Centre Number

Student Number

CSSA
CATHOLIC SECONDARY SCHOOLS
ASSOCIATION OF NSW

2013
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

Morning Session
Monday, 5 August 2013

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided on a separate sheet
- In Questions 11–16, show relevant mathematical reasoning and/or calculations
- Write your Centre Number and Student Number at the top of this page

Total marks – 100

Section I Pages 2–6

10 marks
- Attempt Questions 1–10
- Allow 15 minutes for this section

Section II Pages 7–14

90 marks
- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

6400-1

STANDARD INTEGRALS

$\int x^n dx$	$= \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$
$\int \frac{1}{x} dx$	$= \ln x, \quad x > 0$
$\int e^{ax} dx$	$= \frac{1}{a} e^{ax}, \quad a \neq 0$
$\int \cos ax \, dx$	$= \frac{1}{a} \sin ax, \quad a \neq 0$
$\int \sin ax \, dx$	$= -\frac{1}{a} \cos ax, \quad a \neq 0$
$\int \sec^2 ax \, dx$	$= \frac{1}{a} \tan ax, \quad a \neq 0$
$\int \sec ax \tan ax \, dx$	$= \frac{1}{a} \sec ax, \quad a \neq 0$
$\int \frac{1}{a^2 + x^2} dx$	$= \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$
$\int \frac{1}{\sqrt{a^2 - x^2}} dx$	$= \sin^{-1} \frac{x}{a}, \quad a > 0, \ -a < x < a$
$\int \frac{1}{\sqrt{x^2 - a^2}} dx$	$= \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$
$\int \frac{1}{\sqrt{x^2 + a^2}} dx$	$= \ln\left(x + \sqrt{x^2 + a^2}\right)$

NOTE: $\ln x = \log_e x, \quad x > 0$

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

1 If $z = 2+3i$ and $w = -5-2i$, what is the value of zw?

(A) -4

(B) $-3+i$

(C) $-4-19i$

(D) $-16-19i$

2 What is the gradient of the tangent to the curve $-2x^2 + y^2 + y = 0$ at the point $(1,1)$?

(A) 1

(B) $\frac{4}{3}$

(C) $\frac{3}{2}$

(D) 2

3 The point $P\left(ct, \frac{c}{t}\right)$ lies on the hyperbola $xy = c^2$.

What is the x-intercept of the tangent to the hyperbola at P?

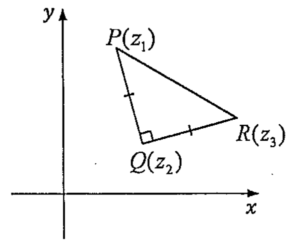
(A) $(2ct, 0)$

(B) $\left(\frac{2c}{t}, 0\right)$

(C) $\left(ct - \frac{c}{t^3}, 0\right)$

(D) $\left(\frac{c}{t} - ct^3, 0\right)$

4 The vertices of the triangle PQR are represented by the complex numbers z_1, z_2 and z_3 respectively. The triangle PQR is isosceles and right-angled at Q, as shown in the diagram.



Which of the following statements is true?

(A) $z_2 - z_1 = i(z_3 - z_2)$

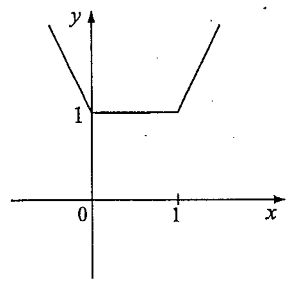
(B) $z_1 - z_2 = i(z_3 - z_2)$

(C) $z_2 - z_1 = i(z_1 - z_3)$

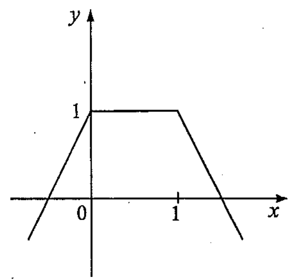
(D) $z_1 - z_2 = i(z_1 - z_3)$

5 Which of the following graphs could represent the graph of $y = |x| + |x-1|$?

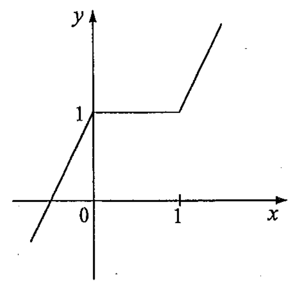
(A)



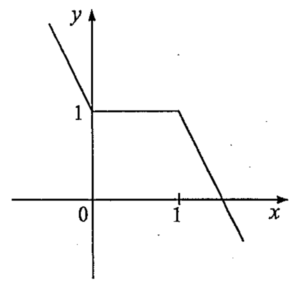
(B)



(C)



(D)



6 Which of the following, for $x > 0$, is an expression for $\int \frac{1}{x^3+x}\,dx$?

(A) $\log_e\left(x\sqrt{x^2+1}\right)+C$

(B) $\log_e\left(x(x^2+1)\right)+C$

(C) $\log_e\left(\frac{x}{\sqrt{x^2+1}}\right)+C$

(D) $\log_e\left(\frac{x}{x^2+1}\right)+C$

7 A stone of mass m is dropped from rest and falls in a medium in which the resistance is directly proportional to the square of the velocity v. Suppose mk is the constant of proportionality and that the displacement downwards from the initial position is x at time t. The acceleration due to gravity is g.

Which of the following is true?

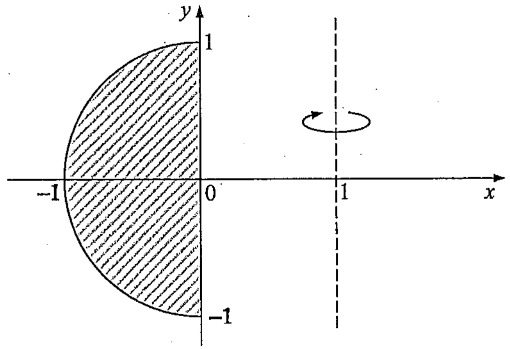
(A) The terminal velocity is $\frac{g}{k}$.

