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MATHEMATICS

2006

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

General Instructions

- Reading time – 5 minutes
- Writing time – 3 hours
- All questions should be attempted
- Total marks available - 120
- All questions are worth 12 marks
- An approved calculator may be used
- All relevant working should be shown for each question

The release date for this exam is Tuesday 15 August 2006. Teachers are asked not to release this trial exam to students until this date except under exam conditions where the trial exams are collected by teachers at the end of the exam.

This paper has been prepared independently of the Board of Studies NSW to provide additional exam preparation for students. It is in no way connected with or endorsed by the Board of Studies NSW.

Question 1 (12 marks) **Marks**

(a) Evaluate $\sqrt{\dfrac{4.3^2-1}{1.7^3}}$ correct to 3 significant figures. **2**

(b) Solve the equation $\dfrac{2x-1}{2}-\dfrac{x+1}{3}=\dfrac{1}{2}$. **2**

(c) Express $4\sqrt{27}-2\sqrt{12}$ in its simplest surd form. **2**

(d) Find a primitive of $x^2+\dfrac{5}{x}$. **2**

(e) The price of a certain textbook is marked \$38.50 including 10% GST. The book is on special with 20% off the marked price. Find the cost of the book after the discount and **excluding** GST. **2**

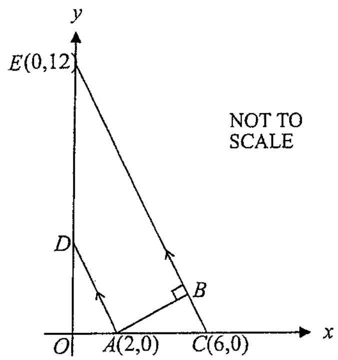
(f) The angle θ is subtended by an arc of length 10 cm in a circle with radius 15 cm. Find the value of θ to the nearest minute. **2**

Question 2 (12 marks) **Marks**

(a) Given $f(x)=\begin{cases}\dfrac{1}{3^x} & \text{for } x<3,\\ (x-3)^2+1 & \text{for } x\geq 3\end{cases}$ **2**

evaluate $2f(4)-f(-1)$.

(b)



In the diagram above, A, C and E are the points $(2,0)$, $(6,0)$ and $(0,12)$ respectively. The line AD is parallel to the line CE and the line AB is perpendicular to the lines AD and CE.

(i) Show that the equation of the line CE is $y=-2x+12$. **2**

(ii) Find coordinates of the point D. **2**

(iii) Show that the perpendicular distance from A to the line CE is $\dfrac{8\sqrt{5}}{5}$. **2**

(iv) Find the coordinates of the point B. **2**

(v) Find the area of the trapezium $ACED$. **2**

Question 3 (12 marks) **Marks**

(a) Differentiate with respect to x:

(i) $\cos^2 3x$ **2**

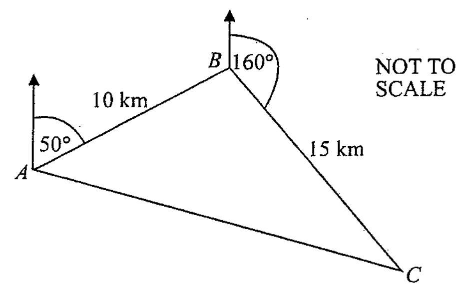
(ii) $\dfrac{2\tan x}{\sqrt{e^x}}$ **2**

(b) Find:

(i) $\int_0^1 e^{\frac{x}{2}}\,dx$ **2**

(ii) $\int \dfrac{\cos 2x}{3+\sin 2x}\,dx$ **2**

(c) A marathon runner runs 10 km from point A to point B on the bearing of 050° in relation to point A. He then runs a further 15 km to point C on the bearing of 160° in relation to point B.



(i) Show $\angle ABC = 70^\circ$. **1**

(ii) Show $AC^2 = 25(13-12\cos 70^\circ)$. **1**

(iii) Find the bearing of A from C. Express your answer to the nearest whole degree. **2**

Question 4 (12 marks) **Marks**

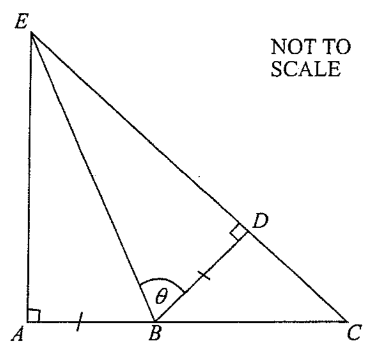
(a) (i) Find all the values of θ for which **2**

$$3\cos\theta - 2 = 0 \text{ where } 0 \leq \theta \leq 2\pi$$

Express your answer(s) in radian measure correct to two decimal places.

(ii) Hence sketch the graph of $y = 3\cos\theta - 2$ for $0 \leq \theta \leq 2\pi$ marking clearly any intercepts. **2**

(b)



In the diagram, ACE is a right-angled triangle. The point B lies on AC and the point D lies on CE. Also $\angle BDE = 90^\circ$, $AB = BD$ and $\angle DBE = \theta$.

(i) Show that $\Delta ABE \equiv \Delta DBE$. **2**

(ii) Show that $\angle ACE = 2\theta - 90^\circ$. **1**

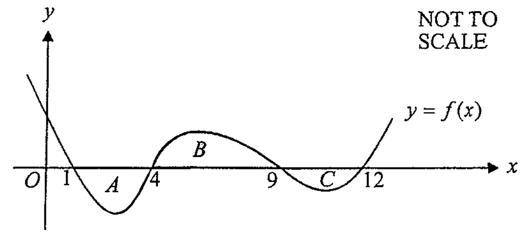
(iii) Show that $\Delta ACE \parallel\!\mid \Delta DCB$. **2**

(iv) Hence show that $EA : AB = CE : CB$. **1**

Question 4 continues on the next page.

Question 4 (cont'd) **Marks**

(c) **2**



The graph of the function f is shown in the diagram above. The shaded area A is equal in area to the shaded area B.

Write an alternative, equivalent expression for $\int_1^{12} f(x)dx$ in terms of the integral of $f(x)$.

(Do not attempt to evaluate any integrals.)

