



THE HILLS GRAMMAR SCHOOL

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

2006

MATHEMATICS

EXTENSION I

Teacher Responsible: Mrs P Singh

General Instructions:

- Reading time – 5 minutes
- Working time – 2 hours
- This paper contains 7 questions.
- ALL questions to be attempted.
- ALL questions are of equal value.
- ALL necessary working should be shown in every question in the booklets provided.
- Start each question in a new booklet.
- A table of standard integrals is supplied at the back of this paper.
- Board approved calculators may be used.
- Hand up your paper in ONE bundle, together with this question paper.

Students are advised that this is a Trial Examination only and cannot in any way guarantee the content or the format of the Higher School Certificate Examination.

(a) Differentiate with respect to x :

(i) $\cos^3 x$

(ii) $e^x \log_e(5x)$

(b) Solve the inequality $\frac{3}{x+4} \geq 1$

(c) A is the point $(1, 2)$ and P is the point $(4, 5)$. P divides the interval AB internally in the ratio 3:5.

Find the co-ordinates of B .

(d) Find the exact value of $\int_0^3 \frac{2 dx}{3+x^2}$.

Examination cont'd over page .../

Question Two

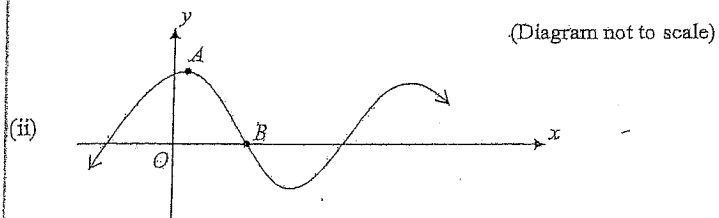
12 marks

(a) Suppose that the cubic $F(x) = x^3 + ax^2 + bx + c$ has a relative minimum at $x = \alpha$ and a relative maximum at $x = \beta$.

(i) By examining the zeroes of $F'(x)$, prove that $\alpha + \beta = -\frac{2}{3}a$. 2

(ii) Deduce that a point of inflexion occurs at $x = \frac{1}{2}(\alpha + \beta)$. 2

(b) (i) Express $\sqrt{3} \sin x + \cos x$ in the form $R \sin(x + \alpha)$ where $R > 0$. 2



The graph of $y = \sqrt{3} \sin x + \cos x$ is shown above. A is a turning point.

Write down the co-ordinates of A and B . 4

(c) Find the size of the acute angle between the lines $3x - 2y + 1 = 0$ and $5x + y - 7 = 0$. 2

Examination cont'd over page .../

Question Three

12 marks

(a) (i) Prove that $\frac{1 - \cos 2x}{1 + \cos 2x} = \sec^2 x - 1$. 3

(ii) Hence evaluate $\int_0^{\pi/4} \frac{1 - \cos 2x}{1 + \cos 2x} dx$. 3

(b) Sand is in a heap in the shape of a right circular cone whose height equals the radius of the base. More sand is added at the rate of $8 \text{ m}^3/\text{min}$. (Volume of cone = $\frac{1}{3} \pi r^2 h$).

When the heap is 10 m high, find:

(i) the rate at which the height is increasing (in terms of π), 3

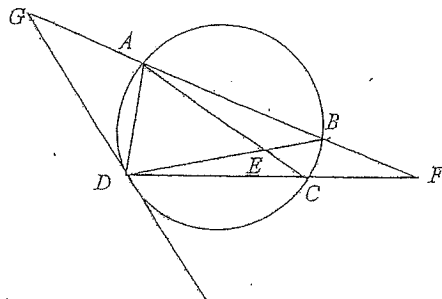
(ii) the rate at which the area of the base is increasing. 3

Examination cont'd over page .../

Question Four

12 marks

(a)



(Diagram not to scale)

DG is a tangent to the circle at D .

$GABF$ and DCF are straight lines.