(B) As $t \to \infty$, $x \to L$ where L is a positive constant.

(C) The equation of motion is given by $v\frac{dv}{dx} = g - kv^2$.

(D) The time for the stone to reach velocity V is given by $\int_0^V g - kv^2\,dv$.

8 The diagram shows the graph $x^2+y^2=1$ for $-1 \le x \le 0$. The region bounded by the graph and the y-axis is rotated about the line $x = 1$ to form a solid.



Which integral represents the volume of the solid?

(A) $2\pi\int_{-1}^{0}(1+x)\sqrt{1-x^2}\,dx$

(B) $2\pi\int_{-1}^{0}(1-x)\sqrt{1-x^2}\,dx$

(C) $4\pi\int_{-1}^{0}(1+x)\sqrt{1-x^2}\,dx$

(D) $4\pi\int_{-1}^{0}(1-x)\sqrt{1-x^2}\,dx$

9 The polynomial $p(x)$ of degree 4 has real coefficients.
$p(x)$ has roots α, β, γ and δ and it is known that $\alpha^2+\beta^2+\gamma^2+\delta^2=-8$.

Which of the following must be true?

(A) $p(x)$ has all its roots real.

(B) $p(x)$ has one real and three imaginary roots.

(C) $p(x)$ has two real and two imaginary roots.

(D) $p(x)$ has at least two imaginary roots.

10 If $f(x)$ is a non-zero odd function with period π, which of the following statements is false?

(A) $\int_0^{2\pi} f(x)\,dx = 0$

(B) $\int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} f(x)\,dx = 2\int_0^{\frac{\pi}{2}} f(x)\,dx$

(C) $\int_0^{\pi} f(x)\,dx = -\int_0^{\pi} f(-x)\,dx$

(D) $\int_{\alpha}^{\alpha+\pi} f(x)\,dx = \int_0^{\pi} f(x)\,dx$ for any real number α.

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z=2+3i$ and $w=-3-4i$.

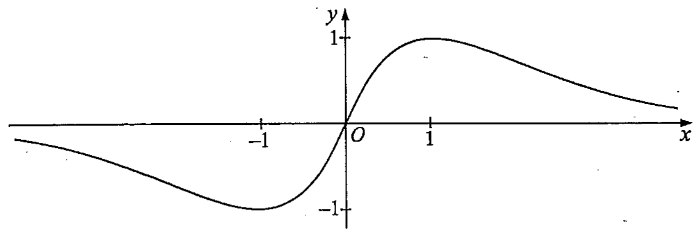
(i) Find $z+\overline{w}$ **1**

(ii) Express $\frac{w}{z}$ in the form $a+ib$, where a and b are real numbers. **2**

(b) Sketch the region in the complex plane which satisfies $\frac{\pi}{6}<\arg(z)<\frac{5\pi}{6}$ and $\frac{1}{2}\le \mathrm{Im}(z)\le 2$. **3**

(c) Find $\int \frac{dx}{(9-x^2)^{\frac{3}{2}}}$ using the substitution $x=3\sin\theta$. Give your answer in terms of x. **4**

(d) The following diagram shows the graph of $f(x)=\frac{2x}{x^2+1}$.



Draw separate one-third page diagrams of the graphs of each of the following.

(i) $y=[f(x)]^2$ **2**

(ii) $y=\sqrt{f(x)}$ **1**

(iii) $y=\frac{1}{f(x)}$ **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Find the square roots of $-24-10i$. **2**

(ii) Hence, or otherwise, solve $x^2-(1-i)x+6+2i=0$. **2**

(b) Use integration by parts to find $\int \frac{\ln x}{x^2}\,dx$. **2**

(c) Using the substitution $t=\tan\frac{x}{2}$, or otherwise, evaluate $\int_0^{\frac{\pi}{2}} \frac{dx}{2+\cos x}$. **4**

(d) An ellipse is defined by the parametric equations:

$$x=2\cos\theta$$
$$y=3\sin\theta$$

for $0\le\theta<2\pi$.

(i) Find the Cartesian equation of the ellipse. **1**

(ii) Find the eccentricity of the ellipse. **1**

(iii) Sketch the ellipse showing the intercepts, foci and directrices. **3**

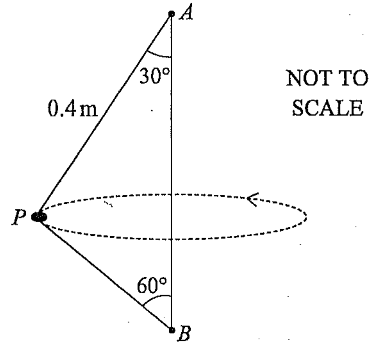
Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) A group of 30 students is to be divided into three groups consisting of 7, 8 and 15 students. In how many ways can this be done? **1**

(b) (i) Find a and b such that $x=2$ is a double root of $p(x)=x^4+ax^3+x^2+b$. **3**

(ii) For the values of a and b above, factorise $p(x)$ over the real numbers. **1**

(c)



A particle P of mass 0.3 kg is attached to one end of each of two light inextensible strings of different lengths. The longer string is also attached to a fixed point A and the shorter string is also attached to a fixed point B, which is vertically below A.

AP makes an angle of 30° with the vertical and is 0.4 m long. PB makes an angle of 60° with the vertical. The particle moves in a horizontal circle with constant angular speed and with both strings taut. The tension in the string AP is 5 N. Assume the acceleration due to gravity is $10\,\text{ms}^{-2}$.