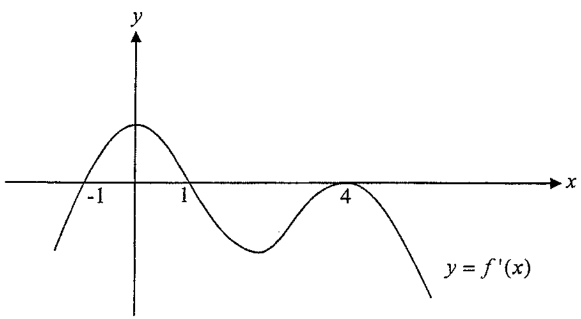
Question 5 (12 marks) **Marks**

(a) Solve $(x^2-2x)^2 = 3+2(x^2-2x)$ **4**

(b) The quadratic equation $5x^2-(21+a)x+a=0$ has two roots where one is the reciprocal of the other. Find the value of a and hence find the two roots. **3**

(c) Find the values of k for which the equation $kx^2-(k+1)x=-1$ has two distinct roots. **2**

(d) Sketch $y=f(x)$ given that it passes through the points $(0,0)$ and $(4,-2)$ given the graph of $y=f'(x)$. Show clearly any turning points or points of inflexion. **3**



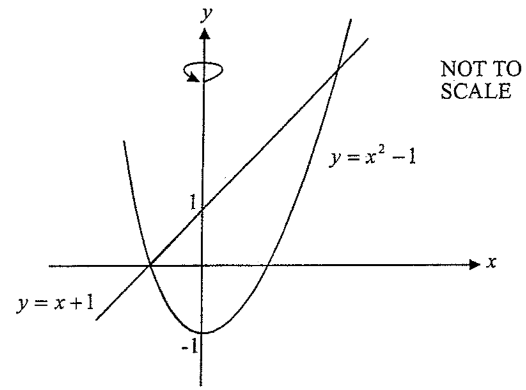
Question 6 (12 marks) **Marks**

(a) Use the change of base formula to evaluate **2**

$$\log_5 7 + 2\log_5 3$$

correct to two decimal places.

(b)



The graph of the line $y=x+1$ and the curve $y=x^2-1$ are shown on the diagram above.

(i) Find the coordinates of the points of intersection of the two graphs. **1**

(ii) The shaded region shown on the graph is rotated about the y-axis. By considering two integrals or otherwise find the volume of the solid formed. **3**

Question 6 continues on the next page.

Question 6 (cont'd) **Marks**

(c) In a game, two coins are used. One of the coins is a large, fair coin and the other is a small, biased coin. The probability of a tail appearing on the small, biased coin is two thirds.
Two players take it in turns to throw the two coins simultaneously.
Throwing a head scores 1 point and throwing a tail scores 0 points.
A game involves two turns by each player.

(i) What is the probability that a player scores 4 points in a game? **2**

(ii) What is the probability that a player scores at least 1 point in a game? **2**

(iii) What is the probability that a player scores 3 points in a game? **2**

Question 7 (12 marks) **Marks**

(a) Consider the geometric series

$$2+2\sin^2 x+2\sin^4 x+\ldots,\ \text{for } 0<x<\frac{\pi}{4}$$

(i) Show that the limiting sum exists. **2**

(ii) Find the limiting sum as $x \to \frac{\pi}{4}$. **2**

(b) Consider the arithmetic series $\log_2 k+2\log_2 k+3\log_2 k+\ldots$. **3**
Find the value of k given the sum of the first ten terms of the series is 165.

(c) The displacement of a certain particle is given by $x=5+2\sin \pi t$ where the displacement x is in metres and time t is in seconds.

(i) Find an expression for the velocity of the particle at any time t. **1**

(ii) At what time is the particle **first** at rest? **1**

(iii) Sketch the graph of x as a function of t for $0 \le t \le 4$. **3**
Hence find the distance travelled for the first 4 seconds.

Question 8 (12 marks) **Marks**

(a) Sketch the graph of the function $y = 2\tan x$ for $0 \le x \le \frac{\pi}{4}$ and state the range. **2**

(b) Let $f(x) = x^2$.

(i) Calculate $f(0)$, $f(2)$, $f(4)$, $f(6)$ and $f(8)$ and use your results to find $\int_0^8 f(x)dx$ using Simpson's rule. **2**

(ii) Hence explain whether or not Simpson's rule gives an approximation or an exact answer for this particular function over the given interval. **1**

(c) Julian borrowed \$20 000 from a finance company to purchase a car. Interest on the loan is calculated quarterly at the rate of 2.5% per quarter and is charged immediately prior to Julian making his quarterly repayment of \$*R*.
Let A_n be the amount in dollars owing on the loan after the n^{th} repayment has been made.

(i) Show that $A_3 = 20000 \times 1 \cdot 025^3 - R(1 + 1 \cdot 025 + 1 \cdot 025^2)$. **1**

(ii) Show that $A_n = 20000 \times 1 \cdot 025^n - 40R(1 \cdot 025^n - 1)$. **2**

(iii) If the loan were to be paid out after 7 years what would the value of *R* be? **2**

(iv) If Julian were to pay \$1282.94 per quarter in repayments, how long would it take to pay out his loan? **2**

Question 9 (12 marks) **Marks**

(a) The function $f(x) = e^x + e^{-x}$ is defined for all real values of *x*.

(i) Show $f(x)$ is an even function. Find the stationary point and its nature. Hence sketch the curve of $y = f(x)$. **3**

(ii) Find the equation of the tangent at $x = 1$. **2**

(iii) Find the area of the region bounded by the curve, the tangent at $x = 1$ and the *y*-axis. (Leave your answer in terms of *e*.) **2**

(b) The number *N* of a certain species is falling according to $N = N_0 e^{-0.03t}$ where *t* is in days and N_0 is the initial number of species present.

(i) Show that $N = N_0 e^{-0.03t}$ is a solution to the differential equation $\frac{dN}{dt} = -0 \cdot 03N$. **1**

(ii) How long, to the nearest day, will it take for the number of species to halve? **1**

(iii) Find, in terms of N_0, the rate of change at the time when the number of species has halved. **1**

(iv) Find the number of days, to the nearest whole number, for the number of species to fall to just below 5% of the initial number. **2**

Question 10 (12 marks) **Marks**

(a) An underground wine cellar is in the shape of a rectangular prism with a floor area of $12\,\text{m}^2$ and a ceiling height of 2 m.
At 2pm one Saturday, water begins to enter the cellar. The rate at which the volume, V, of water in the cellar changes over time t hours, is given by

$$\frac{dV}{dt} = \frac{24t}{t^2 + 15}$$

where $t = 0$ represents 2pm on Saturday and where V is measured in cubic metres.

The cellar is initially dry.

(i) Show that the volume of water in the cellar at time t is given by **2**

$$V = 12\ln\left(\frac{t^2 + 15}{15}\right), t > 0$$

(ii) Find the time when the cellar will be completely filled with water if the water continues to enter the cellar at the given rate. **2**
Express your answer to the nearest minute.

(iii) The owners return to the house and manage to simultaneously stop the water entering the cellar and start the pump in the cellar. This occurs at 6pm on Saturday. **2**

The rate at which the water is pumped out of the cellar is given by

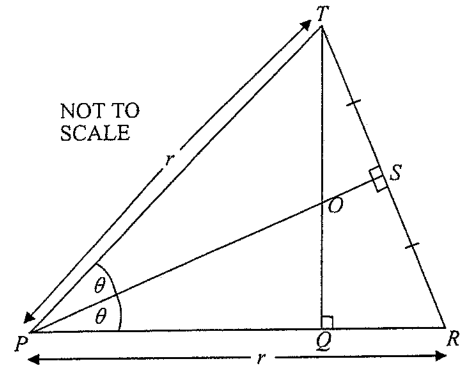
$$\frac{dV}{dt} = \frac{t^2}{k} \text{ where } k \text{ is a constant.}$$

At exactly 8pm the cellar is emptied of water.