- (i) Copy the diagram into your answer booklet. 4
- (ii) By letting $\angle GDA = x$, prove that $2\angle ADG = \angle BEC + \angle BFC$. 4
- (b) (i) Find the points of intersection of the graphs $y = 10 - x^2$ and $y = \frac{9}{x^2}$. 3
- (ii) Sketch these two graphs on the same set of axes. 3
- (iii) Using your graphs, or otherwise, solve $10 - x^2 > \frac{9}{x^2}$. 2

Examination cont'd over page .../

Question Five

12 marks

- (a) Prove by mathematical induction that $7^n - 1$ is divisible by 3 for any positive integer n . 4
- (b) Consider the function given by $f(x) = \sin^{-1}x + \cos^{-1}x$, $0 \leq x \leq 1$.
- (i) Evaluate $f(0)$. 1
- (ii) Find $f'(x)$. 1
- (iii) Sketch the graph of $y = f(x)$. 2
- (c) If $x = \tan \theta + \sec \theta$, use the t -formulae (i.e. $t = \tan \frac{\theta}{2}$) to show that $\frac{x^2 - 1}{x^2 + 1} = \sin \theta$. 4

Examination cont'd over page .../

Question Six

12 marks

- (a) A loan of \$50,000 is to be repaid over 20 years by equal monthly instalments. Interest is calculated monthly on the balance owing at a rate of 12% p.a.

(i) Write down an expression for A_1 , the amount owing at the end of the first month. 1

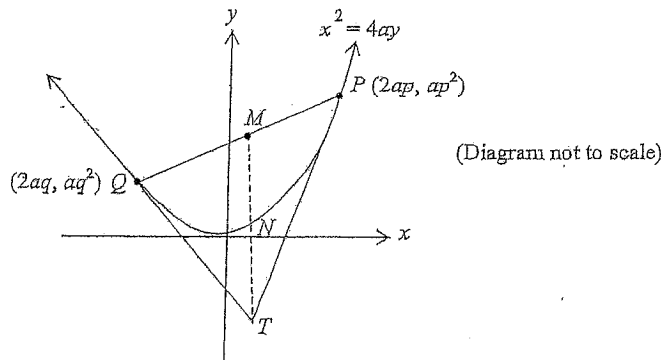
(ii) Show that A_n , the amount owing at the end of n months, is given by

$$A_n = 50000(1.01)^n - 100M(1.01^n - 1)$$

And hence find the monthly repayment, M . 2

(iii) The borrower decides to pay twice the minimum amount each month, i.e. $\$2M$ each month. How long would it then take to pay off the loan? 3

- (b) P and Q are points on the parabola $x^2 = 4ay$ with parameters p and q . The tangents at P and Q meet at T . M is the midpoint of PQ and MT is drawn to cut the parabola at N .



(i) Prove that MT is parallel to the axis of the parabola. 3

(ii) Prove that N is the midpoint of MT . 3

Examination cont'd over page .../

Question Seven

12 marks

- (a) The 'half-life' of a radio active substance is the time it takes a given amount of the substance to lose one-half of its mass. It is given that the 'half-life' of radium is 1700 years.

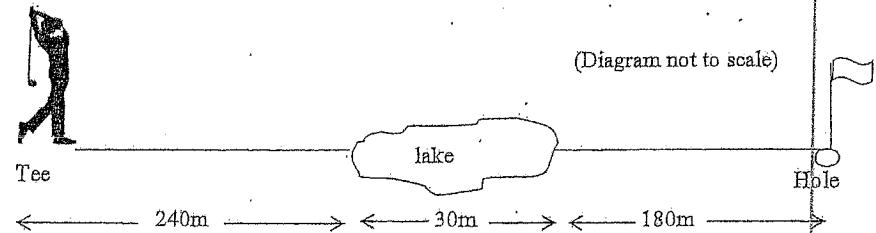
Assuming that radium decays according to the law $M = M_0 e^{-kt}$ where M measures mass, t measures time and M_0 and k are constants, find how long it would take for an amount of radium to lose 80% of its mass. (Answer to nearest 100 years). 5

- (b) A projectile is fired from the origin O at a speed $v \text{ ms}^{-1}$ and an angle of elevation of α (where $\alpha \neq \pi/2$).