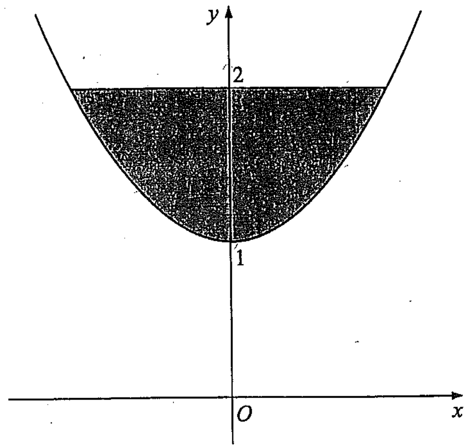
(i) Find the tension in the string PB. **2**

(ii) Calculate the angular velocity of the particle P correct to 1 decimal place. **3**

Question 13 continues of page 10

Question 13 (continued)

(d) The base of a solid, S, is the region enclosed by the parabola $y = x^2 + 1$ and the line $y = 2$.



Each cross section of S perpendicular to the y-axis is a rectangle. The height of the rectangle that is y units from the origin is $\frac{1}{2}y$ units.

(i) Show that the area of the rectangle y units from the origin is given by $y\sqrt{y-1}$ square units. **2**

(ii) Hence find the volume of the solid S. **3**

End of Question 13

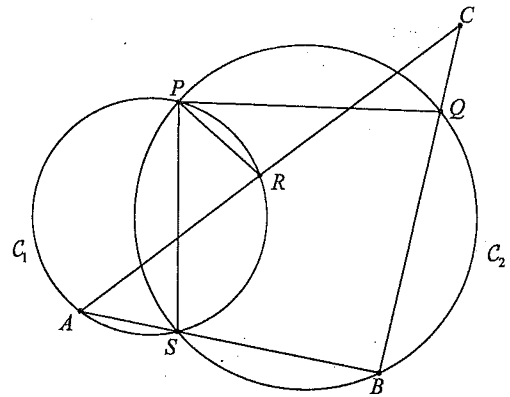
Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Consider the polynomial $p(x) = x^3 - x^2 - 21x + 45$ with roots α, β and γ.

(i) Find the monic polynomial with roots $\alpha - 3$, $\beta - 3$, $\gamma - 3$. **3**

(ii) Hence solve $p(x) = 0$. **1**

(b) Two circles C_1 and C_2 meet at P and S. Points A and R lie on C_1 and points B and Q lie on C_2. AB passes through S and AR produced meets BQ produced at C, as shown in the diagram.



(i) Prove that $\angle PRA = \angle PQB$. **2**

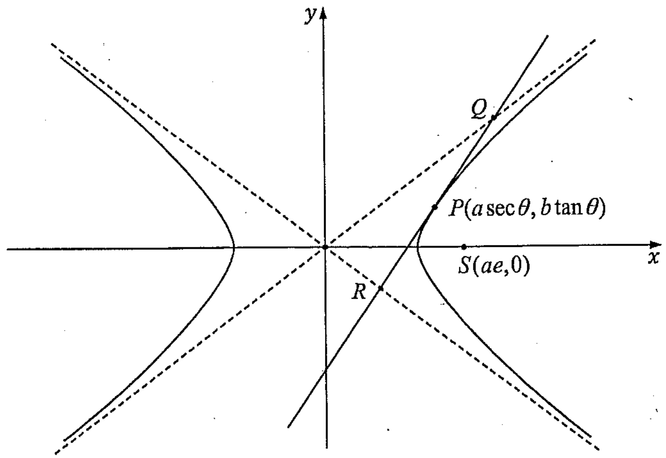
(ii) Prove that the points P, R, Q and C are concyclic. **2**

Question 14 continues on page 12

Question 14 (continued)

(c) The point $P(a\sec\theta, b\tan\theta)$ lies on the hyperbola $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$ with focus $S(ae,0)$.

The tangent to the hyperbola at P meets the asymptotes of the hyperbola at Q and R, as shown in the diagram.



(i) Show that the equation of the tangent to the hyperbola at P is given by **2**

$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1.$$

(ii) Show that Q has coordinates $\left(\dfrac{a}{\sec\theta-\tan\theta},\dfrac{b}{\sec\theta-\tan\theta}\right)$. **2**

The coordinates of R are given by $\left(\dfrac{a}{\sec\theta+\tan\theta},\dfrac{-b}{\sec\theta+\tan\theta}\right)$. (Do NOT prove this).

(iii) Prove that $\tan\angle QSR=\dfrac{b}{a}$. **3**

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Evaluate $\displaystyle\int_0^{\frac{\pi}{4}}\tan x\,dx$. **2**

(ii) Suppose $I_n=\displaystyle\int_0^{\frac{\pi}{4}}\tan^n x\,dx$. **2**

Given that $I_n=\dfrac{1}{n-1}-I_{n-2}$ for any integer $n\geq 2$, find the value of

$$\int_0^{\frac{\pi}{4}}\tan^5 x\,dx.$$

(b) (i) Find the five fifth roots of $z^5=1$ and plot these on an Argand diagram. **2**

(ii) Express z^5-1 as the product of real linear and quadratic factors. **2**

(iii) Prove that $\cos\dfrac{2\pi}{5}+\cos\dfrac{4\pi}{5}=-\dfrac{1}{2}$. **2**

(c) A projectile starting from the origin has acceleration given by

$$\frac{d^2x}{dt^2}=-6\frac{dx}{dt}-9x$$

where x is the displacement from the origin at time t.

The solution to this equation is known to be of the form, $x(t)=f(t)e^{-3t}$, for some function $f(t)$.

(i) Show that $f(t)=At+B$, where A and B are constants. **2**

(ii) Find the value of B. **1**

(iii) Assuming that A is positive, at what time is the displacement a maximum? You do not need to prove a maximum is attained. **2**

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) The points P and Q represent the complex numbers z and w on an Argand diagram, where $w=\dfrac{3z+2}{z-1}$ and P moves on the circle $|z|=1$. **3**

By writing an expression for z in terms of w, or otherwise, sketch the locus of w.

