SOUTH SYDNEY HIGH SCHOOL

Number: .....

2013

Year 12

TRIAL HSC

Tuesday 30th July 2013

(ALSO NSW INDEPENDENT SCHOOLS PAPER)

TRIAL HSC

Mathematics Ext 2

Weighting 40%

Working time: 3 hours

Total marks: 100

Outcomes:

E1 – E8

Topics examined:

All topics

Question	Mark
1 -10	/10
11	/15
12	/15
13	/15
14	/15
15	/15
16	/15
TOTAL	

General Instructions:

- Write using blue or black pen
- Board-approved calculators and templates may be used
- All necessary working should be shown in every question
- Questions are of equal value
- Full marks may not be awarded for careless or badly arranged work
- Questions are not necessarily arranged in order of difficulty
- Allow 15 minutes for section 1
- Allow 2 hours 45 minutes for section 2

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n<0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a>0, \ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE: $\ln x = \log_e x, \quad x>0$

Student name / number ....

Marks

Section 1

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10

1 What is the value of $(\log_a b)(\log_b c)(\log_c d)$? **1**

(A) $\dfrac{1}{\log_d a}$

(B) $\log_e \dfrac{d}{a}$

(C) $\log_e \dfrac{a}{d}$

(D) $\log_d a$

2 $y = \sin^{-1} e^x$. Which of the following is an expression for $\dfrac{dy}{dx}$? **1**

(A) $\operatorname{cosec} y$

(B) $\cot y$

(C) $\sec y$

(D) $\tan y$

3 What is the number of asymptotes on the graph of $y = \dfrac{x^3}{x^2-1}$? **1**

(A) 1

(B) 2

(C) 3

(D) 4

4 Which of the following graphs is the locus of the point P representing the complex number z moving in an Argand diagram such that $|z-2i| = 2 + \operatorname{Im} z$? **1**

(A) a circle

(B) a hyperbola

(C) a parabola

(D) a straight line

Student name / number ....

Marks

5 Which of the following is an expression for the eccentricity of the ellipse $\dfrac{x^2}{k} + \dfrac{y^2}{k-1} = 1$ where $k > 1$? **1**

(A) $\dfrac{\sqrt{2k-1}}{k}$

(B) $\dfrac{1}{\sqrt{k}}$

(C) $\sqrt{\dfrac{2k-1}{k}}$

(D) $\dfrac{\sqrt{2k^2-2k+1}}{k}$

6 Which of the following is an expression for $\int x^3 \log_e x \, dx$? **1**

(A) $\frac{1}{4}x^4 \log_e x - \frac{1}{4}x^4 + c$

(B) $\frac{1}{4}x^4 \log_e x - \frac{1}{16}x^4 + c$

(C) $\frac{1}{4}x^4 \log_e x + \frac{1}{16}x^4 + c$

(D) $\frac{1}{4}x^4 \log_e x + \frac{1}{4}x^4 + c$

7 The base of a solid is the circle $x^2 + y^2 = 1$. Every cross section of the solid taken perpendicular to the x axis is a right-angled, isosceles triangle with its hypotenuse lying in the base of the solid. Which of the following is an expression for the volume V of the solid? **1**

(A) $\int_{-1}^{1} (1-x^2)\, dx$

(B) $2\int_{-1}^{1} (1-x^2)\, dx$

(C) $4\int_{-1}^{1} (1-x^2)\, dx$

(D) $8\int_{-1}^{1} (1-x^2)\, dx$

Student name / number ....

Marks

8 What is the multiplicity of the root $x=1$ of the equation $3x^5-5x^4+5x-3=0$? 1

(A) 1

(B) 2

(C) 3

(D) 4

9 A particle of mass m is moving horizontally in a straight line. Its motion is opposed by a force of magnitude $mk(v+v^2)$ Newtons when its speed is v ms^{-1} (where k is a positive constant). At time t seconds the particle has displacement x metres from a fixed point O on the line and velocity v ms^{-1}. Which of the following is an expression for x in terms of v ? 1

(A) $\frac{1}{k}\int\frac{1}{1+v}\,dv$

(B) $\frac{1}{k}\int\frac{1}{v(1+v)}\,dv$

(C) $-\frac{1}{k}\int\frac{1}{v(1+v)}\,dv$

(D) $-\frac{1}{k}\int\frac{1}{1+v}\,dv$

10 If n fair dice are rolled together, what is the probability that the product of the n scores is an even number? 1

(A) $\frac{1}{6^n}$

(B) $\frac{1}{2^n}$

(C) $\frac{2^n-1}{2^n}$

(D) $\frac{6^n-1}{6^n}$

Section II

Student name / number ....

Marks

90 marks
Attempt Question 11 – 16
Allow about 2 hours 45 minutes for this section

Answer the questions on your own paper, or writing booklets if provided.
Start each question on a new page.
All necessary working should be shown in every question.

Question 11 (15 marks) Use a SEPARATE writing booklet.

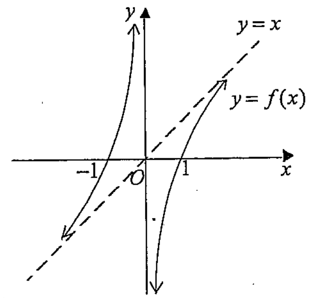
(a) Find $\frac{dy}{dx}$ in simplest form if $x=t+\frac{1}{t}$ and $y=\frac{1}{3}t^3-t$. 2

(b) Find $\int\frac{\cos x}{1+\cos x}\,dx$. 2

(c) Use the substitution $u=e^x+1$ to find $\int\frac{e^{2x}}{\sqrt{e^x+1}}\,dx$. 2

(d) Use the substitution $t=\tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}}\frac{1}{4+5\sin x}\,dx$ in simplest exact form. 4

(e) The diagram shows the graph of the curve $y=f(x)$ where $f(x)=x-\frac{1}{x}$.



On separate diagrams sketch the graphs of the following curves showing any intercepts on the axes and the equations of any asymptotes.

(i) $y=\left|f(x)\right|$ 1

(ii) $y=\frac{1}{f(x)}$ 2

(iii) $y=e^{f(x)}$ 2

Marks

Question 12 (15 marks) Use a SEPARATE writing booklet.