You may assume that $x = vt \cos \alpha$ and $y = vt \sin \alpha - \frac{1}{2}gt^2$ where x and y are horizontal and vertical displacements of the projectile in metres from O at time t seconds after firing and g is the acceleration due to gravity.

Show that the horizontal range of the projectile is $\frac{v^2 \sin 2\alpha}{g}$ m. 4

- (c)



Vijay Singh hits a golf ball at an initial speed of 50 ms^{-1} from a tee on a level fairway towards a hole 450 m away. There is a lake 30 m wide which is situated 240 m from the tee. Vijay knows that if the ball lands within 1 m either side of the lake, it will roll into the lake. At what angle should he attempt to hit the ball in order for it to travel the furthest possible distance towards the hole, without landing in the lake? (Take $g = 10 \text{ ms}^{-2}$ and answer to nearest degree. You may use the result from (b) if you wish). 3

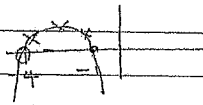
END OF EXAMINATION

Question 1

a) (i) $y = \cos^3 x$
 $y' = 3 \cos^2 x \cdot (-\sin x)$
 $= -3 \sin x \cos^2 x$

(ii) $y = e^x \log 5x$
 $y' = e^x \log 5x + e^x \cdot \frac{1}{x}$
 $= e^x (\log 5x + \frac{1}{x})$

b) $\frac{3}{x+4} \geq 1$ $x \neq -4$
 $3(x+4) \geq (x+4)^2$
 $3(x+4) - (x+4)^2 \geq 0$
 $(x+4)(3-x-4) \geq 0$
 $(x+4)(-x-1) \geq 0$
 $-4 < x \leq -1$



c) A(1,2) P($\frac{4}{3}, \frac{5}{3}$)
 B(x,y)
 $3x+5 = 4$ $4 = \frac{3x+5}{8}$
 $32 = 12x+20$ $32 = 3x+5$
 $x = 9$

$5 = \frac{3y+10}{8}$
 $56 = 6y+20$ $40 = 3y+10$
 $6y = 36$ $y = 10$
 $y = 6$

$\therefore B = (9, 10)$
 B(9,10)

d) $\int \frac{4}{x+5} dx$
 $u = x+5$
 $\frac{du}{dx} = 1$
 $x = 4, u = 2$
 $x = 1, u = 1$
 $= 2 \int_1^2 \frac{1}{u+1} \cdot \frac{1}{2} du$
 $= 2 \int_1^2 \frac{1}{u+1} du$
 $= 2 [\ln(u+1)]_1^2$
 $= 2 (\ln 3 - \ln 2)$
 $= 2 \ln \frac{3}{2}$

e) $\int_0^3 \frac{2 dx}{3+4x^2}$
 $= 2 \int_0^3 \frac{1}{\sqrt{3} \tan^{-1} \frac{x}{\sqrt{3}}} \cdot \frac{1}{\sqrt{3}} dx$
 $= \frac{2}{\sqrt{3}} (\tan^{-1} \frac{3}{\sqrt{3}} - \tan^{-1} 0)$
 $= \frac{2\sqrt{3}}{3}$

QUESTION 2

(a) (i) $F'(x) = 3x^2 + 2ax + b$
 $x+B = \frac{-2a}{3}$

(ii) $F''(x) = 6x + 2a$
 $6x = -2a$
 $x = -\frac{1}{3}a$

$-\frac{2a}{3} = (x+B)$
 $\therefore -\frac{a}{3} = \frac{(x+B)}{2}$
 as reqd.