(b) (i) For any positive integer k, show that **1**

$$\frac{1}{2^k+1}+\frac{1}{2^k+2}+\frac{1}{2^k+3}+\ldots+\frac{1}{2^k+2^k}\geq\frac{1}{2}.$$

(ii) Prove by induction that, for integers $n\geq 1$, **3**

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^n}\geq\frac{1}{2}(n+1).$$

(c) Let $x=\cos\theta+i\sin\theta$ for $0<\theta<2\pi$, and let n be a positive integer.

(i) Show that $x^k+\dfrac{1}{x^k}=2\cos k\theta$, for any positive integer k. **2**

(ii) Show that **3**

$$\left(x+\frac{1}{x}\right)^{2n}=\left(x^{2n}+\frac{1}{x^{2n}}\right)+\binom{2n}{1}\left(x^{2n-2}+\frac{1}{x^{2n-2}}\right)+\binom{2n}{2}\left(x^{2n-4}+\frac{1}{x^{2n-4}}\right)+$$
$$\ldots+\binom{2n}{n-1}\left(x^2+\frac{1}{x^2}\right)+\binom{2n}{n}.$$

(iii) Deduce that **1**

$$2^{2n-1}\cos^{2n}\theta=\cos 2n\theta+\binom{2n}{1}\cos(2n-2)\theta+\binom{2n}{2}\cos(2n-4)\theta+$$
$$\ldots+\binom{2n}{n-1}\cos 2\theta+\frac{1}{2}\binom{2n}{n}.$$

(iv) Hence show that $\displaystyle\int_0^{2\pi}\cos^{2n}\theta\,d\theta=\frac{\pi}{2^{2n-1}}\binom{2n}{n}$. **2**

End of paper

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CSSA 2013 EXT 2 TRIAL HSC SOLUTIONS

Section I
10 marks

Questions 1-10 (1 mark each)

Question 1 (1 mark)

Solution	Answer	Mark
$(2+3i)(-5-2i)$ $=-10-4i-15i-6i^2$ $=-4-19i$	C	1

Question 2 (1 mark)

Solution	Answer	Mark
Using implicit differentiation $-4x+2y\frac{dy}{dx}+\frac{dy}{dx}=0$ $\frac{dy}{dx}=\frac{4x}{2y+1}$ Substituting (1, 1), $\frac{dy}{dx}=\frac{4}{3}$	B	1

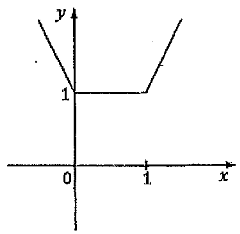
Question 3 (1 mark)

Solution	Answer	Mark
$\frac{dy}{dx}=\frac{-c^2}{x^2}$. At $P\left(ct,\frac{c}{t}\right)$, $\frac{dy}{dx}=\frac{-c^2}{c^2t^2}=\frac{-1}{t^2}$. Equation of the tangent at P: $y-\frac{c}{t}=\frac{-1}{t^2}(x-ct)$ $x+t^2y=2ct$ The x-intercept is $(2ct,0)$.	A	1

Question 4 (1 mark)

Solution	Answer	Mark
$\overrightarrow{QP}=i\times\overrightarrow{QR}$ $z_1-z_2=i(z_3-z_2)$	B	1

Question 5 (1 mark)

Solution	Answer	Mark
	A	1

Question 6 (1 mark)

Solution	Answer	Mark
$\int\frac{1}{x^3+x}dx=\int\left(\frac{1}{x}+\frac{-x}{x^2+1}\right)dx$ $=\log_e x-\frac{1}{2}\log_e(x^2+1)+C$ $=\log_e\frac{x}{\sqrt{x^2+1}}+C$	C	1

Question 7 (1 mark)

Solution	Answer	Mark
The equation of motion is $m\ddot{x}=mg-mkv^2$ $\ddot{x}=g-kv^2$ Since $\ddot{x}=v\frac{dv}{dx}$, $v\frac{dv}{dx}=g-kv^2$.	C	1

Question 8 (1 mark)

Solution	Answer	Mark
$V = 2\pi\int_{-1}^{0}(1-x)\times 2\sqrt{1-x^2}\,dx$ $= 4\pi\int_{-1}^{0}(1-x)\sqrt{1-x^2}\,dx$	D	1

Question 9 (1 mark)

Solution	Answer	Mark
$p(x)$ has at least 2 imaginary roots.	D	1

Question 10 (1 mark)

Solution	Answer	Mark
$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\,dx \neq 2\int_{0}^{\frac{\pi}{2}} f(x)\,dx$	B	1

Section II
60 marks

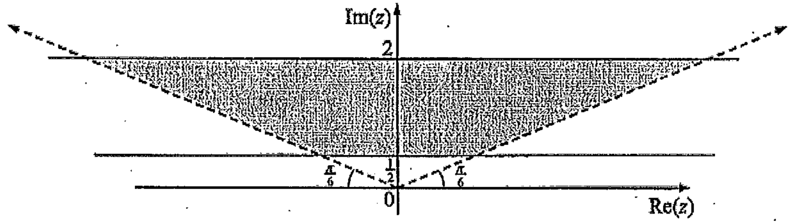
Question 11 (15 marks)
(a) (i) (1 mark)

$$z + \overline{w} = 2 + 3i + (-3 + 4i)$$
$$= -1 + 7i$$

(a) (ii) (2 marks)

$$\frac{-3-4i}{2+3i} = \frac{-3-4i}{2+3i} \times \frac{2-3i}{2-3i}$$
$$= \frac{-18+i}{13}$$
$$= \frac{-18}{13} + \frac{i}{13}$$