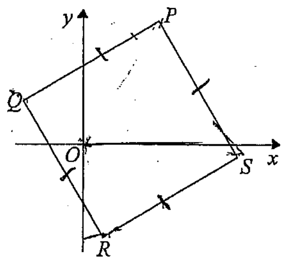
(a) If $z=1+2i$ and $w=3-i$, express in the form $a+ib$ (a and b real numbers)

(i) $2z-w$ **1**

(ii) $z\overline{w}$ **1**

(b) If $z=4\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$, express in modulus/argument form the two values of $z^{\frac{1}{2}}$. **2**

(c) In the Argand diagram below vectors $\overrightarrow{OP}$, $\overrightarrow{OQ}$, $\overrightarrow{OR}$, $\overrightarrow{OS}$ represent the complex numbers p, q, r, s respectively where $PQRS$ is a square. Show that $s+ip=q+ir$. **2**



(d) Consider the function $f(x)=x^3-px+q$, where $p>0$ and q are real numbers.

(i) Sketch the graph of $y=f(x)$ showing the coordinates of the turning points. **2**

(ii) Show that the equation $x^3-px+q=0$ has exactly one real root if and only if $27q^2>4p^3$. **2**

(e) The region bounded by the curve $y=\tan^{-1}x$ and the x axis between $x=0$ and $x=1$ is rotated through one complete revolution about the line $x=1$.

(i) Use the method of cylindrical shells to show that the volume V of the solid formed is given by $V=2\pi\int_0^1(1-x)\tan^{-1}x\,dx$. **2**

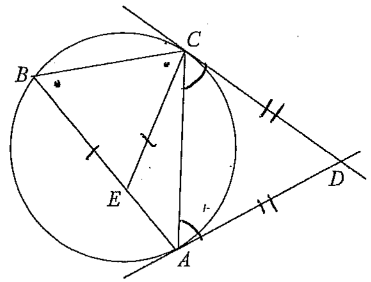
(ii) Hence find the value of V in simplest exact form. **3**

Marks

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) $P\left(2\sqrt{2}, 2\sqrt{3}\right)$ and $Q(4,6)$ are two points on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ where $a>b>0$. Find the values of a and b. **3**

(b) In the diagram below, ABC is a triangle inscribed in a circle. Tangents to the circle at A and C meet at D. E is the point on AB such that $CE=BE$.



(i) Show $DAEC$ is a cyclic quadrilateral. **3**

(ii) Hence show $DE \parallel CB$. **2**

(c)(i) Show that the area of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is πab. **2**

(ii) Show that if the hyperbola $xy=c^2$ and the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (where $a>b>0$) intersect at $(a\cos\theta, b\sin\theta)$ then $2c^2=ab\sin 2\theta$. **3**

By considering the graph of $y=\sin 2\theta$ for $-\pi<\theta\le\pi$, show that if $2c^2<ab$ the ellipse and hyperbola intersect at four points (two in the first quadrant and two in the third quadrant), while if $2c^2=ab$ the curves touch at two points (one in the first quadrant and one in the third quadrant).

(iii) The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (where $a>b>0$) has eccentricity e and foci S and S'. **2**

This ellipse touches the hyperbola $xy=\frac{1}{2}ab$ at points P and Q. Show that the ratio of the area of the quadrilateral $PSQS'$ to the area of the ellipse is $e\sqrt{2}:\pi$.

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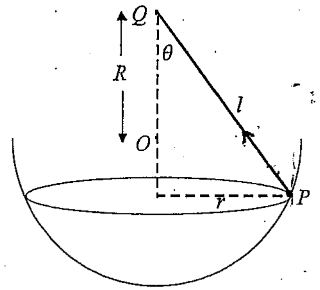
Marks

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a)



In the diagram, particle P of mass m kg is moving in a horizontal circle of radius r metres with constant angular velocity ω radians/second on the inside of a smooth hemispherical bowl with centre O and radius R metres. The particle is fastened to one end of a light, inextensible string of length l metres, the other end of which is suspended from a point Q at a distance R metres vertically above O. The string makes an angle θ with the downward vertical through Q. The tension in the string is T Newtons while the normal reaction between P and the surface of the bowl is N Newtons. The acceleration due to gravity is g ms^{-2}.

(i) Show that $N\cos 2\theta + T\cos\theta = mg$ and $N\sin 2\theta + T\sin\theta = mr\omega^2$. **2**

(ii) Hence show that $N = ml\omega^2\cos\theta - mg$ and find a similar expression for T. **3**

(iii) Hence show that if the particle remains in contact with the bowl then **1**

$\omega \geq \sqrt{\dfrac{g}{l\cos\theta}}$.

(b) A particle of mass m kg is dropped from rest in a medium where the resistance is mkv^2 Newtons when the speed of the particle is v ms^{-1} and the terminal velocity is V ms^{-1}. After t seconds the particle has fallen x metres. The acceleration due to gravity is g ms^{-2}.

(i) Explain why $\ddot{x} = \dfrac{g}{V^2}\left(V^2 - v^2\right)$. **2**

(ii) By finding x and t as functions of v, show that $Vt - x = \dfrac{V^2}{g}\log_e\left(1 + \dfrac{v}{V}\right)$. **4**

(iii) Express v as a function of t. **2**

(iv) Find the limiting difference as $t \to \infty$ between the distance travelled at constant speed V ms^{-1} for t seconds and the distance fallen by the particle in this medium over t seconds. **1**

Student name / number ....

Marks

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) The equation $x^3 + kx + 1 = 0$ has roots α, β and γ.

(i) Find the monic cubic equation with roots $\dfrac{1}{\alpha^2}$, $\dfrac{1}{\beta^2}$ and $\dfrac{1}{\gamma^2}$. **3**

(ii) Find the value of $\alpha^4 + \beta^4 + \gamma^4$. **2**

(b) A sequence of numbers T_n, $n = 1, 2, 3\ldots$ is such that $T_1 = 2$, $T_2 = -4$ and $T_n = 2T_{n-1} - 4T_{n-2}$ for $n = 3, 4, 5, 6\ldots$. Use Mathematical induction to show that $T_n = \left(1+\sqrt{3}\,i\right)^n + \left(1-\sqrt{3}\,i\right)^n$ for $n \geq 1$. **5**

(c) Consider the function $f(x) = \sum_{k=1}^{n}\left(\sqrt{a_k}\,x - \dfrac{1}{\sqrt{a_k}}\right)^2$ where $a_1, a_2, \ldots, a_n$ are positive real numbers.

(i) By expressing $f(x)$ as a quadratic function of x, show that **3**

$\left(a_1 + a_2 + \ldots + a_n\right)\left(\dfrac{1}{a_1} + \dfrac{1}{a_2} + \ldots + \dfrac{1}{a_n}\right) \geq n^2$.

