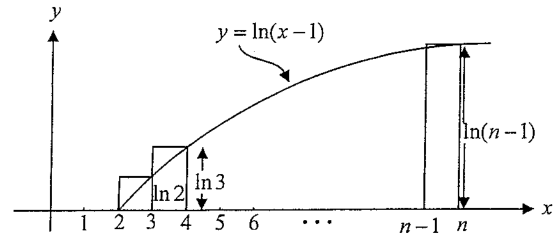
From part (i), the x coordinates $\pm\dfrac{ab}{\sqrt{a^2+b^2}}$ are the points where $\mathscr{E}_1$ and $\mathscr{E}_2$ intersect. Since $P(x_0, y_0)$ lies outside $\mathscr{E}_1$ and $\mathscr{E}_2$ then $x_0 \neq \pm\dfrac{ab}{\sqrt{a^2+b^2}}$

So Q cannot lie outside both $\mathscr{E}_1$ and $\mathscr{E}_2$.

3 marks	Correctly reasoned answer
2 marks	Obtaining $x_0 = \pm\dfrac{ab}{\sqrt{a^2+b^2}}$ and stating that Q lies outside $\mathscr{E}_1$ and $\mathscr{E}_2$ if this holds
1 mark	Obtaining $x_0 = \pm\dfrac{ab}{\sqrt{a^2+b^2}}$ with no explanation

(b) (i) $\ln(n-1)! = \ln\{(n-1)\times(n-2)\times\ldots\times 2\}, \quad n \geq 3$

$= \ln(n-1) + \ln(n-2) + \ldots + \ln 2$



$\ln(n-1)!$ is represented by the sum of the area of each of the rectangles with height $\ln 2, \ln 3, \ldots \ln(n-1)$ and each with width 1 unit.
The area between the curve $y = \ln(x-1)$ and the x-axis, between $x = 2$ and $x = n$, is $\int_2^n \ln(x-1)\,dx$.

Clearly from the graph, $\ln(n-1)! > \int_2^n \ln(x-1)\,dx$ for $n \geq 3$.

3 marks	Correct explanation
2 marks	Correctly identifying the sum of the area of rectangles of width 1 unit and height $\ln 2, \ln 3, \ldots \ln(n-1)$ as representing $\ln(n-1)!$
1 mark	Correctly identifying on a graph the area represented by $\int_2^n \ln(x-1)\,dx$ OR stating that $\ln(n-1)! = \ln(n-1) + \ln(n-2) + \ldots + \ln 2$

(ii) Using integration by parts, we have

$$\int_2^n \ln(x-1)\,dx = [x\ln(x-1)]_2^n - \int_2^n x \times \frac{1}{x-1}\,dx$$

$$= n\ln(n-1) - \int_2^n \left(1 + \frac{1}{x-1}\right)dx$$

$$= n\ln(n-1) - [x + \ln(x-1)]_2^n$$

$$= n\ln(n-1) - \{(n + \ln(n-1)) - (2 + \ln 1)\}$$

$$= n\ln(n-1) - n - \ln(n-1) + 2$$

$$= (n-1)\ln(n-1) - n + 2$$

From part (i), $\ln(n-1)! > \int_2^n \ln(x-1)\,dx, \quad n \geq 3$

So, $\ln(n-1)! > (n-1)\ln(n-1) - n + 2$

$$e^{\ln(n-1)!} > e^{(n-1)\ln(n-1)-n+2}$$

$$(n-1)! > \frac{e^{\ln(n-1)^{n-1}} \times e^2}{e^n}$$

$$(n-1)! > \frac{(n-1)^{n-1} \times e^2}{e^n}$$

So, $n! > n(n-1)^{n-1} e^{2-n}$

as required.

4 marks	Correct derivation
3 marks	Obtaining $e^{\ln(n-1)!} > e^{(n-1)\ln(n-1)-n+2}$
2 marks	Correct integration by parts
1 mark	Reasonable attempt at integration by parts