Find the value of k. Express your answer correct to 4 significant figures.

Question 10 (cont'd) **Marks**

(b)



In the diagram above, PRT is an isosceles triangle with $PR = PT = r$. Also, PS is a perpendicular bisector of TR, TQ meets PR at right angles and $\angle TPS = \angle RPS = \theta$, where $0 < \theta < \frac{\pi}{4}$.

(i) Show that $PQ = r(1 - 2\sin^2\theta)$ and $OQ = r\tan\theta(1 - 2\sin^2\theta)$. **2**

(ii) The area of ΔPQO is given by $A = \frac{r^2\tan\theta}{2}(1 - 2\sin^2\theta)^2$. **3**

Show that

$$\frac{dA}{d\theta} = \frac{r^2}{2\cos^2\theta}(1 - 2\sin^2\theta)(8\cos^4\theta - 6\cos^2\theta - 1).$$

(iii) It is known that the equation $8a^2 - 6a - 1 = 0$ has solutions $a = 0\cdot 89$ and $a = -0\cdot 14$ (each correct to 2 decimal places). It is also known that the function $A = \frac{r^2\tan\theta}{2}(1 - 2\sin^2\theta)^2$ has only one local maximum and no minimum turning points for $0 < \theta < \frac{\pi}{4}$. **1**
Use this information to find the value of θ for which A is a maximum. Express your answer in radians correct to 3 decimal places.

END OF EXAM

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MATHEMATICS HIGHER SCHOOL CERTIFICATE TRIAL EXAMINATION SOLUTIONS 2006

Question 1 (12 marks)

(a) $\sqrt{\dfrac{4.3^2-1}{1.7^3}}=1.88678...$

$=1.89$ correct to 3 significant figures

2 marks	Correct answer
1 mark	Correct answer with no rounding

(b)
$$\frac{2x-1}{2}-\frac{x+1}{3}=\frac{1}{2}$$
$$\frac{3(2x-1)-2(x+1)}{6}=\frac{3}{6}$$
$$6x-3-2x-2=3$$
$$4x-5=3$$
$$4x=8$$
$$x=2$$

2 marks	Correct answer
1 mark	Obtaining the line $6x-3-2x-2=3$

(c)
$$4\sqrt{27}-2\sqrt{12}=4\times3\sqrt{3}-2\times2\sqrt{3}$$
$$=12\sqrt{3}-4\sqrt{3}$$
$$=8\sqrt{3}$$

2 marks	Correct answer
1 mark	Obtaining $4\times3\sqrt{3}-2\times2\sqrt{3}$

(d) A primitive of $x^2+\dfrac{5}{x}$

is $\dfrac{x^3}{3}+5\ln x$.

2 marks	Correct answer
1 mark	Obtaining one correctly integrated term

(e) cost of book after discount (including GST) $=38\cdot50\times80\%$ $=\$30\cdot80$

cost of book after discount (excluding GST) $=30\cdot80\times\dfrac{100}{110}$ $=\$28$

2 marks	Correct answer
1 mark	Obtaining \$30.80

(f)
$$l=r\theta$$
$$10=15\theta$$
$$\theta=\frac{2^c}{3}$$
$$\theta=\frac{2^c}{3}\times\frac{180^\circ}{\pi^c}$$
$$=\left(\frac{120}{\pi}\right)^\circ$$
$$=(38.197...)^\circ$$
$$=38^\circ12'\text{(to the nearest minute)}$$

2 marks	Correct answer
1 mark	Using $l=r\theta$

Question 2 (12 marks)

(a)
$$f(4) = (4-3)^2 + 1$$
$$= 2$$
$$f(-1) = \frac{1}{3^{-1}} = 3$$
$$2f(4) - f(-1) = 2(2) - 3$$
$$= 1$$

2 marks	Correct answer
1 mark	Finding correctly either $f(4)$ or $f(-1)$

(b) (i) Gradient of CE is $\frac{12-0}{0-6} = -2$

Equation of CE is $y - 0 = -2(x-6)$
$$y = -2x + 12$$
(Alternatively $y - 12 = -2(x-0)$
$$y = -2x + 12)$$

2 marks	Deriving equation correctly
1 mark	Finding gradient correctly

(ii) gradient of AD = gradient of CE = -2

The x coordinate of D is zero.

Let D be the point $(0, y_1)$

then $\frac{y_1 - 0}{0-2} = -2$
$$y_1 = 4$$
D has coordinates $(0,4)$

2 marks	Correct answer
1 mark	Finding one correct coordinate

(iii) The equation of CE is $2x + y - 12 = 0$.

The perpendicular distance from $A(2,0)$

to $CE = \frac{|2\times 2 + 1\times 0 - 12|}{\sqrt{2^2 + 1^2}}$
$$= \frac{|-8|}{\sqrt{5}}$$
$$= \frac{8}{\sqrt{5}}$$
$$= \frac{8\sqrt{5}}{5}$$

2 marks	Correctly deriving answer
1 mark	Making a reasonable attempt to substitute into the correct formula

(iv) Find B.

Equation of AB is $y - 0 = \frac{1}{2}(x-2)$
$$y = \frac{1}{2}(x-2)$$
Note that the gradient of AB is $\frac{1}{2}$ since the gradient of CE is -2.

B is the point of intersection of CE and AB.
$$y = -2x + 12$$
$$y = \frac{x}{2} - 1$$
So $-2x + 12 = \frac{x}{2} - 1$
$$-4x + 24 = x - 2$$
$$-5x = -26$$
$$x = \frac{26}{5}$$
$$y = -2 \times \frac{26}{5} + 12$$
$$= \frac{8}{5}$$
B is the point $\left(5\frac{1}{5}, 1\frac{3}{5}\right)$.

2 marks	Correct answer
1 mark	Finding the equation of AB

(v) Area of $ACED = AB\left(\frac{AD+CE}{2}\right)$

Now, $AB = \frac{8\sqrt{5}}{5}$ from part (iii).

$$A(2,0),\ C(6,0),\ D(0,4),\ E(0,12)$$

$$\begin{aligned} AD &= \sqrt{(2-0)^2+(0-4)^2} \\ &= \sqrt{20} \\ &= 2\sqrt{5} \end{aligned}$$

$$\begin{aligned} CE &= \sqrt{(6-0)^2+(0-12)^2} \\ &= \sqrt{180} \\ &= 6\sqrt{5} \end{aligned}$$

$$\begin{aligned} \text{Area of } ACED &= \frac{8\sqrt{5}}{5}\left(\frac{2\sqrt{5}+6\sqrt{5}}{2}\right) \\ &= \frac{8\sqrt{5}}{5}\times 4\sqrt{5} \\ &= 32 \text{ square units} \end{aligned}$$

2 marks	Correct answer
1 mark	Finding correct lengths for AD and CE

Question 3 (12 marks)