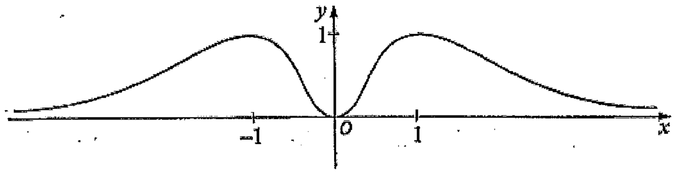
(b) (3 marks)



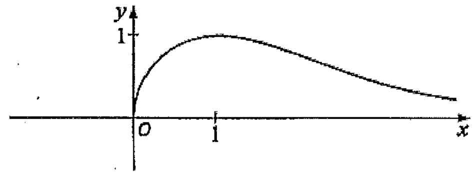
(c) (4 marks)

$$\int \frac{dx}{(9-x^2)^{\frac{3}{2}}} = \int \frac{3\cos\theta\,d\theta}{(9-9\sin^2\theta)^{\frac{3}{2}}}$$
$$= \int \frac{\cos\theta\,d\theta}{9\cos^3\theta}$$
$$= \frac{1}{9}\int \sec^2\theta\,d\theta$$
$$= \frac{1}{9}\tan\theta + C$$
$$= \frac{x}{9\sqrt{9-x^2}} + C$$

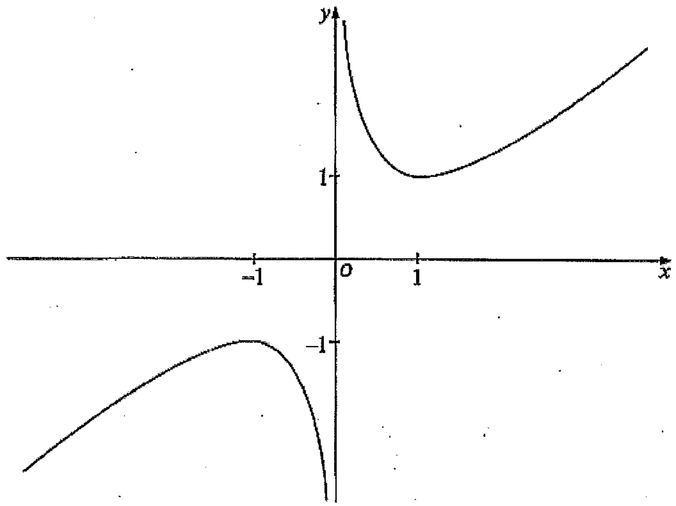
(d) (i) (2 marks)



(d) (ii) (1 mark)



(d) (iii) (2 marks)



Question 12 (15 marks)

(a) (i) (2 marks)

Let $-24-10i=(a+ib)^2$ where a and b are real.

Then, $a^2-b^2+2abi=-24-10i$

Equating real and imaginary parts: $a^2-b^2=-24$ and $2ab=-10$

Substituting $b=\frac{-5}{a}$ into $a^2-b^2=-24$, gives

$a^4+24a^2-25=0$

$(a^2+25)(a^2-1)=0$

$a=\pm 1$ since a is real.

$\therefore b=\mp 5$

$\therefore$ The square roots of $-24-10i$ are $1-5i, -1+5i$.

(a) (ii) (2 marks)

$$x=\frac{1-i\pm\sqrt{(1-i)^2-4(6+2i)}}{2}$$
$$=\frac{1-i\pm\sqrt{-24-10i}}{2}$$
$$=\frac{1-i\pm(1-5i)}{2}$$
$$=1-3i,\ 2i$$

(b) (2 marks)

$$\int\frac{\ln x}{x^2}dx=\int \ln x.x^{-2}dx$$
$$=\ln x\left(-x^{-1}\right)-\int\left(-x^{-1}\right)\frac{1}{x}dx$$
$$=-\frac{\ln x}{x}+\int x^{-2}\,dx$$
$$=-\frac{\ln x}{x}-\frac{1}{x}+C$$

(c) (4 marks)

$$\int_0^{\frac{\pi}{2}}\frac{dx}{2+\cos x}=\int_0^1\frac{1}{2+\frac{1-t^2}{1+t^2}}\times\frac{2dt}{1+t^2}$$
$$=\int_0^1\frac{2dt}{t^2+3}$$
$$=\left[\frac{2}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}}\right]_0^1$$
$$=\frac{\sqrt{3}\pi}{9}$$

(d) (i) (1 mark)

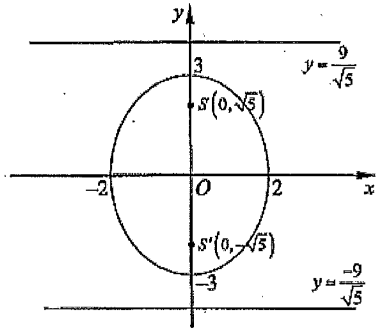
$$\frac{x^2}{4}+\frac{y^2}{9}=1$$

(d) (ii) (1 mark)

$4 = 9(1 - e^2)$

$e = \dfrac{\sqrt{5}}{3}$

(d) (iii) (3 marks)



Question 13 (15 marks)

(a) (1 mark)

$\dfrac{30!}{7!8!15!}$ or $\dbinom{30}{7}\dbinom{23}{8}\dbinom{15}{15}$

(b) (i) (3 marks)

$p(x) = x^4 + ax^3 + x^2 + b$

$p'(x) = 4x^3 + 3ax^2 + 2x$

$p'(2) = 0 \Rightarrow 32 + 12a + 4 = 0 \Rightarrow a = -3$

$p(2) = 0 \Rightarrow 16 - 24 + 4 + b = 0 \Rightarrow b = 4$

$\therefore a = -3$ and $b = 4$

(b) (ii) (1 mark)

$p(x) = x^4 - 3x^3 + x^2 + 4$

$(x^4 - 3x^3 + x^2 + 4) \div (x^2 - 4x + 4) = x^2 + x + 1$

$\therefore p(x) = (x-2)^2(x^2 + x + 1)$

(c) (i) (2 marks)