(ii) Hence show that $1 + \dfrac{1}{2} + \dfrac{1}{3} + \ldots + \dfrac{1}{n} \geq \dfrac{2n}{n+1}$. **2**

Student name / number ....

Marks

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) The equation $z^5 - 1 = 0$ has roots $1,\ \alpha,\ \alpha^2,\ \alpha^3,\ \alpha^4$ where $\alpha = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$. **3**
Find the monic quadratic equation with integer coefficients whose roots are $(\alpha + \alpha^4)$ and $(\alpha^2 + \alpha^3)$.

(b)(i) If a and b are positive real numbers, show that $a^2 - ab + b^2 \geq \left(\frac{a+b}{2}\right)^2$. **2**

(ii) In ΔABC, if $\angle BCA \geq 60°$ show that $c^2 \geq a^2 - ab + b^2$ and hence deduce that $2c^3 \geq a^3 + b^3$ with equality if and only if ΔABC is equilateral. **3**

(c) Let $I_n = \int_1^2 \left(1 - \frac{1}{x}\right)^n dx$ for $n = 1, 2, 3, \ldots$.

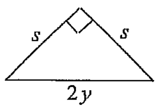
(i) Show that $\frac{1}{n+1} I_{n+1} = \frac{1}{n} I_n - \frac{1}{n(n+1)2^n}$ for $n = 1, 2, 3, \ldots$. **2**

(ii) Hence show that $\frac{1}{n+1} I_{n+1} = I_1 - \sum_{r=1}^{n} \frac{1}{r(r+1)2^r}$. **2**

(iii) Show that $\sum_{r=1}^{n} \frac{1}{r(r+1)2^r} = (1 - \log_e 2) - \frac{1}{n+1} I_{n+1}$ and hence find the limiting sum **3**
of the series $\frac{1}{1\times2\times2^1} + \frac{1}{2\times3\times2^2} + \frac{1}{3\times4\times2^3} + \ldots$

Section I

Questions 1-10 (1 mark each)

Question	Answer	Solution	Outcomes
1.	A	$(\log_a b)(\log_b c)(\log_c d)=\frac{\log_d b}{\log_d a}\cdot\frac{\log_d c}{\log_d b}\cdot\frac{\log_d d}{\log_d c}=\frac{1}{\log_d a}$	H3
2.	D	$y=\sin^{-1}e^x$ $\quad\therefore \sin y=e^x$ $\quad \cos y\frac{dy}{dx}=e^x$ $\quad\therefore \frac{dy}{dx}=\frac{\sin y}{\cos y}=\tan y$	HE4
3.	C	$y=\frac{x^3}{x^2-1}=\frac{x(x^2-1)+x}{x^2-1}=x+\frac{x}{x^2-1}$, where $\frac{x}{x^2-1}\to 0$ as $x\to\infty$ $\therefore y=x,\ x=1$ and $x=-1$ are the asymptotes on the graph.	E6
4.	C	The distance from P to the fixed point $(0,2)$ (the focus) is equal to the vertical distance to the horizontal line $y=-2$ (the directrix).	E3
5.	B	$k-1=k(1-e^2)$ $\quad\therefore 1-\frac{1}{k}=1-e^2$ $\quad\therefore e=\frac{1}{\sqrt{k}}$	E4
6.	B	$\int x^3\log_e x\,dx=\frac{1}{4}x^4\log_e x-\frac{1}{4}\int x^3\,dx=\frac{1}{4}x^4\log_e x-\frac{1}{16}x^4+c$	E8
7.	A	 $A=\frac{1}{2}s^2=\frac{1}{4}(s^2+s^2)=\frac{1}{4}(2y)^2$ $\therefore A=y^2=1-x^2$ $\therefore V=\int_{-1}^{1}(1-x^2)\,dx$	E7
8.	C	$P(x)=3x^5-5x^4+5x-3 \Rightarrow P(1)=0$ $P'(x)=15x^4-20x^3+5 \Rightarrow P'(1)=0$ $P''(x)=60x^3-60x^2 \Rightarrow P''(1)=0$ $P'''(x)=180x^2-120x \Rightarrow P'''(1)\neq 0$	E4
9.	D	$v\frac{dv}{dx}=-k(v+v^2)$ $\quad\therefore \frac{dx}{dv}=-\frac{1}{k(1+v)}$ $\quad\therefore x=-\frac{1}{k}\int\frac{1}{1+v}\,dv$	E5
10.	C	$P(\text{product even})=P(\text{at least one even})=1-P(\text{all odd})$ $\therefore P(\text{product even})=1-\left(\frac{1}{2}\right)^n=\frac{2^n-1}{2^n}$	HE3

Question 11

a. Outcomes assessed : E6

Marking Guidelines

Criteria	Marks
• finds $\frac{dy}{dt}$ and $\frac{dx}{dt}$	1
• finds the quotient	1

Answer

$$y=\tfrac{1}{3}t^3-t \Rightarrow \frac{dy}{dt}=t^2-1$$

$$x=t+\frac{1}{t} \Rightarrow \frac{dx}{dt}=1-\frac{1}{t^2}=\frac{t^2-1}{t^2}$$

$$\therefore \frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=t^2$$

b. Outcomes assessed : E8

Marking Guidelines

Criteria	Marks
• rearranges integrand into appropriate form	1
• writes primitive function	1

Answer

$$\frac{\cos x}{1+\cos x}=1-\frac{1}{1+\cos x}=1-\frac{1}{2\cos^2(\frac{1}{2}x)}=1-\tfrac{1}{2}\sec^2(\tfrac{1}{2}x)$$

$$\therefore \int\frac{\cos x}{1+\cos x}\,dx=x-\tan(\tfrac{1}{2}x)+c$$

c. Outcomes assessed : E8

Marking Guidelines

Criteria	Marks
• performs substitution and finds primitive as a function of u	1
• finds primitive as a function of x	1

Answer

$u=e^x+1$

$du=e^x\,dx$

$$\int\frac{e^{2x}}{\sqrt{e^x+1}}\,dx=\int\frac{e^x}{\sqrt{e^x+1}}\,e^x dx=\int\frac{u-1}{\sqrt{u}}\,du=\tfrac{2}{3}u^{\frac{3}{2}}-2u^{\frac{1}{2}}+c$$

$$\therefore \int\frac{e^{2x}}{\sqrt{e^x+1}}\,dx=\tfrac{2}{3}\sqrt{e^x+1}\left(e^x-2\right)+c$$