(a) (i) $\frac{d}{dx}(\cos^2 3x)$

$$\begin{aligned} &= \frac{d}{dx}\left((\cos 3x)^2\right) \\ &= 2(\cos 3x)(-3\sin 3x) \\ &= -6\cos 3x \sin 3x \end{aligned}$$

2 marks	Correct answer
1 mark	Obtaining $2(\cos 3x)(-3\sin 3x)$

(ii) $\frac{d}{dx}\left(\frac{2\tan x}{\sqrt{e^x}}\right)$

$$\begin{aligned} &= \frac{d}{dx}\left(\frac{2\tan x}{e^{\frac{x}{2}}}\right) \\ &= \frac{(2\sec^2 x)e^{\frac{x}{2}} - \frac{e^{\frac{x}{2}}}{2}(2\tan x)}{\left(e^{\frac{x}{2}}\right)^2} \\ &= \frac{e^{\frac{x}{2}}(2\sec^2 x - \tan x)}{e^x} \\ &= \frac{(2\sec^2 x - \tan x)}{e^{\frac{x}{2}}} \end{aligned}$$

2 marks	Correct answer
1 mark	Obtaining $\frac{(2\sec^2 x)e^{\frac{x}{2}} - \frac{e^{\frac{x}{2}}}{2}(2\tan x)}{\left(e^{\frac{x}{2}}\right)^2}$

(b) (i) $\int_0^1 e^{\frac{x}{2}}dx = 2\left[e^{\frac{x}{2}}\right]_0^1$

$$\begin{aligned} &= 2\left(e^{\frac{1}{2}} - e^0\right) \\ &= 2\left(e^{\frac{1}{2}} - 1\right) \end{aligned}$$

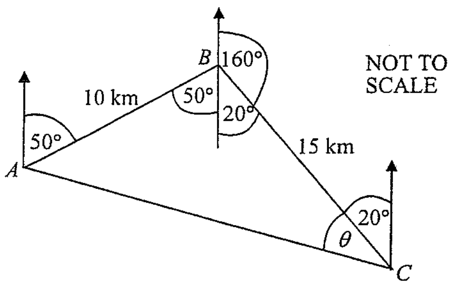
2 marks	Correct answer
1 mark	Obtaining $2\left[e^{\frac{x}{2}}\right]_0^1$

(ii) $\int \frac{\cos 2x}{3+\sin 2x}dx = \frac{1}{2}\int \frac{2\cos 2x}{3+\sin 2x}dx$

$= \frac{1}{2}\log_e(3+\sin 2x)+c$

2 marks	Correct answer
1 mark	Recognizing that a log function was required

(c)



(i) $\angle ABC = 50^\circ + 20^\circ$

$= 70^\circ$ (as shown in diagram)

1 mark	Correct answer

(ii) $AC^2 = 15^2 + 10^2 - 2(10)(15)\cos 70^\circ$

$= 225 + 100 - 300\cos 70^\circ$

$= 325 - 300\cos 70^\circ$

$= 25(13 - 12\cos 70^\circ)$

1 mark	Correct answer

(iii) Let $\angle ACB = \theta$

$\frac{\sin\theta}{10} = \frac{\sin 70^\circ}{AC}$

$\sin\theta = \frac{10\sin 70^\circ}{\sqrt{25(13-12\cos 70^\circ)}}$ where AC is from part (ii)

$\theta = 39^\circ 4'$ (nearest minute)

Now, $360^\circ - (39^\circ 4') - 20^\circ = 300^\circ 56'$

The bearing of A from C is 301° (to the nearest degree).

2 marks	Correct answer
1 mark	Finding θ correctly.

Question 4 (12 marks)

(a) (i) $3\cos\theta - 2 = 0$

$\cos\theta = \frac{2}{3}$ $\quad 0 \le \theta \le 2\pi$

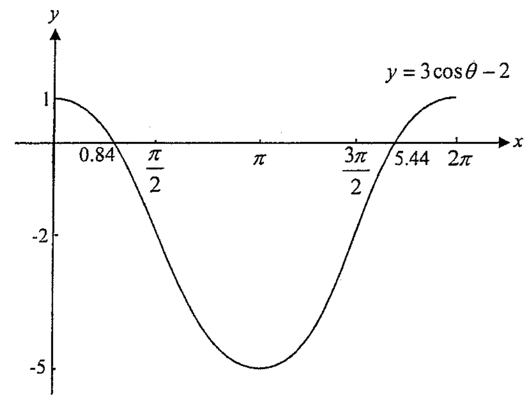
S	A
T	C

$\theta = 0.84106...,\ 2\pi - 0.84106...$

$= 0.84,\ 5.44$ (correct to2 decimal places)

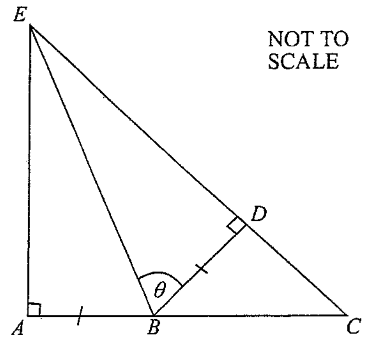
2 marks	Correct answers
1 mark	One correct answer

(ii)



2 marks	Correct graph with all intercepts marked
1 mark	Correct graph with no intercepts marked

(b)



(i) Δ's ABE and DBE are both right angled triangles.
Also, $AB = DB$ (given)
EB is a shared hypotenuse of Δ's ABE and DBE.
So $\Delta ABE \equiv \Delta DBE$ (If two right-angled triangles have equal hypotenuses and one other pair of sides equal in length they are congruent.)

2 marks	Correctly reasoned answer
1 mark	Making some attempt at showing the RHS condition for congruency

(ii) $\angle ABE = \theta$ ($\Delta ABE \equiv \Delta DBE$ from part (i) and $\angle ABE$ is the corresponding angle to $\angle DBE$.)

$\angle BDC + \angle ACE = \angle DBA$ (The sum of two angles in a triangle equals the exterior opposite angle.)

So $90^\circ + \angle ACE = \angle DBE + \angle EBA$

$90^\circ + \angle ACE = 2\theta$

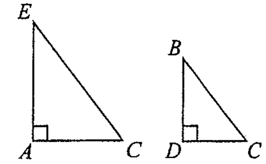
$\angle ACE = 2\theta - 90^\circ$

1 mark	Correctly reasoned answer

(iii) $\angle CAE = \angle BDC = 90^\circ$ (given)
$\angle ACE = \angle BCD$ (common angle)
So $\Delta ACE \parallel\!\mid \Delta DCB$ (Two angles of one triangle are equal respectively to two angles of another triangle.)

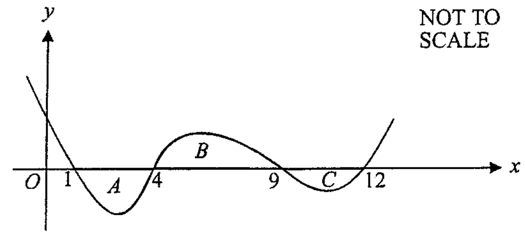
2 marks	Correctly reasoned answer
1 mark	Making some reasonable attempt at showing similarity

(iv) Since $\Delta ACE \parallel\!\mid \Delta DCB$,
$EA : BD = CE : CB$
Since $BD = AB$ (given),
$EA : AB = CE : CB$



1 mark	Correctly reasoned answer

(c)



$\int_1^{12} f(x)dx =$ area B - area A - area C

because $\int_1^4 f(x)dx$ and $\int_9^{12} f(x)dx$ are both negative.