Let T be the tension in the string PB.

$$5\cos 30° = T\cos 60° + mg$$
$$\frac{5\sqrt{3}}{2} = \frac{T}{2} + 0.3 \times 10$$
$$T = 5\sqrt{3} - 6 \text{ Newtons}$$
$$(\approx 2.66 \text{ Newtons})$$

(c) (ii) (3 marks)

Let P move in a horizontal circle of radius r metres at ω radians/second.

$$r = 0.4\sin 30° = 0.2 \text{ metres}$$

Resolving forces horizontally

$$5\sin 30° + T\sin 60° = m\omega^2 r$$
$$\frac{5}{2} + (5\sqrt{3} - 6)\frac{\sqrt{3}}{2} = 0.3 \times \omega^2 \times 0.2$$
$$\omega = \sqrt{\frac{50(10 - 3\sqrt{3})}{3}}$$
$$\approx 8.9 \text{ radians / second (1 d.p.)}$$

(d) (i) (2 marks)

Parabola has equation $y = x^2 + 1 \Rightarrow x = \pm\sqrt{y-1}$

$\therefore$ y units from the origin, the base of the rectangular cross-section has length $2\sqrt{y-1}$.

The height of the rectangular cross-section is $\frac{1}{2}y$ (given).

Hence, the area of the rectangle y units from the origin is $2\sqrt{y-1} \times \dfrac{1}{2}y = y\sqrt{y-1}$ square units.

(d) (ii) (3 marks)

$V=\int_1^2 y\sqrt{y-1}\,dy$

Using the substitution $y=u+1$

$V=\int_0^1 (u+1)\sqrt{u}\,du$

$=\int_0^1 \left(u^{\frac{3}{2}}+u^{\frac{1}{2}}\right)du$

$=\left[\frac{2u^{\frac{5}{2}}}{5}+\frac{2u^{\frac{3}{2}}}{3}\right]_0^1$

$=\frac{16}{15}$

$\therefore$ The volume of the solid S is $\frac{16}{15}$ cubic units.

Question 14 (15 marks)

(a) (i) (3 marks)

$p(x)=x^3-x^2-21x+45=(x-\alpha)(x-\beta)(x-\gamma)$ has roots α, β and γ.

$\therefore$ The monic polynomial with roots $\alpha-3$, $\beta-3$ and $\gamma-3$ is given by

$$\left((x+3)-\alpha\right)\left((x+3)-\beta\right)\left((x+3)-\gamma\right)$$
$$=(x+3)^3-(x+3)^2-21(x+3)+45$$
$$=x^3+8x^2.$$

(a) (ii) (1 mark)

$x^3+8x^2=x^2(x+8)$ has roots $\alpha-3=0, \beta-3=0, \gamma-3=-8$

$\therefore \alpha=3, \beta=3, \gamma=-5$.

$\therefore x=3,-5$ are the solutions to $p(x)=0$.

(b) (i) (2 marks)

$\angle PRA=\angle PSA$ (angles in the same segment are equal)

$\angle PSA=\angle PQB$ (exterior angle of cyclic quadrilateral $PSBQ$ equals the opposite interior angle)

$\therefore \angle PRA=\angle PQB$

(b) (ii) (2 marks)

$\angle PRA=\angle PQB$ (from part i)

$180°-\angle PRA=180°-\angle PQB$

Therefore, $\angle PRC=\angle PQC$

$\therefore$ P, R, Q and C are concyclic as PC subtends equal angles at R and Q (on the same side of PC)

(c) (i) (2 marks)

$x=a\sec\theta \Rightarrow \frac{dx}{d\theta}=a\sec\theta\tan\theta$

$y=b\tan\theta \Rightarrow \frac{dy}{d\theta}=b\sec^2\theta$

$\therefore \frac{dy}{dx}=\frac{b\sec\theta}{a\tan\theta}$

The equation of the tangent at P:

$$y-b\tan\theta=\frac{b\sec\theta}{a\tan\theta}(x-a\sec\theta)$$
$$ay\tan\theta-ab\tan^2\theta=bx\sec\theta-ab\sec^2\theta$$
$$bx\sec\theta-ay\tan\theta=ab(\sec^2\theta-\tan^2\theta)$$
$$\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$$

(c) (ii) (2 marks)

For the coordinates of Q solve $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$ simultaneously with the equation of the asymptote $y=\frac{b}{a}x$.

$\frac{x}{a}\sec\theta-\frac{\frac{b}{a}x}{b}\tan\theta=1$

$x\left(\frac{\sec\theta}{a}-\frac{\tan\theta}{a}\right)=1$

$x=\frac{a}{\sec\theta-\tan\theta}$

Substituting into $y=\frac{b}{a}x$:

$y=\frac{b}{a}\left(\frac{a}{\sec\theta-\tan\theta}\right)$

$=\frac{b}{\sec\theta-\tan\theta}$

Hence, the coordinates of Q are $\left(\frac{a}{\sec\theta-\tan\theta},\frac{b}{\sec\theta-\tan\theta}\right)$.