Q11(cont)

d. Outcomes assessed : E8

Marking Guidelines

Criteria	Marks
• converts into a definite integral in terms of t	1
• rearranges integrand into partial fractions	1
• finds the primitive function	1
• evaluates in simplest exact form	1

Answer

$$t = \tan\frac{x}{2}$$
$$dt = \tfrac{1}{2}\sec^2\tfrac{x}{2}\,dx$$
$$dx = \frac{2}{1+t^2}\,dt$$

$$x = 0 \Rightarrow t = 0$$
$$x = \tfrac{\pi}{2} \Rightarrow t = 1$$

$$4 + 5\sin x = \frac{4(1+t^2)+10t}{1+t^2}$$

$$\int_0^{\frac{\pi}{2}} \frac{1}{4+5\sin x}\,dx = \int_0^1 \frac{1}{2t^2+5t+2}\,dt$$
$$= \int_0^1 \frac{1}{(2t+1)(t+2)}\,dt$$
$$= \tfrac{1}{3}\int_0^1 \left(\frac{2}{2t+1} - \frac{1}{t+2}\right)dt$$
$$= \tfrac{1}{3}\left[\log_e \frac{2t+1}{t+2}\right]_0^1$$
$$= \tfrac{1}{3}\left(\log_e 1 - \log_e \tfrac{1}{2}\right)$$
$$= \tfrac{1}{3}\log_e 2$$

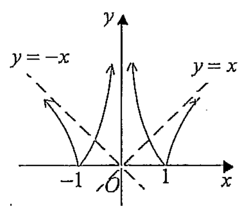
e. Outcomes assessed : E6

Marking Guidelines

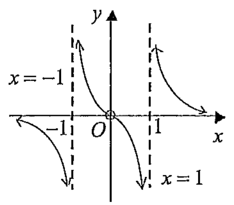
Criteria	Marks
i • reflects sections of graph below x axis in x axis, giving equations of oblique asymptotes	1
ii • sketches curve for $\lvert x\rvert > 1$ giving equations of vertical asymptotes	1
• sketches curve for $\lvert x\rvert < 1$ with origin excluded	1
iii • sketches curve for $x < 0$ through $(-1,1)$	1
• sketches curve for $x > 0$ through $(1, 1)$ with origin excluded	1

Answer

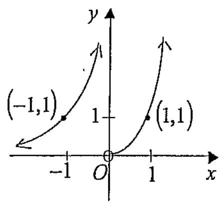
i. $y = \lvert f(x)\rvert$



ii. $y = \dfrac{1}{f(x)}$



iii. $y = e^{f(x)}$



Question 12

a. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
i • subtracts and simplifies	1
ii • expands product and simplifies	1

Answer

i. $2z - w = (2+4i) - (3-i) = -1+5i$

ii. $z\overline{w} = (1+2i)(3+i) = 3+7i+2i^2 = 1+7i$

b. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
• writes one value in modulus/argument form	1
• writes second value in modulus/argument form	1

Answer

The two square roots of $z = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ have modulus 2 and arguments $\frac{\pi}{6}$ and $\frac{\pi}{6} - \pi$.

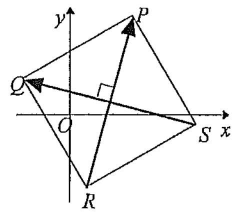
The two values of $z^{\frac{1}{2}}$ are $2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$ and $2\left(\cos\frac{-5\pi}{6} + i\sin\frac{-5\pi}{6}\right)$

c. Outcomes assessed : E3

Marking Guidelines

Criteria	Marks
• uses the geometrical properties of a square to compare vectors along the diagonals	1
• translates this into an appropriate statement about p, q, r, s and rearranges	1

Answer



The diagonals of a square are equal and meet at right angles.

Hence the vector $\overrightarrow{SQ}$, representing $(q-s)$ is the rotation anticlockwise by $\frac{\pi}{2}$ of the vector $\overrightarrow{RP}$, representing $(p-r)$.

$\therefore q - s = i(p-r)$.

$s + ip = q + ir$.

Q12 (cont)
d. Outcomes assessed : E4

Marking Guidelines

Criteria	Marks
i • finds x coordinates for stationary points and determines their nature	1
• sketches curve showing coordinates of turning points	1
ii • deduces y coordinates of turning points have same sign for exactly one real root	1
• uses product of y coordinates of turning points is positive to deduce $27q^2 > 4p^3$	1

Answer

i.

$f(x) = x^3 - px + q$

$f'(x) = 3x^2 - p \Rightarrow f'(x) = 0$ for $x = \pm\sqrt{\frac{p}{3}}$

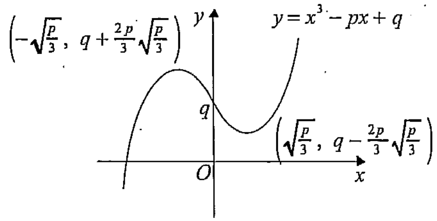
$f''(x) = 6x \quad \therefore f''(x) > 0$ for $x > 0$

$f''(x) < 0$ for $x < 0$

$\left(\sqrt{\frac{p}{3}},\ q - \frac{2p}{3}\sqrt{\frac{p}{3}}\right)$ is a min. turning point

$\left(-\sqrt{\frac{p}{3}},\ q + \frac{2p}{3}\sqrt{\frac{p}{3}}\right)$ is a max. turning point

$(0, q)$ is a point of inflexion.



ii. The equation has exactly one real root if and only if the curve cuts through the x axis exactly once, and this occurs when the y coordinates of the turning points are either both positive or both negative and therefore have a product which is positive.

Hence the equation has exactly one real root if and only if

$$\left(q - \tfrac{2p}{3}\sqrt{\tfrac{p}{3}}\right)\left(q + \tfrac{2p}{3}\sqrt{\tfrac{p}{3}}\right) > 0$$

$$q^2 - \left(\tfrac{2p}{3}\sqrt{\tfrac{p}{3}}\right)^2 > 0$$

$$q^2 > \tfrac{4p^3}{27}$$

$$27q^2 > 4p^3$$

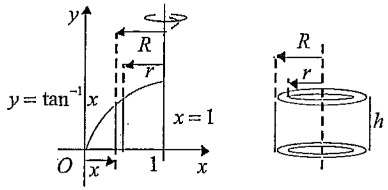
Q12 (cont)
e. Outcomes assessed : E7, E8

Marking Guidelines

Criteria	Marks
i • finds the volume of a typical cylindrical shell in terms of x	1
• takes the limiting sum of cylindrical shells to deduce the required expression for V	1
ii • applies integration by parts to simplify the definite integral	1
• finds the primitive function	1
• evaluates in simplest exact form	1

Answer

i.