Since area A = area B,

$\int_1^{12} f(x)dx = -$ area C

$= \int_9^{12} f(x)dx$

2 marks	Correct answer
1 mark	Some attempt made at arriving at correct answer

Question 5 (12 marks)

(a) $(x^2-2x)^2 = 3+2(x^2-2x)$

let $m = x^2 - 2x$

so $m^2 = 3+2m$

$m^2 - 2m - 3 = 0$

$(m-3)(m+1) = 0$

$m = 3, m = -1$

$x^2 - 2x = 3 \qquad x^2 - 2x = -1$

$x^2 - 2x - 3 = 0 \qquad x^2 - 2x + 1 = 0$

$(x-3)(x+1) = 0 \qquad (x-1)^2 = 0$

$x = 3, x = -1 \qquad x = 1$

4 marks	Correct answers
3 marks	Finding $x=3$ and $x=-1$ as solutions OR $x=1$ as solution
2 marks	Finding $x^2-2x-3=0$ or $x^2-2x+1=0$
1 mark	Solving correctly $m^2-2m-3=0$

(b) $5x^2 - (21+a)x + a = 0$

let $\alpha, \frac{1}{\alpha}$ be the roots of the equation

so $\alpha\left(\frac{1}{\alpha}\right) = \frac{a}{5}$

$a = 5$

$5x^2 - 26x + 5 = 0$

$(5x-1)(x-5) = 0$

$x = \frac{1}{5}, x = 5$

The two roots are $\frac{1}{5}$ and 5.

3 marks	Correct answer
2 marks	Finding a correctly and one of the roots
1 mark	Finding a correctly

Question 5 (cont'd)

(c) $kx^2 - (k+1)x + 1 = 0$

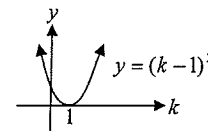
$\Delta = (k+1)^2 - 4\times k\times 1$

$= k^2 + 2k + 1 - 4k$

$= k^2 - 2k + 1$

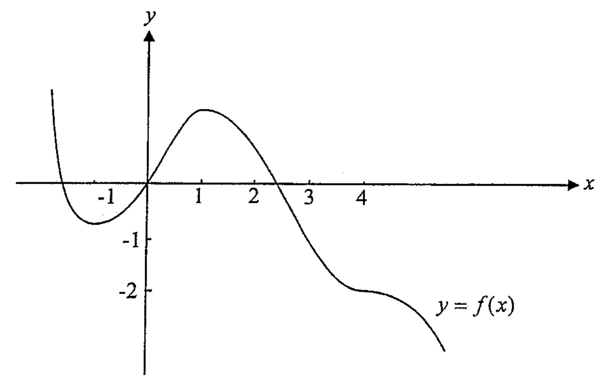
$= (k-1)^2$

For 2 distinct roots we require $(k-1)^2 > 0$; that is we require k real, $k \neq 1$.



2 marks	Correct answer
1 mark	Obtaining $(k-1)^2 > 0$

(d) The graph of $y = f(x)$ has stationary points where $x=-1$, $x=1$ and $x=4$. It must pass through $(0,0)$ and $(4,-2)$.



3 marks	Correct graph
2 marks	Showing correct turning points and point of inflexion
1 mark	Showing one correct turning point or point of inflexion

Question 6 (12 marks)

(a) $\log_5 7 + 2\log_5 3 = \log_5 7 + \log_5 3^2$

$= \log_5 7 \times 3^2$

$= \log_5 63$

$= \dfrac{\log_e 63}{\log_e 5}$ (using change of base formula)

$= 2 \cdot 57$ (correct to 2 decimal places)

Alternately, $\log_5 63$

$= \dfrac{\log_{10} 63}{\log_{10} 5}$

$= 2 \cdot 57$ (correct to 2 decimal places)

2 marks	Correct answer
1 mark	Obtaining $\log_5 63$

(b) (i) $y = x + 1$

$y = x^2 - 1$

$x^2 - 1 = x + 1$

$x^2 - x - 2 = 0$

$(x-2)(x+1) = 0$

$x = 2$ or $x = -1$

The points of intersection are $(2,3)$ and $(-1,0)$.

1 mark	Correct answer

(ii) $\text{Volume} = \pi \int_0^1 (y-1)^2\,dy + \pi \int_{-1}^0 (y+1)\,dy$

$= \pi \int_0^1 (y^2 - 2y + 1)\,dy + \pi \left[\dfrac{y^2}{2} + y\right]_{-1}^0$

$= \pi \left[\dfrac{y^3}{3} - y^2 + y\right]_0^1 + \pi\left\{0 - \left(\dfrac{1}{2} - 1\right)\right\}$

$= \pi\left\{\left(\dfrac{1}{3} - 1 + 1\right) - (0)\right\} + \dfrac{\pi}{2}$

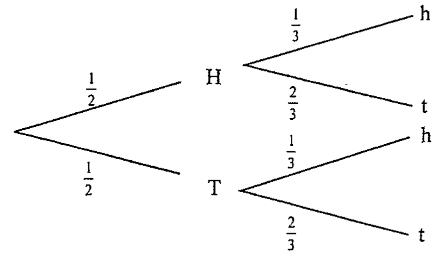
$= \dfrac{\pi}{3} + \dfrac{\pi}{2}$

$= \dfrac{5\pi}{6}$ cubic units

3 marks	Correct answer
2 marks	Correct integrals with an integration or arithmetic mistake
1 mark	Obtaining correct integrand with incorrect terminal(s) OR correct terminals with incorrect integrand(s)

Question 6 (cont'd)

(c) (i) To score 4 points a player must have thrown 2 heads on each turn. The tree diagram below shows the probabilities for one turn.