(b)(i) $\sqrt{3} \sin x + \cos x$
 $= R \sin(x+\alpha), R > 0$
 $R = \sqrt{3+1}$
 $= 2$

$2 \sin(x+\alpha)$
 $= 2 \sin x \cos \alpha + 2 \cos x \sin \alpha$
 $2 \cos \alpha = \sqrt{3}$
 $2 \sin \alpha = 1$
 $\therefore \tan \alpha = \frac{1}{\sqrt{3}}$
 $\alpha = \frac{\pi}{6}$

$\sqrt{3} \sin x + \cos x$
 $= 2 \sin(x + \frac{\pi}{6})$

(ii) At max TP
 $\sin(x + \frac{\pi}{6}) = 1$
 $x + \frac{\pi}{6} = \frac{\pi}{2}$
 $\therefore x = \frac{\pi}{3}$

$\therefore A$ is point $(\frac{\pi}{3}, 2)$
 Bis point where curve cuts x axis ie $y=0$

$\therefore \sin(x + \frac{\pi}{6}) = 0$
 $x + \frac{\pi}{6} = 0, \pi, \dots$
 $\therefore x = -\frac{\pi}{6}, \frac{5\pi}{6}, \dots$
 $\therefore B$ is $(-\frac{\pi}{6}, 0)$

(c) $y_1 = \frac{3}{2}x + \frac{1}{2}$
 $y_2 = -5x + 7$
 $m_1 = \frac{3}{2}$ $m_2 = -5$

$\therefore \text{angle} = \left| \frac{\frac{3}{2} + 5}{1 + (\frac{3}{2} \cdot -5)} \right|$
 $= 1$
 $\therefore \theta = 45^\circ$

(ii)

Q3

(a) LHS $\frac{1 - \cos 2x}{1 + \cos 2x}$
 $= \frac{1 - (1 - 2\sin^2 x)}{1 + (2\cos^2 x - 1)}$
 $= \frac{2\sin^2 x}{2\cos^2 x}$
 $= \tan^2 x$
 $= \sec^2 x - 1$ (3)
 $= \text{RHS}$

(ii) $A = \pi r^2 = \pi h$
 $\frac{dA}{dt} = 2\pi r$
 $\frac{dA}{dt} = \frac{dA}{dh} \times \frac{dh}{dt}$
 $= 2\pi r \times \frac{2}{25\pi}$

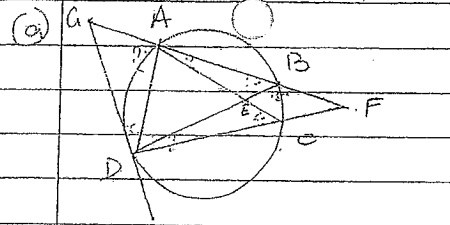
when $r = 10$, $\frac{dA}{dt} = 20\pi \times \frac{2}{25\pi}$
 $= \frac{40}{25} = \frac{8}{5}$ (3)

\therefore area of bases \uparrow at rate of $8/25 \text{ m}^2/\text{min}$.

(ii) $\int_0^{\pi/4} \frac{1 - \cos 2x}{1 + \cos 2x} dx$
 $= \int_0^{\pi/4} (\sec^2 x - 1) dx$
 $= [\tan x - x]_0^{\pi/4}$
 $= \tan \pi/4 - \pi/4$ (3)
 $= 1 - \pi/4$