(c) (iii) (3 marks)

$$m_{QS}=\frac{\frac{b}{\sec\theta-\tan\theta}}{\frac{a}{\sec\theta-\tan\theta}-ae}=\frac{b}{a(1-e\sec\theta+e\tan\theta)}$$

$$m_{RS}=\frac{\frac{-b}{\sec\theta+\tan\theta}}{\frac{a}{\sec\theta+\tan\theta}-ae}=\frac{-b}{a(1-e\sec\theta-e\tan\theta)}$$

$$\tan(\angle QRS)=\left|\frac{m_{QS}-m_{RS}}{1+m_{QS}m_{RS}}\right|$$

$$=\left|\frac{\frac{b}{a(1-e\sec\theta+e\tan\theta)}-\frac{-b}{a(1-e\sec\theta-e\tan\theta)}}{1+\frac{b}{a(1-e\sec\theta+e\tan\theta)}\times\frac{-b}{a(1-e\sec\theta-e\tan\theta)}}\right|$$

$$=\left|\frac{ab(1-e\sec\theta-e\tan\theta+1-e\sec\theta+e\tan\theta)}{a^2(1-e\sec\theta+e\tan\theta)(1-e\sec\theta-e\tan\theta)-b^2}\right|$$

$$=\left|\frac{2ab(1-e\sec\theta)}{a^2-2a^2e\sec\theta+a^2e^2\sec^2\theta-a^2e^2\tan^2\theta-b^2}\right|$$

$$=\left|\frac{2ab(1-e\sec\theta)}{a^2-2a^2e\sec\theta+a^2e^2(\sec^2\theta-\tan^2\theta)-(a^2e^2-a^2)}\right|$$

$$=\left|\frac{2ab(1-e\sec\theta)}{2a^2(1-e\sec\theta)}\right|$$

$$=\left|\frac{b}{a}\right|$$

$$=\frac{b}{a}\text{ since }a,b>0$$

Question 15 (15 marks)

(a) (i) (2 marks)

$$\int_0^{\frac{\pi}{4}}\tan x\,dx=\int_0^{\frac{\pi}{4}}\frac{\sin x}{\cos x}dx$$

$$=\left[-\ln(\cos x)\right]_0^{\frac{\pi}{4}}$$

$$=\ln\sqrt{2}$$

(a) (ii) (2 marks)

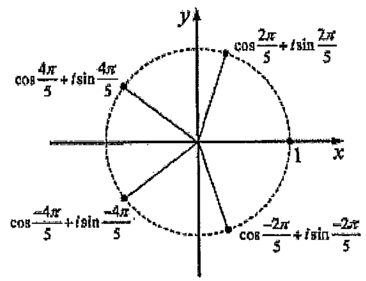
$$I_5=\frac{1}{4}-I_3$$

$$=\frac{1}{4}-\left(\frac{1}{2}-I_1\right)$$

$$=\frac{-1}{4}+\ln\sqrt{2}$$

(b) (i) (2 marks)

$z=1,\ \cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5},\ \cos\frac{-2\pi}{5}+i\sin\frac{-2\pi}{5},\ \cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5},\ \cos\frac{-4\pi}{5}+i\sin\frac{-4\pi}{5}$



(b) (ii) (2 marks)

The quadratic with roots $\alpha=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5},\ \beta=\cos\frac{-2\pi}{5}+i\sin\frac{-2\pi}{5}=\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}$

is $z^2-(\alpha+\beta)z+\alpha\beta=z^2-\left(2\cos\frac{2\pi}{5}\right)z+1$.

Similarly, the quadratic with roots $\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5},\ \cos\frac{-4\pi}{5}+i\sin\frac{-4\pi}{5}$ is $z^2-\left(2\cos\frac{4\pi}{5}\right)z+1$.

Hence, $z^5-1=(z-1)\left(z^2-\left(2\cos\frac{2\pi}{5}\right)z+1\right)\left(z^2-\left(2\cos\frac{4\pi}{5}\right)z+1\right)$.

(b) (iii) (2 marks)

sum of roots $= \frac{-b}{a} = 0$

$1+\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}+\cos\frac{-2\pi}{5}+i\sin\frac{-2\pi}{5}+\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}+\cos\frac{-4\pi}{5}+i\sin\frac{-4\pi}{5}=0$

$1+\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}+\cos\frac{2\pi}{5}-i\sin\frac{2\pi}{5}+\cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}+\cos\frac{4\pi}{5}-i\sin\frac{4\pi}{5}=0$

$1+2\cos\frac{2\pi}{5}+2\cos\frac{4\pi}{5}=0$

$\cos\frac{2\pi}{5}+\cos\frac{4\pi}{5}=\frac{-1}{2}$

(c) (i) (2 marks)

$x(t)=f(t)e^{-3t}$

$x'(t)=\left(-3f(t)+f'(t)\right)e^{-3t}$

$x''(t)=\left(9f(t)-3f'(t)\right)e^{-3t}+\left(-3f'(t)+f''(t)\right)e^{-3t}$

$\quad=\left(9f(t)-6f'(t)+f''(t)\right)e^{-3t}$

Since $\frac{d^2x}{dt^2}=-6\frac{dx}{dt}-9x$,

$\left(9f(t)-6f'(t)+f''(t)\right)e^{-3t}=-6\left(\left(-3f(t)+f'(t)\right)e^{-3t}\right)-9\left(f(t)e^{-3t}\right)$

$\left(9f(t)-6f'(t)+f''(t)\right)e^{-3t}=\left(9f(t)-6f'(t)\right)e^{-3t}$

Hence, $f''(t)=0$

$\therefore f(t)$ is linear, i.e. $f(t)=At+B$ where A and B are constants

(c) (ii) (1 mark)

$x(t)=\left(At+B\right)e^{-3t}$

When $t=0, x=0$ (as the projectile is starting from the origin)

$\left(A\times 0+B\right)e^{-3\times 0}=0$

$B=0$

(c) (iii) (2 marks)

$x(t)=Ate^{-3t}$

$x'(t)=Ae^{-3t}-3Ate^{-3t}$

$\quad=Ae^{-3t}\left(1-3t\right)$

Displacement is a maximum when $x'(t)=0$, i.e. $t=\frac{1}{3}$

Therefore, the particle's displacement is at maximum after $\frac{1}{3}$ seconds.

Note: since $x''\left(\frac{1}{3}\right)=-3Ae^{-1}<0$ since $A>0$, a maximum is attained.