Sequence	Probability
Hh	$\dfrac{1}{2} \times \dfrac{1}{3} = \dfrac{1}{6}$
Ht	$\dfrac{1}{2} \times \dfrac{2}{3} = \dfrac{1}{3}$
Th	$\dfrac{1}{2} \times \dfrac{1}{3} = \dfrac{1}{6}$
Tt	$\dfrac{1}{2} \times \dfrac{2}{3} = \dfrac{1}{3}$

$P(\text{4 points}) = P(\text{2 heads}) \times P(\text{2 heads})$

$= \dfrac{1}{6} \times \dfrac{1}{6}$

$= \dfrac{1}{36}$

2 marks	Correct answer
1 mark	Find correctly the probability of throwing 2 heads

(ii) $P(\text{at least 1 point})$

$= 1 - P(\text{0 points})$

$= 1 - P(\text{4 tails})$

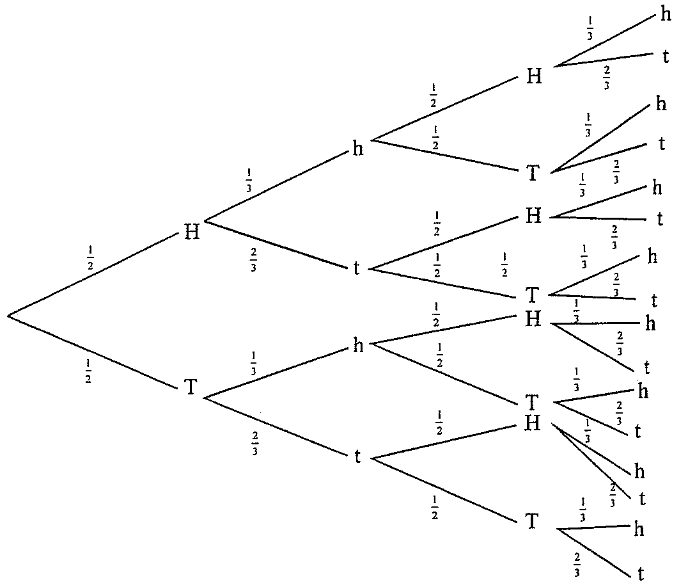
$= 1 - \left(\dfrac{1}{3} \times \dfrac{1}{3}\right)$

$= \dfrac{8}{9}$

2 marks	Correct answer
1 mark	Use of the complementary approach

Question 6 (cont'd)

(iii) To score three points in a game a player must throw exactly 3 heads. The tree diagram below shows the probability for two turns. There are 16 possible outcomes for a player in a game. We are interested in those that have exactly 3 heads (or exactly 1 tail).



$$P(3\text{ heads}) = P(HhHt) + P(HhTh) + P(HtHh) + P(ThHh)$$
$$= \frac{1}{18} + \frac{1}{36} + \frac{1}{18} + \frac{1}{36}$$
$$= \frac{6}{36}$$
$$= \frac{1}{6}$$

2 marks	Correct answer
1 mark	Making a reasonable attempt to find how many outcomes have 3 heads appearing

Question 7 (12 marks)

(a) (i) $r = \frac{2\sin^2 x}{2} = \sin^2 x$

For $0 < x < \frac{\pi}{4}$, $\sin^2 0 < r < \sin^2 \frac{\pi}{4}$

$0 < r < \frac{1}{2}$

Since $-1 < r < 1$ the limiting sum exists.

2 marks	Correctly reasoned answer
1 mark	Finding correct values of r

(ii) $S_\infty = \frac{2}{1-\sin^2 x}$

$= \frac{2}{\cos^2 x}$

As $x \to \frac{\pi}{4}$, $S_\infty \to \frac{2}{\cos^2 \frac{\pi}{4}}$

so $S_\infty \to 4$

2 marks	Correct answer
1 mark	Finding $S_\infty = \frac{2}{\cos^2 x}$

(b) $a = \log_2 k,\ d = \log_2 k$

$S_{10} = \frac{10}{2}[2\log_2 k + 9\log_2 k] = 165$

$11\log_2 k = \frac{165}{5} = 33$

$\log_2 k = 3$

$k = 2^3$

so $k = 8$

3marks	Correct answer
2 marks	Obtaining $\log_2 k = 3$
1 mark	Obtaining correct expression for S_{10}

Question 7 (cont'd)

(c) (i) $x = 5 + 2\sin \pi t$

$v = 2\pi \cos \pi t$

1 mark	Correct answer

(ii) Given $v = 0$, find t.

$2\pi \cos \pi t = 0$

$\cos \pi t = 0$

$\pi t = \cos^{-1} 0$

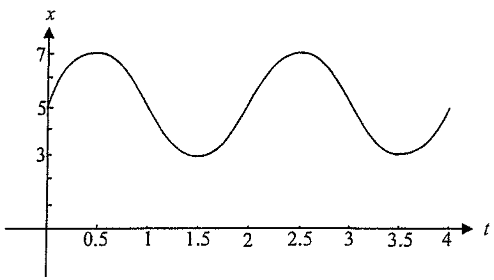
$\pi t = \frac{\pi}{2}$

The particle is first at rest at $t = \frac{1}{2}$ second.

1 mark	Correct answer

(iii) $x = 5 + 2\sin \pi t$

amplitude = 2, period = $\frac{2\pi}{\pi} = 2$, graph is translated 5 units up



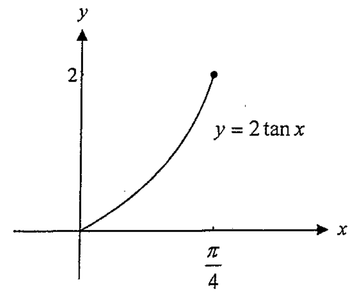
distance travelled $= 2 + 3 \times 4 + 2$

$= 16$ metres

3 marks	Correct graph and distance travelled
2 marks	Correct graph
1 mark	Finding correct period, amplitude and translation

Question 8 (12 marks)

(a)



range = $[0,2]$ or $0 \le y \le 2$

2 marks	Correct sketch and answer
1 mark	One of the above

(b) (i)

$f(x) = x^2$

$f(0) = 0, f(2) = 4, f(4) = 16, f(6) = 36$ and $f(8) = 64$

$\int_0^8 f(x)dx = \int_0^4 f(x)dx + \int_4^8 f(x)dx$

$= \frac{1}{6} \times 4(0 + 4 \times 4 + 16) + \frac{1}{6} \times 4(16 + 4 \times 36 + 64)$

$= \frac{128}{6} + \frac{896}{6}$

$= 170\frac{2}{3}$

2 marks	Correct answer using Simpson's rule
1 mark	Correct method with one arithmetic mistake

(ii)

$\int_0^8 x^2 dx = \left[\frac{x^3}{3}\right]_0^8$

$= \frac{512}{3}$

$= 170\frac{2}{3}$

Since the same answer is obtained by integrating, we see that for this particular function over this interval, Simpson's rule provides an exact answer rather than an approximation.

1 mark	Correct explanation

(c) (i) $A_1 = 20\,000 \times 1\cdot025 - R$

$A_2 = A_1 \times 1\cdot025 - R$

$= 20\,000 \times 1\cdot025^2 - 1\cdot025R - R$

$A_3 = A_2 \times 1.025 - R$

$= 20\,000 \times 1\cdot025^3 - 1\cdot025^2 R - 1\cdot025R - R$

$= 20\,000 \times 1\cdot025^3 - R(1 + 1\cdot025 + 1\cdot025^2)$

1 mark	Correct derivation

(ii) $A_n = 20\,000 \times 1\cdot025^n - R(1 + 1\cdot025 + 1\cdot025^2 + ... + 1\cdot025^{n-1})$

For the series in the brackets, $a = 1$, $r = 1\cdot025$ and there are n terms.