$R = 1 - x$

$r = 1 - x - \delta x$

$h = \tan^{-1} x$

$\delta V = \pi(R^2 - r^2)h$

$= \pi(R + r)(R - r)h$

$= \pi\{2(1-x) - \delta x\}\delta x .\tan^{-1} x$

Ignoring terms in $(\delta x)^2$,

$$V = \lim_{\delta x \to 0} \sum_{x=0}^{x=1} 2\pi(1-x)\tan^{-1} x\, \delta x$$

$$= 2\pi\int_0^1 (1-x)\tan^{-1} x\, dx$$

ii.

$$V = 2\pi\int_0^1 (1-x)\tan^{-1} x\, dx$$

$$= 2\pi\left\{\left[-\tfrac{1}{2}(1-x)^2\tan^{-1} x\right]_0^1 + \tfrac{1}{2}\int_0^1 (1-x)^2 \frac{1}{1+x^2}\, dx\right\}$$

$$= \pi\left\{0 + \int_0^1 \frac{1+x^2-2x}{1+x^2}\, dx\right\}$$

$$= \pi\int_0^1 \left(1 - \frac{2x}{1+x^2}\right) dx$$

$$= \pi\left[x - \log_e(1+x^2)\right]_0^1$$

$$= \pi(1 - \log_e 2)$$

Question 13

a. Outcomes assessed : E4

Marking Guidelines

Criteria	Marks
• writes a pair of simultaneous equations in a^2 and b^2	1
• finds the value of one pronumeral	1
• finds the value of the second pronumeral	1

Answer

$$\frac{8}{a^2}-\frac{12}{b^2}=1 \quad (1) \qquad 2\times(1)-(2) \Rightarrow \frac{12}{b^2}=1 \quad \therefore b=2\sqrt{3}$$

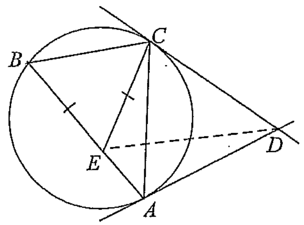
$$\frac{16}{a^2}-\frac{36}{b^2}=1 \quad (2) \qquad 3\times(1)-(2) \Rightarrow \frac{8}{a^2}=2 \quad \therefore a=2$$

b. Outcomes assessed : PE3

Marking Guidelines

Criteria	Marks
i • deduces $\angle DCA=\angle CBA=\angle CAD$ using alternate segment theorem	1
• finds either $\angle CEA$ or $\angle CEB$ in terms of $\angle CBA$	1
• applies an appropriate test for a cyclic quadrilateral to complete the proof	1
ii • deduces $\angle CED=\angle CAD$	1
• applies appropriate test for parallel lines to complete proof	1

Answer



i. $\angle DCA=\angle CBA=\angle CAD$ (*$\angle$ between a tangent and a chord is equal to $\angle$ subtended by the chord in the alternate segment*)

$\therefore \angle CDA+2\angle CBA=180°$ (*$\angle$ sum of ΔCDA is 180°*)

But $\angle ECB=\angle CBE$ (*$\angle$'s opp. equal sides in ΔEBC are equal*)

$\therefore \angle CEA=2\angle CBA$ (*ext. $\angle$ is sum of int. opp. $\angle$'s in ΔEBC*)

$\therefore \angle CDA+\angle CEA=180°$

$\therefore$ quadrilateral $DAEC$ is cyclic (*one pair of opp. $\angle$'s supplementary*)

ii. $\angle CED=\angle CAD$ (*$\angle$'s in the same segment subtended by the same arc CD in circle DAEC are equal*)

But $\angle CAD=\angle CBA=\angle ECB$ (*proven in* i.)

$\therefore \angle CED=\angle ECB$

$\therefore DE \parallel CB$ (*equal alt. $\angle$'s on transversal EC*)

Q13 (cont)

c. Outcomes assessed : H8, E3

Marking Guidelines

Criteria	Marks
i • expresses the area as a definite integral	1
• evaluates the integral	1
ii • shows $2c^2=ab\sin 2\theta$ if curves intersect at $(a\cos\theta, b\sin\theta)$	1
• deduces result for $2c^2<ab$	1
• deduces result for $2c^2=ab$	1
iii • finds the coordinates of P, Q	1
• finds the area of $PSQS'$ and compares with area of ellipse	1

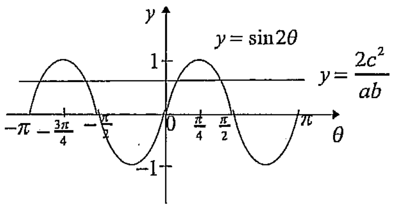
Answer

i. $y=\pm\frac{b}{a}\sqrt{a^2-x^2}$. Hence the area of the ellipse is $2\frac{b}{a}\int_{-a}^{a}\sqrt{a^2-x^2}\,dx=2\frac{b}{a}\left(\frac{1}{2}\pi a^2\right)=\pi ab$

(Using the fact that the area of a semi-circle of radius a is given by the definite integral)

ii. $(a\cos\theta, b\sin\theta)$ lies on the ellipse. If it also lies on the hyperbola, then $(a\cos\theta)(b\sin\theta)=c^2$.

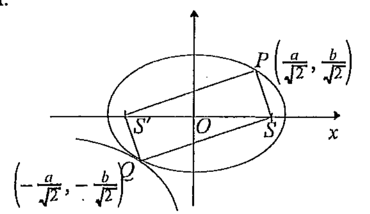
Hence $2c^2=ab(2\sin\theta\cos\theta)=ab\sin 2\theta$



If $2c^2<ab$ the line cuts the sine curve in two points where $0<\theta<\frac{\pi}{2}$, giving two points of intersection $(a\cos\theta, b\sin\theta)$ of the ellipse and hyperbola in the first quadrant, and the line cuts the sine curve in two further points where $-\pi<\theta<-\frac{\pi}{2}$, giving a further two points of intersection $(a\cos\theta, b\sin\theta)$ of the ellipse and hyperbola in the third quadrant.