Question 16 (15 marks)

(a) (3 marks)

$w=\frac{3z+2}{z-1}$

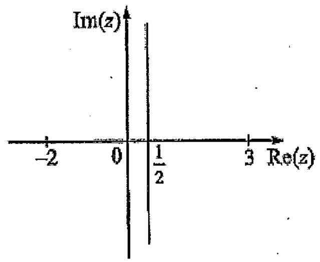
$wz-w=3z+2$

$z=\frac{w+2}{w-3}$

Since $|z|=1$,

$\left|\frac{w+2}{w-3}\right|=1 \Rightarrow |w+2|=|w-3|$

Thus w is equidistant from -2 and 3.



(b) (i) (1 mark)

$$\frac{1}{2^k+1}+\frac{1}{2^k+2}+\frac{1}{2^k+3}+\ldots+\frac{1}{2^k+2^k} \geq \frac{1}{2^k+2^k}+\frac{1}{2^k+2^k}+\frac{1}{2^k+2^k}+\ldots+\frac{1}{2^k+2^k}$$
$$= 2^k \times \frac{1}{2(2^k)}$$
$$= \frac{1}{2}$$

(b) (ii) (3 mark)

Let $P(n)$ be the given proposition. $P(1)$ is true since $1+\frac{1}{2}=\frac{3}{2}\geq\frac{1}{2}(1+1)=1$.

Assume $P(k)$ is true for some positive integer k.

i.e. $1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^k}\geq\frac{1}{2}(k+1)$

Prove $P(k+1)$ is true:

$$1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^{k+1}}=\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{2^k}\right)+\left(\frac{1}{2^k+1}+\frac{1}{2^k+2}+\frac{1}{2^k+3}+\ldots+\frac{1}{2^k+2^k}\right)$$
$$\geq\frac{1}{2}(k+1)+\frac{1}{2} \text{ using the assumption and the result in part i}$$
$$=\frac{1}{2}((k+1)+1)$$

$\therefore$ By the Principle of Mathematical Induction, $P(n)$ is true for integers $n\geq 1$.

(c) (i) (2 marks)

By DeMoivre's theorem,

$$x^k=(\cos\theta+i\sin\theta)^k=\cos k\theta+i\sin k\theta$$
$$x^{-k}=(\cos\theta+i\sin\theta)^{-k}=\cos(-k\theta)+i\sin(-k\theta)=\cos k\theta-i\sin k\theta$$

$$x^k+x^{-k}=\cos k\theta+i\sin k\theta+\cos k\theta-i\sin k\theta$$
$$=2\cos k\theta$$

(c) (ii) (3 marks)

$$\left(x+\frac{1}{x}\right)^{2n}=\binom{2n}{0}x^{2n}+\binom{2n}{1}x^{2n-2}+\binom{2n}{2}x^{2n-4}+\ldots+\binom{2n}{n}+\ldots+\binom{2n}{2n-2}\frac{1}{x^{2n-4}}+\binom{2n}{2n-1}\frac{1}{x^{2n-2}}+\binom{2n}{2n}\frac{1}{x^{2n}}$$
$$=\binom{2n}{0}x^{2n}+\binom{2n}{1}x^{2n-2}+\binom{2n}{2}x^{2n-4}+\ldots+\binom{2n}{n}+\ldots+\binom{2n}{2}\frac{1}{x^{2n-4}}+\binom{2n}{1}\frac{1}{x^{2n-2}}+\binom{2n}{0}\frac{1}{x^{2n}}$$

since $\binom{2n}{2n-k}=\binom{2n}{k}$

$$\left(x+\frac{1}{x}\right)^{2n}=\binom{2n}{0}\left(x^{2n}+\frac{1}{x^{2n}}\right)+\binom{2n}{1}\left(x^{2n-2}+\frac{1}{x^{2n-2}}\right)+\binom{2n}{2}\left(x^{2n-4}+\frac{1}{x^{2n-4}}\right)+\ldots+\binom{2n}{n-1}\left(x^2+\frac{1}{x^2}\right)+\binom{2n}{n}$$

Note $\binom{2n}{0}=1$

(c) (iii) (1 mark)

Using the result from part i, $x^k+\frac{1}{x^k}=2\cos k\theta$ in the identity from part ii:

$$(2\cos\theta)^{2n}=\binom{2n}{0}(2\cos 2n\theta)+\binom{2n}{1}(2\cos(2n-2)\theta)+\binom{2n}{2}(2\cos(2n-4)\theta)+\ldots+\binom{2n}{n-1}(2\cos 2\theta)+\binom{2n}{n}$$

Dividing both sides by 2:

$$2^{2n-1}\cos^{2n}\theta=\cos 2n\theta+\binom{2n}{1}\cos(2n-2)\theta+\binom{2n}{2}\cos(2n-4)\theta+\ldots+\binom{2n}{n-1}\cos 2\theta+\frac{1}{2}\binom{2n}{n}$$

(c) (iv) (2 marks)

$$\int_0^{2\pi}2^{2n-1}\cos^{2n}\theta\,d\theta=\int_0^{2\pi}\cos 2n\theta+\binom{2n}{1}\cos(2n-2)\theta+\binom{2n}{2}\cos(2n-4)\theta+\ldots+\binom{2n}{n-1}\cos 2\theta+\frac{1}{2}\binom{2n}{n}\,d\theta$$

Since $\int_0^{2\pi}\cos k\theta\,d\theta=0$ for all even integers k, all the integrals on the right hand side are zero except for the constant term.

$$\int_0^{2\pi}2^{2n-1}\cos^{2n}\theta\,d\theta=\int_0^{2\pi}\frac{1}{2}\binom{2n}{n}d\theta$$
$$=\left[\frac{1}{2}\binom{2n}{n}\theta\right]_0^{2\pi}$$
$$=\pi\binom{2n}{n}$$

Dividing both sides by 2^{2n-1},

$$\int_0^{2\pi}\cos^{2n}\theta\,d\theta=\frac{\pi}{2^{2n-1}}\binom{2n}{n}.$$