$S_n = \dfrac{(1\cdot025^n - 1)}{0\cdot025}$

$= 40(1\cdot025^n - 1)$

$A_n = 20\,000 \times 1\cdot025^n - 40R(1\cdot025^n - 1)$

2 marks	Correctly derived answer including the series finishing with $1\cdot025^{n-1}$ and not $1\cdot025^n$
1 mark	Obtaining the first line in the working

(iii) If the loan is paid off after 7 years or $7 \times 4 = 28$ quarters, then $A_n = 0$ when $n = 28$

So $0 = 1\cdot025^{28} \times 20\,000 - 40R(1\cdot025^{28} - 1)$

$40R(1\cdot025^{28} - 1) = 1\cdot025^{28} \times 20\,000$

$R = \dfrac{1\cdot025^{28} \times 20\,000}{40(1\cdot025^{28} - 1)}$

$= \$1001\cdot76$

2 marks	Correct answer
1 mark	Obtaining the equation $0 = 1\cdot025^{28} \times 20\,000 - 40R(1\cdot025^{28} - 1)$

Question 8 (Cont'd.)

(iv) If Julian pays \$1 282.94 per quarter in repayments then

$A_n = 20\,000 \times 1\cdot025^n - 40R(1\cdot025^n - 1)$

becomes $0 = 20\,000 \times 1\cdot025^n - 40 \times 1282\cdot94(1\cdot025^n - 1)$

so $20\,000 \times 1\cdot025^n = 40 \times 1282\cdot94(1\cdot025^n - 1)$

$= 51317\cdot6 \times 1\cdot025^n - 51317\cdot6$

$31317\cdot6 \times 1\cdot025^n = 51317\cdot6$

$1\cdot025^n = 1\cdot6386...$

Method 1 – trial and error

$1\cdot025^n = 1\cdot6386...$

If $n = 30$, $LS = 2\cdot0975...$TOO HIGH

$n = 10$, $LS = 1\cdot2800...$TOO LOW

$n = 20$, $LS = 1\cdot6386...$

So it would take 20 quarters or five years.

Method 2 – change of base

$1\cdot025^n = 1\cdot6386...$

$\log_{1.025}(1\cdot6386...) = n$

So $n = \dfrac{\log_{10} 1.6386...}{\log_{10} 1.025}$

$= 19.9995...$

So it would take 20 quarters or five years.

Method 3 – take $\log_{10}$ or $\log_e$ of both sides

$1\cdot025^n = 1\cdot6386...$

$\log_{10} 1.025^n = \log_{10} 1.6386...$

$n \log_{10} 1.025 = \log_{10} 1.6386...$

$n = \dfrac{\log_{10} 1.6386...}{\log_{10} 1.025}$

$= 19.9995...$

So it would take 20 quarters or five years.

2 marks	Correct answer
1 mark	Obtaining $1\cdot025^n = 1\cdot6386...$

Question 9 (12 marks)

(a) (i) $f(x)=e^{x}+e^{-x}$

$f(-x)=e^{-x}+e^{x}$

$=f(x)$

so $f(x)$ is an even function

$f'(x)=e^{x}-e^{-x}$

For the stationary point $f'(x)=0$:

$e^{x}-e^{-x}=0$

$e^{x}-\frac{1}{e^{x}}=0$

$e^{2x}=1$

$2x=\log_e 1$

$=0$

so $x=0$

$f(0)=e^{0}+e^{0}$

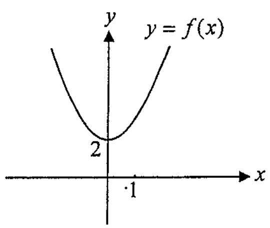
$=2$

The stationary point is $(0, 2)$.

Nature:

x	-1	0	1
$f'(x)$	-ve	0	+ve

The point $(0,2)$ is a local minimum.



3 marks	Correctly sketched graph with stationary point given and having shown $f(x)=f(-x)$.
2 marks	Correctly sketched graph with stationary point given OR correctly shown $f(x)=f(-x)$ and correct stationary point.
1 mark	Showing $f(x)=f(-x)$ OR finding the stationary point

Question 9 (cont'd)

(ii) At $x=1$, $f(1)=e+e^{-1}$

$=e+\frac{1}{e}$

At $x=1$, $f'(1)=e-e^{-1}$

$=e-\frac{1}{e}$

The equation of the tangent at $\left(1, e+\frac{1}{e}\right)$ is given by

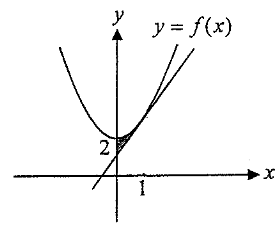
$y-\left(e+\frac{1}{e}\right)=\left(e-\frac{1}{e}\right)(x-1)$

$y-e-\frac{1}{e}=\left(e-\frac{1}{e}\right)x-e+\frac{1}{e}$

so $y=\left(e-\frac{1}{e}\right)x+\frac{2}{e}$

2 marks	Correct equation
1 mark	Finding gradient of tangent

(iii)



$$\text{Area}=\int_0^1\left(e^{x}+e^{-x}-\left(\left(e-\frac{1}{e}\right)x+\frac{2}{e}\right)\right)dx$$

$$=\left[e^{x}-e^{-x}-\frac{1}{2}\left(e-\frac{1}{e}\right)x^{2}-\frac{2}{e}x\right]_0^1$$

$$=\left(e-e^{-1}-\frac{1}{2}e+\frac{1}{2e}-\frac{2}{e}\right)-\left(e^{0}-e^{0}-(0-0)\right)$$

$$=\frac{1}{2}e-\frac{5}{2e}-(1-1)$$

$$=\frac{1}{2}\left(e-\frac{5}{e}\right)\text{ square units}$$

2 marks	Correct answer
1 mark	Giving correct integral

Question 9 (cont'd)

(b) (i) $N = N_0 e^{-0.03t}$

$$\frac{dN}{dt} = -0.03N_0 e^{-0.03t}$$
$$= -0.03N$$

1 mark	Correctly showing it to be a solution

(ii) Given that $N = \frac{1}{2}N_0$, find t.

$$\frac{1}{2}N_0 = N_0 e^{-0.03t}$$
$$\frac{1}{2} = e^{-0.03t} \quad N_0 \neq 0$$
$$-0.03t = \ln\frac{1}{2}$$
$$t = \frac{\ln 2}{0.03}$$
$t = 23$ days (to the nearest day)

It will take 23 days.

1 mark	Correctly finding number of days

(iii) Also, when

$$N = \frac{1}{2}N_0,$$
$$\frac{dN}{dt} = -0.03\left(\frac{1}{2}N_0\right)$$
$$= -0.015N_0$$

1 mark	Correctly finding rate

(iv) When $N < 5\%N_0$, find t.

$$N_0 e^{-0.03t} < \frac{5N_0}{100}$$
$$e^{-0.03t} < \frac{1}{20}$$
$$-0.03t < \ln\frac{1}{20}$$
$$t > \frac{\ln\frac{1}{20}}{-0.03} = 99.9 \text{ (1 dec)}$$

So $t = 100$ days (to the nearest whole number) for the number of species to fall just below 5%.