If $2c^2=ab$ the line touches the sine curve at $\theta=\frac{\pi}{4}$ and $\theta=-\frac{3\pi}{4}$, giving two points where the ellipse touches the hyperbola, $\left(\frac{a}{\sqrt{2}},\frac{b}{\sqrt{2}}\right)$ in the first quadrant and $\left(-\frac{a}{\sqrt{2}},-\frac{b}{\sqrt{2}}\right)$ in the third quadrant.

iii.



By symmetry,

$Area\ PSQS'=2\times Area\ \Delta PSS'$

$=2\times\frac{1}{2}\times 2ae\times\frac{b}{\sqrt{2}}$

$=e\sqrt{2}\,ab$

$\therefore Area\ PSQS' : Area\ ellipse = e\sqrt{2}:\pi$

Question 14

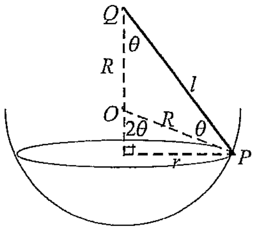
a. Outcomes assessed : H5, E5

Marking Guidelines

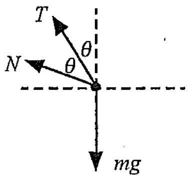
Criteria	Marks
i • shows forces on P in a diagram and justifies angle made by N with the vertical through P	1
• applies Newton's second law, resolving forces vertically and horizontally	1
ii • eliminates T to find an expression for N	1
• uses trigonometry to simplify expression for N	1
• applies similar process to find an expression for T	1
iii • deduces $N \geq 0$ to find inequality for ω	1

Answer

i.



Forces on P



ΔOPQ is isosceles with equal angles θ at P and Q, so that the radius PO makes angle 2θ with the vertical (by the exterior angle theorem). The normal to the surface at P is directed along the radius PO.

The resultant force on the particle is directed horizontally toward the centre of the circle of motion with magnitude $mr\omega^2$. Hence by Newton's second law, resolving forces on P vertically and horizontally,

$N\cos 2\theta + T\cos\theta = mg$ (1) (vertical component of resultant force is 0)

$N\sin 2\theta + T\sin\theta = mr\omega^2$ (2)

ii. $(2)\times\cos\theta - (1)\times\sin\theta \Rightarrow N\left(\sin 2\theta\cos\theta - \cos 2\theta\sin\theta\right) = mr\omega^2\cos\theta - mg\sin\theta$

$$\therefore N\sin\theta = m(l\sin\theta)\omega^2\cos\theta - mg\sin\theta$$

$$N = ml\omega^2\cos\theta - mg$$

$(1)\times\sin 2\theta - (2)\times\cos 2\theta \Rightarrow T\left(\sin 2\theta\cos\theta - \cos 2\theta\sin\theta\right) = mg\sin 2\theta - mr\omega^2\cos 2\theta$

$$\therefore T\sin\theta = mg\sin 2\theta - m(l\sin\theta)\omega^2\cos 2\theta$$

$$T = 2mg\cos\theta - ml\omega^2\cos 2\theta$$

iii. If particle remains in contact with the bowl, $N \geq 0$ and hence $l\omega^2\cos\theta \geq g$. $\therefore \omega \geq \sqrt{\dfrac{g}{l\cos\theta}}$.

Q14 (cont)

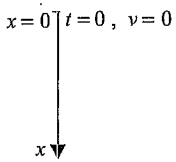
b. Outcomes assessed : H3, E5

Marking Guidelines

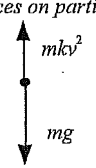
Criteria	Marks
i • shows forces on particle and uses Newton's 2nd law to find equation of motion	1
• investigates the terminal velocity to substitute for k, obtaining required form of $\ddot{x}$	1
ii • uses appropriate expression for $\ddot{x}$ to find primitive function for x in terms of v	1
• uses appropriate expression for $\ddot{x}$ to find primitive function for t in terms of v	1
• evaluates constants of integration in both cases using initial conditions	1
• uses log laws to establish required relation	1
iii • removes logarithm from expression for t in terms of v by applying exponential function	1
• rearranges to find v as a function of t	1
iv • takes an appropriate limit as $t \to \infty$	1

Answer

i.



Forces on particle



Applying Newton's second law,

$$m\ddot{x} = mg - mkv^2$$

$$\ddot{x} = g - kv^2$$

$$\therefore \ddot{x} = \frac{g}{V^2}\left(V^2 - v^2\right)$$

$\ddot{x} \to 0$ as $kv^2 \to g$

$\therefore kV^2 = g$

ii.

$$\frac{1}{2}\frac{dv^2}{dx} = \frac{g}{V^2}\left(V^2 - v^2\right)$$

$$-\frac{2g}{V^2}\frac{dx}{d(v^2)} = \frac{-1}{V^2 - (v^2)}$$

$$-\frac{2g}{V^2}x = \log_e\left(V^2 - v^2\right) + c_1$$

$$\left.\begin{matrix} x = 0 \\ v = 0 \end{matrix}\right\} \Rightarrow 0 = \log_e V^2 + c_1$$

$$-\frac{2g}{V^2}x = \log_e\left(\frac{V^2 - v^2}{V^2}\right)$$

$$x = -\frac{V^2}{2g}\log_e\frac{(V-v)(V+v)}{V^2}$$

$$\frac{dv}{dt} = \frac{g}{V^2}\left(V^2 - v^2\right)$$

$$\frac{g}{V^2}\frac{dt}{dv} = \frac{1}{V^2 - v^2}$$

$$\frac{2g}{V}\frac{dt}{dv} = \frac{1}{V-v} + \frac{1}{V+v}$$

$$\frac{2g}{V}t = \log_e\frac{V+v}{V-v} + c_2$$

$$\left.\begin{matrix} t = 0 \\ v = 0 \end{matrix}\right\} \Rightarrow 0 = \log_e 1 + c_2$$

$$\frac{2g}{V}t = \log_e\frac{V+v}{V-v}$$

$$t = \frac{V}{2g}\log_e\frac{V+v}{V-v}$$

$$\therefore Vt - x = \frac{V^2}{2g}\log_e\left(\frac{V+v}{V}\right)^2 = \frac{V^2}{g}\log_e\frac{V+v}{V} = \frac{V^2}{g}\log_e\left(1 + \frac{v}{V}\right)$$ (using log laws for products and powers)

iii. $\dfrac{V-v}{V+v} = e^{-\frac{2g}{V}t}$ $\quad\therefore v\left(1 + e^{-\frac{2g}{V}t}\right) = V\left(1 - e^{-\frac{2g}{V}t}\right)$ $\quad\therefore v = V\left(\dfrac{1 - e^{-\frac{2g}{V}t}}{1 + e^{-\frac{2g}{V}t}}\right)$

iv. $\lim\limits_{t\to\infty}\left(Vt - x\right) = \dfrac{V^2}{g}\lim\limits_{t\to\infty}\log_e\left(1 + \dfrac{v}{V}\right) = \dfrac{V^2}{g}\log_e 2$