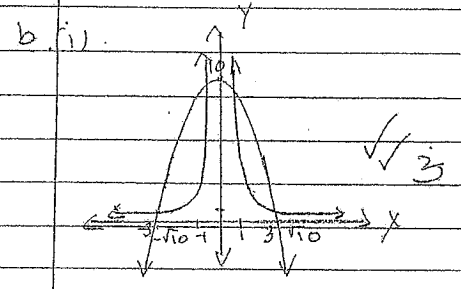
(b) (i) $V = \frac{1}{3} \pi r^2 h$
 $= \frac{1}{3} \pi h^3$ ($r=h$)
 $\frac{dV}{dh} = \pi h^2$
 and $\frac{dh}{dt} = 8$ (3)
 $\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$
 $= \pi h^2 \times 8$
 when $h = 10$, $\frac{dV}{dt} = \frac{100\pi \times 8}{1}$
 $= \frac{800\pi}{1}$
 $\therefore h$ increasing at $\frac{800\pi}{25\pi} \text{ m/min}$

Q4



(i) from graph.
 $10 - x^2 > 9/x^2$
 for $-3 < x < 1$, (2)
 $-1 < x < 3$

(ii) $\angle ACD = \angle ADB = x^\circ$
 (\angle bet \tan & cd) (iv)
 $\angle ABD = x$ (\angle 's subt by same arc)
 $\therefore \angle FBE = \angle FCE = 180^\circ - x$
 (suppl \angle 's)
 $\therefore \angle BEC + \angle BFC = 360^\circ - x(180^\circ - x)$
 $= 2x$ (\angle 's in quad)
 $\therefore \angle BEC + \angle BFC = 2\angle ADB$



(i) By substitution
 $x = 1, y = 9$
 \therefore intersect at $x = 1$
 $x = 3, y = 1$
 \therefore intersect also $x = 3$
 By sym $x = -1, x = -3$
 will also be pt. of inter.
 OR solve
 $x^2 = 10 - x^2$ or
 $x^4 - 10x^2 + 9 = 0$
 $(x^2 - 9)(x^2 - 1) = 0$
 $x = \pm 3 \pm 1$

Q5

(a) Test $n=1$

$$7^1 - 1 = 6$$

\therefore true for $n=1$

Assume true for $n=k$

$$\text{i.e. } 7^k - 1 = M$$

(where M is $\equiv 3$)

$$\text{i.e. } 7^{k+1} - 1 = 3M$$

Prove true for $n=k+1$

$$\text{i.e. } 7^{k+1} - 1$$

$$= 7 \cdot 7^k - 1 = 7 \cdot 7^k - 7 + 6$$

$$= 7(7^k - 1) + 6$$

$$= 7(3M) + 6$$

$$= 3(7M) + 6$$

$$= 3(7M+2)$$

which is divisible by 3

\therefore true for $n=k+1$

\therefore true for all n

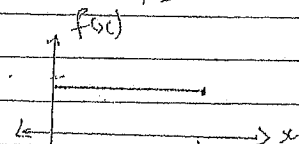
(b) (i) $f(x) = \sin^{-1}x + \cos^{-1}x$

(i) $f(0) = \sin^{-1}0 + \cos^{-1}0$
 $= \pi/2$

(ii) $f'(x) = \frac{1}{\sqrt{1-x^2}} + \frac{-1}{\sqrt{1-x^2}}$
 $= 0$

(iii) $f'(x) = 0$ for all x

\therefore graph is a horizontal line



(c) $x^2 - 1$
 $x^2 + 1$

$$= \frac{(\tan \theta + \sec \theta)^2 - 1}{(\tan \theta + \sec \theta)^2 + 1}$$

$$= \left(\frac{2t}{1-t^2} + \frac{1+t^2}{1-t^2} \right)^2 - 1$$

$$\left(\frac{2t}{1-t^2} + \frac{1+t^2}{1-t^2} \right) + 1$$

$$= \left(\frac{2t + t^2 + 1}{1-t^2} \right)^2 - 1 + t^2$$

Q6 (i) $U_n = 50000(1.01)^n - 1000M$

(a) (i) $M = \frac{PR^N(R-1)}{R^N - 1}$
 $= \frac{50000(1.01)^{240}(0.01)}{1.01^{240} - 1}$
 $= \$550.54$

(ii) The amt owing would never be paid off.