2 marks	Correct answer
1 mark	Obtaining $N_0 e^{-0.03t} < \frac{5N_0}{100}$

Question 10 (12 marks)

(a) (i) $\frac{dV}{dt} = \frac{24t}{t^2+15}$

$$V = 12\int \frac{2t}{t^2+15}dt$$
$$V = 12\log_e(t^2+15)+c$$
$$t = 0, V = 0$$
$$0 = 12\log_e(15)+c$$
$$c = -12\log_e(15)$$
$$V = 12\log_e\left(\frac{t^2+15}{15}\right)$$

Have shown

2 marks	Correctly derived answer
1 mark	Obtaining $V = 12\log_e(t^2+15)+c$

(ii) The volume of the cellar is $12\text{m}^2 \times 2\text{m} = 24\text{m}^3$

So $24 = 12\log_e\left(\frac{t^2+15}{15}\right)$

$$2 = \log_e\left(\frac{t^2+15}{15}\right)$$
$$15e^2 = t^2+15$$
$$t = \sqrt{15e^2-15}, \; t > 0$$
$$= 9\cdot7895...$$

The cellar will fill with water at 11.47 pm on Saturday (to the nearest minute).

2 marks	Correct answer
1 mark	Obtaining $2 = \log_e\left(\frac{t^2+15}{15}\right)$

Question 10 (cont'd)

(iii) 6 pm Saturday corresponds to $t = 4$ for the function $V = 12\log_e\left(\frac{t^2+15}{15}\right)$ (when the cellar is filling with water).

So $V = 12\log_e\left(\frac{16+15}{15}\right)$

$= 8\cdot7112...$

So at 6pm when the water stops flowing in and the pump starts,
$V = 8\cdot711244\text{m}^3$ (correct to 6 decimal places).

Now, $\frac{dV}{dt} = \frac{t^2}{k}$ where $t = 0$ corresponds to 6pm

$$V = \frac{1}{k}\int t^2\,dt$$

$$V = \frac{t^3}{3k} + c$$

When $t = 0$, $V = 8\cdot711244$

$8\cdot711244 = c$

$$V = \frac{t^3}{3k} + 8\cdot711244$$

At 8pm; that is, at $t = 2$, $V = 0$.

So, $0 = \frac{8}{3k} + 8\cdot711244$

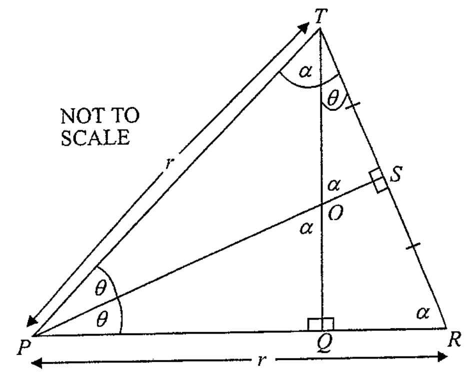
$$k = \frac{8}{3\times -8\cdot711244}$$

$= -0\cdot3061$ (correct to 4 significant figures)

2 marks	Correct answer
1 mark	Obtaining $V = 8\cdot7112$ at $t = 0$ AND $V = \frac{t^3}{3k} + c$

Question 10 (cont'd)

(b) (i)



Now, in ΔPSR, $\sin\theta = \frac{RS}{r}$, so $RS = r\sin\theta$.

Since $RS = r\sin\theta$, $TR = 2r\sin\theta$.

So in ΔTQR, $\sin\theta = \frac{RQ}{2r\sin\theta}$ so $RQ = 2r\sin^2\theta$

Now $PQ = PR - RQ$

$= r - 2r\sin^2\theta$

$= r(1-2\sin^2\theta)$

In ΔPQO, $\tan\theta = \frac{OQ}{r(1-2\sin^2\theta)}$

$$OQ = r\tan\theta(1-2\sin^2\theta)$$

as required.

2 marks	Correctly reasoned answers
1 mark	Obtaining one correctly reasoned answer

Question 10 (cont'd)

(ii)

$$A=\frac{r^2\tan\theta}{2}\left(1-2\sin^2\theta\right)^2$$

$$\frac{dA}{d\theta}=\frac{r^2}{2}\sec^2\theta\left(1-2\sin^2\theta\right)^2+\frac{r^2}{2}\tan\theta\times 2\left(1-2\sin^2\theta\right)\times -4\sin\theta\cos\theta$$

$$=\frac{r^2}{2}\sec^2\theta\left(1-2\sin^2\theta\right)^2-4r^2\sin^2\theta\left(1-2\sin^2\theta\right)$$

$$=r^2\left(1-2\sin^2\theta\right)\left\{\frac{\sec^2\theta}{2}\left(1-2\sin^2\theta\right)-4\sin^2\theta\right\}$$

$$=\frac{r^2}{2}\left(1-2\sin^2\theta\right)\left\{\sec^2\theta\left(1-2\sin^2\theta\right)-8\sin^2\theta\right\}$$

$$=\frac{r^2}{2}\left(1-2\sin^2\theta\right)\left\{\sec^2\theta-2\sin^2\theta\sec^2\theta-8\sin^2\theta\right\}$$

$$=\frac{r^2}{2}\left(1-2\sin^2\theta\right)\left\{\frac{1}{\cos^2\theta}\left(1-2\left(1-\cos^2\theta\right)\right)-8\left(1-\cos^2\theta\right)\right\}$$

$$=\frac{r^2}{2}\left(1-2\sin^2\theta\right)\left(\frac{1}{\cos^2\theta}\left(-1+2\cos^2\theta\right)-8+8\cos^2\theta\right)$$

$$=\frac{r^2}{2}\left(1-2\sin^2\theta\right)\left(8\cos^2\theta-6-\frac{1}{\cos^2\theta}\right)$$

$$=\frac{r^2}{2\cos^2\theta}\left(1-2\sin^2\theta\right)\left(8\cos^4\theta-6\cos^2\theta-1\right)$$

as required.

(iii) A maximum occurs when $\frac{dA}{d\theta}=0$.

If $1-2\sin^2\theta=0$, then $\sin\theta=\pm\frac{1}{\sqrt{2}}$ and $\theta=\frac{\pi}{4}$ is a solution.

This solution must be rejected though because $0<\theta<\frac{\pi}{4}$.

For $8a^2-6a-1=0$, $a=0\cdot 89$ or $a=-0\cdot 14$.

So, for $8\cos^4\theta-6\cos^2\theta-1=0$,

$\cos^2\theta=0\cdot 89$ or $\cos^2\theta=-0\cdot 14$ which has no real solutions

$\cos\theta=\pm 0\cdot 9433...$

$\theta=0\cdot 3380...$ or $\theta=2\cdot 8035...$ reject since $0<\theta<\frac{\pi}{4}$

Since we are told that just one maximum and no minimum exist for $0<\theta<\frac{\pi}{4}$, then the maximum occurs when $\theta=0\cdot 338$ (correct to 3 decimal places).

1 mark	Correct answer