Question 15

a. Outcomes assessed : E4

Marking Guidelines

Criteria	Marks
i • applies a technique to find a new equation whose roots are squares of the original roots	1
• applies a technique to find a new equation whose roots are reciprocals of previous roots	1
• combines the two techniques and rearranges to find required equation	1
ii • expresses the sum of the 4th powers in terms of the sums of lower powers	1
• uses relations between roots and coefficients to evaluate required sum	1

Answer

i. α, β, γ roots of $x^3+kx+1=0$. $\quad\therefore \alpha^2,\ \beta^2,\ \gamma^2$ satisfy $\left(x^{\frac{1}{2}}\right)^3+k\left(x^{\frac{1}{2}}\right)+1=0$

$$\left(x^{\frac{3}{2}}+kx^{\frac{1}{2}}\right)^2=\left(-1\right)^2$$

$$x^3+2kx^2+k^2x=1$$

$$\therefore \alpha^2,\ \beta^2,\ \gamma^2 \text{ are roots of } x^3+2kx^2+k^2x-1=0$$

Now $\frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$ satisfy $\left(\frac{1}{x}\right)^3+2k\left(\frac{1}{x}\right)^2+k^2\left(\frac{1}{x}\right)-1=0$

$\therefore \frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$ are roots of $x^3-k^2x^2-2kx-1=0$

ii. α, β, γ satisfy $x\left(x^3+kx+1\right)=0$

$$\therefore \alpha^4+k\alpha^2+\alpha=0$$
$$\beta^4+k\beta^2+\beta=0$$
$$\gamma^4+k\gamma^2+\gamma=0$$

$$\therefore \alpha^4+\beta^4+\gamma^4+k\left(\alpha^2+\beta^2+\gamma^2\right)+\left(\alpha+\beta+\gamma\right)=0$$
$$\alpha^4+\beta^4+\gamma^4+k\left(-2k\right)+0=0$$
$$\therefore \alpha^4+\beta^4+\gamma^4=2k^2$$

b. Outcomes assessed : HE2

Marking Guidelines

Criteria	Marks
• defines an appropriate sequence of statements and verifies that the first is true	1
• verifies that the second statement is also true	1
• considers the (k+1)th statement, using the recurrence relation, conditional on $S(n)$ true, $n\le k$	1
• regroups and removes common factors, recognizing the perfect square expansions from $S(2)$	1
• completes the rearrangement of T_{k+1} and completes the induction process.	1

Answer

Let $S(n)$, $n=1,2,3,\ldots$ be the sequence of statements defined by $S(n)$: $T_n=\left(1+\sqrt{3}\,i\right)^n+\left(1-\sqrt{3}\,i\right)^n$.

Consider $S(1)$: $\left(1+\sqrt{3}\,i\right)^1+\left(1-\sqrt{3}\,i\right)^1=2=T_1$ $\quad\therefore S(1)$ is true

$S(2)$: $\left(1+\sqrt{3}\,i\right)^2+\left(1-\sqrt{3}\,i\right)^2=\left(-2+2\sqrt{3}i\right)+\left(-2-2\sqrt{3}i\right)=-4=T_2$ $\quad\therefore S(2)$ is true

Q15 b (cont)

If $S(n)$ is true for $n\le k$ where $k\ge 2$, then $T_n=\left(1+\sqrt{3}\,i\right)^n+\left(1-\sqrt{3}\,i\right)^n$, $n=1,2,3,\ldots,k$ **

Consider $S(k+1)$, where $k\ge 2$:

$$T_{k+1}=2T_k-4T_{k-1}$$
$$=2\left(1+\sqrt{3}i\right)^k+2\left(1-\sqrt{3}i\right)^k-4\left(1+\sqrt{3}i\right)^{k-1}-4\left(1-\sqrt{3}i\right)^{k-1} \quad \text{if } S(n) \text{ true for } n\le k$$
$$=\left(1+\sqrt{3}i\right)^{k-1}\left(-2+2\sqrt{3}i\right)+\left(1-\sqrt{3}i\right)^{k-1}\left(-2-2\sqrt{3}i\right)$$
$$=\left(1+\sqrt{3}i\right)^{k-1}\left(1+\sqrt{3}i\right)^2+\left(1-\sqrt{3}i\right)^{k-1}\left(1-\sqrt{3}i\right)^2$$
$$=\left(1+\sqrt{3}i\right)^{k+1}+\left(1-\sqrt{3}i\right)^{k+1}$$

Hence for $k\ge 2$, if $S(n)$ is true for $n\le k$ then $S(k+1)$ is true. But $S(1)$ and $S(2)$ are true, hence $S(3)$ is true, and then $S(4)$ is true and so on.

Hence by Mathematical induction, $T_n=\left(1+\sqrt{3}\,i\right)^n+\left(1-\sqrt{3}\,i\right)^n$, $n\ge 1$

c. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • rearranges $f(x)$ as a quadratic function of x by expansion	1
• recognises that $f(x)$ is never negative and hence has discriminant $\Delta\le 0$	1
• uses this fact to obtain required inequality	1
ii • applies result when $a_k=k$, $k=1,2,3,\ldots,n$	1
• uses the sum of an AP to complete the proof	1

Answer

i. $f(x)=\sum_{k=1}^{n}\left(\sqrt{a_k}\,x-\frac{1}{\sqrt{a_k}}\right)^2=\sum_{k=1}^{n}\left(a_k\,x^2-2x+\frac{1}{a_k}\right)=\left(\sum_{k=1}^{n}a_k\right)x^2-2nx+\left(\sum_{k=1}^{n}\frac{1}{a_k}\right)$