(ii) $M = \frac{PR^N(R-1)}{R^N - 1}$

$$MR^N - M = PR^N - PR^N$$

$$\text{i.e. } R^N(M - PR + P) = M$$

$$R^N = \frac{M}{M - PR + P}$$

Want $M > 2 \times 550.54$

$$P = 50000$$

$$R = 1.01$$

$$\therefore 1.01^N = \frac{2 \times 550.54}{2 \times 550.54 - 50000 \times 1.01 + 50000}$$

$$= 1.832$$

$$\therefore N = 60.8 \text{ yrs.}$$

(b)

(i) Midpt $M = \left(\frac{2a+2a}{2}, \frac{ap^2+ap^2}{2} \right)$
 $= \left(a(ptq), a(ptq)^2 \right)$

M & T have same abscissa
 $\therefore MT \parallel Y$ axis (axis of parab)

(ii) $x^2 = 4ay$

$$4ay = \frac{a^2(ptq)^2}{4}$$

$$\therefore N \text{ is } \left[a(ptq), a(ptq)^2 \right]$$

$$\text{midpt of } MT = \left[a(ptq), \frac{a(ptq)^2}{2} \right]$$

$$= \left(a(ptq), \frac{a(ptq)^2}{4} \right)$$

= coord of N

$\therefore N$ is midpt. of MT

(iii) $y = \frac{x^2}{4a}$
 $y' = \frac{x}{2a}$

$$\text{at } N = \left(\frac{a(ptq)}{2a}, \frac{a(ptq)^2}{4} \right)$$

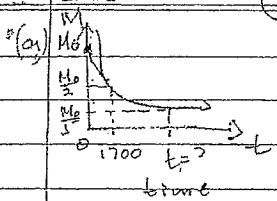
$$= \left(\frac{ptq}{2}, \frac{a(ptq)^2}{4} \right)$$

$$= m \cdot PQ$$

$\therefore PQ \perp$ tangent at N

Q7

(12)

when $t = \frac{2V \sin \alpha}{g}$

$$x = V \left(\frac{2V \sin \alpha}{g} \right) \cos \alpha$$

$$= \frac{2V^2 \sin \alpha \cos \alpha}{g} \quad \checkmark \quad (4)$$

$$M = M_0 e^{-kt}$$

when $t=0$, $M = M_0$ $\therefore M_0$ rep orig mass.

$$t = 1700, M = \frac{M_0}{2}$$

$$\therefore \text{horizontal range} = \frac{v^2 \sin 2\alpha}{g}$$

$$\frac{M_0}{2} = M_0 e^{-k \cdot 1700}$$

(c) To achieve max range, Vijay should hit ball at 45°

$$0.5 = e^{-1700k}$$

$$k = 0.0004077$$

$$\therefore \text{horizontal range} = \frac{50^2 \sin 90^\circ}{10}$$

$$= 250 \text{ m.} \quad \checkmark$$

$$\therefore M = M_0 e^{-0.0004077t}$$

Need t when only 20% of M_0 remains but this will land in lakeof M_0 remains

$$\text{i.e. } \frac{M_0}{5} = M_0 e^{-0.0004077t}$$

 \therefore Vijay should try to hit the ball so that horizontal range is $\leq 239 \text{ m.} \quad (3)$

$$\text{i.e. } 0.2 = e^{-0.0004077t}$$

$$\therefore t = 3900 \text{ yrs} \quad (5)$$

(to nearest 100 yrs)

$$1b) \quad x = vt \cos \alpha$$

$$y = vt \sin \alpha - \frac{1}{2}gt^2$$

horizontal range

$$\Rightarrow y = 0$$

$$\text{when } y = 0, vt \sin \alpha - \frac{1}{2}gt^2 = 0$$

$$t(v \sin \alpha - \frac{1}{2}gt) = 0$$

$$\therefore t = 0 \quad t = \frac{2v \sin \alpha}{g}$$

initial condition