Clearly $f(x)\ge 0$ for all real x. Hence $\Delta\le 0$. $\therefore 4n^2-4\left(\sum_{k=1}^{n}a_k\right)\left(\sum_{k=1}^{n}\frac{1}{a_k}\right)\le 0$

$$\therefore \left(a_1+a_2+\ldots+a_n\right)\left(\frac{1}{a_1}+\frac{1}{a_2}+\ldots+\frac{1}{a_n}\right)\ge n^2$$

ii. Let $a_k=k$, $k=1,2,3,\ldots,n$. Then $\left(1+2+3+\ldots+n\right)\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)\ge n^2$

$$\tfrac{1}{2}n\left(n+1\right)\left(1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\tfrac{1}{n}\right)\ge n^2$$

$$\left(1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots+\tfrac{1}{n}\right)\ge\frac{2n}{n+1}$$

Question 16

a. Outcomes assessed : E4

Marking Guidelines

Criteria	Marks
• finds the sum of the roots of the required quadratic equation	1
• finds the product of the roots of the required quadratic equation	1
• uses these values to determine the coefficients and hence write down the quadratic equation	1

Answer

$z^5-1=0$ has roots $1,\ \alpha,\ \alpha^2,\ \alpha^3,\ \alpha^4$ where $\alpha=\cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}$. Then $1+\alpha+\alpha^2+\alpha^3+\alpha^4=0$ and $\alpha^5=1$.

$\therefore(\alpha+\alpha^4)+(\alpha^2+\alpha^3)=-1$ and $(\alpha+\alpha^4)(\alpha^2+\alpha^3)=\alpha^3+\alpha^4+\alpha^6+\alpha^7=\alpha^3+\alpha^4+\alpha+\alpha^2=-1$

Hence required quadratic equation is $x^2+x-1=0$.

b. Outcomes assessed : HE3

Marking Guidelines

Criteria	Marks
i • writes a^2-ab+b^2 in terms of the squares of $(a+b)$ and $(a-b)$	1
• uses the fact that the square of a real number is non-negative to deduce the result	1
ii • uses the cosine rule to show $c^2\ge a^2-ab+b^2$ if $\angle BCA\ge 60°$	1
• combines this inequality with that from i. to produce required inequality for c^3	1
• justifies the condition for equality	1

Answer

i. $a^2-ab+b^2=\frac{1}{4}(a+b)^2+\frac{3}{4}(a-b)^2$, where $\frac{3}{4}(a-b)^2\ge 0$ since a and b are real.

$\therefore a^2-ab+b^2\ge\left(\frac{a+b}{2}\right)^2$, with equality if and only if $a=b$

ii. In ΔABC, using the cosine rule, $c^2=a^2+b^2-2ab\cos\angle BCA$

If $\angle BCA\ge 60°$, then $\cos\angle BCA\le\frac{1}{2}$. $\therefore c^2\ge a^2+b^2-2ab\times\frac{1}{2}$

$\therefore c^2\ge a^2-ab+b^2$, with equality if and only if $\angle BCA=60°$.

Then, using i., $c\ge\sqrt{a^2-ab+b^2}\ge\frac{a+b}{2}$, with equality if and only if both $\angle BCA=60°$ and $a=b$.

(i.e. if and only if ΔABC is equilateral)

Hence if $\angle BCA\ge 60°$, $c^3\ge\left(\frac{a+b}{2}\right)(a^2-ab+b^2)=\frac{a^3+b^3}{2}$

$2c^3\ge a^3+b^3$, with equality if and only if ΔABC is equilateral.

Q16 (cont)

c. Outcomes assessed : H5, E8

Marking Guidelines

Criteria	Marks
i • applies integration by parts	1
• rearranges to obtain required recurrence relation	1
ii • takes the sum of both sides over integer values from 1 to n	1
• simplifies to obtain required result	1
iii • evaluates I_1	1
• shows that $\frac{1}{n+1}I_{n+1}\to 0$ as $n\to\infty$	1
• states limiting sum of series	1

Answer

i. For $n=1,2,3,\ldots$

$$I_{n+1}=\int_1^2\left(1-\frac{1}{x}\right)^{n+1}dx$$

$$=\left[x\left(1-\frac{1}{x}\right)^{n+1}\right]_1^2-(n+1)\int_1^2 x\left(1-\frac{1}{x}\right)^n\left(\frac{1}{x^2}\right)dx$$

$$=\left(\frac{1}{2^n}-0\right)+(n+1)\int_1^2\left\{\left(1-\frac{1}{x}\right)-1\right\}\left(1-\frac{1}{x}\right)^n dx$$

$$\therefore\ I_{n+1}=\frac{1}{2^n}+(n+1)(I_{n+1}-I_n)$$

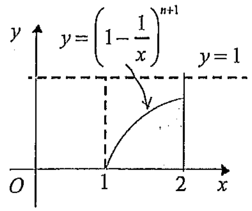
$$-nI_{n+1}=\frac{1}{2^n}-(n+1)I_n$$

$$\frac{1}{n+1}I_{n+1}=\frac{1}{n}I_n-\frac{1}{n(n+1)2^n}$$

ii. $\sum_{r=1}^{n}\frac{1}{r+1}I_{r+1}=\sum_{r=1}^{n}\frac{1}{r}I_r-\sum_{r=1}^{n}\frac{1}{r(r+1)2^r}\ \Rightarrow\ \frac{1}{2}I_2+\ldots+\frac{1}{(n+1)}I_{n+1}=I_1+\frac{1}{2}I_2+\ldots+\frac{1}{n}I_n-\sum_{r=1}^{n}\frac{1}{r(r+1)2^r}$

$$\therefore\ \frac{1}{n+1}I_{n+1}=I_1-\sum_{r=1}^{n}\frac{1}{r(r+1)2^r}$$

iii. $I_1=\int_1^2\left(1-\frac{1}{x}\right)dx=\left[x-\log_e x\right]_1^2=1-\log_e 2$ $\quad\therefore\sum_{r=1}^{n}\frac{1}{r(r+1)2^r}=(1-\log_e 2)-\frac{1}{n+1}I_{n+1}$



Considering areas, $0<I_{n+1}<1$

$$0<\frac{1}{(n+1)}I_{n+1}<\frac{1}{n+1}$$

$$\therefore\frac{1}{(n+1)}I_{n+1}\to 0\ \text{ as }\ n\to\infty$$

Hence $\frac{1}{1\times2\times2^1}+\frac{1}{2\times3\times2^2}+\frac{1}{3\times4\times2^3}+\ldots$ has a limiting sum $(1-\log_e 2)$.