St Catherine's School
Waverley

Student Number: ____________

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

August 2013

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- All necessary working should be shown in every question
- Complete each section in a separate booklet

- All questions are of equal value
- **Total Marks – 100**
- Attempt Questions 1 – 16
- Questions 1 to 10 are multiple choice questions
- Questions 11 to 16 are worth 15 marks each.

TABLE OF STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}; \quad n \neq -1$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

Note: $\ln x = \log_e x, \quad x > 0$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

1 $w = 1 + i$ and $z = -1 + \sqrt{3}\,i$. The complex number $\frac{w}{z}$ in modulus argument form is

A. $\frac{1}{\sqrt{2}} \text{cis}\left(-\frac{5\pi}{12}\right)$

B. $\frac{1}{\sqrt{2}} \text{cis}\left(\frac{5\pi}{12}\right)$

C. $\sqrt{2}\, \text{cis}\left(\frac{5\pi}{12}\right)$

D. $\frac{1}{\sqrt{2}} \text{cis}\left(\frac{7\pi}{12}\right)$

2 $\lim_{x \to 0} \frac{x}{\sin\frac{\pi x}{2}}$

A. $\frac{\pi}{2}$

B. $\frac{2}{\pi}$

C. 0

D. ∞

3. The eccentricity of the general hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, is given by $e^2 = 1 + \frac{b^2}{a^2}$.
The limiting position of the hyperbola as the eccentricity tends to infinity is

A. Circle

B. The y axis

C. The x axis

D. Ellipse

4 $\int \cos^3\theta \, d\theta$ is

A. $\frac{\sin^4\theta}{4} + C$

B. $\frac{\cos^4\theta}{4} + C$

C. $\sin\theta - \frac{\sin^3\theta}{4} + C$

D. $-\cos\theta + +C$

5 $2 + i$ is a root of the equation $x^2 + bx + c = 0$.

The values of b and c are:

A. $b = 4$ and $c = 5$

B. $b = -4$ and $c = -5$

C. $b = -4$ and $c = 5$

D. $b = 2$ and $c = 1$

6. The polynomial equation $x^3 - 3x^2 - x + 2 = 0$ has roots α, β and γ. Which of the following polynomial equations have roots $\frac{1}{\alpha}$, $\frac{1}{\beta}$ and $\frac{1}{\gamma}$?

(A) $x^3 - x^2 - 3x + 1 = 0$

(B) $x^3 - 2x^2 - 3x + 1 = 0$

(C) $2x^3 - x^2 - 3x + 1 = 0$

(D) $2x^3 - 2x^2 - 3x + 1 = 0$

7 What is the derivative of $\sin^{-1} x - \sqrt{1 - x^2}$?

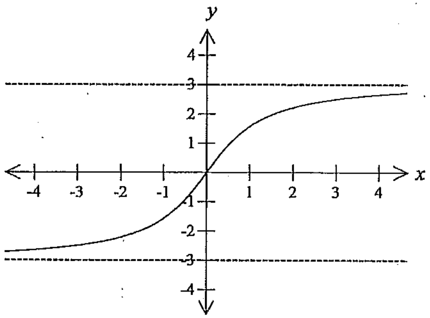
(A) $\frac{\sqrt{1+x}}{\sqrt{1-x}}$

(B) $\frac{\sqrt{1+x}}{1-x}$

(C) $\frac{1+x}{\sqrt{1-x}}$

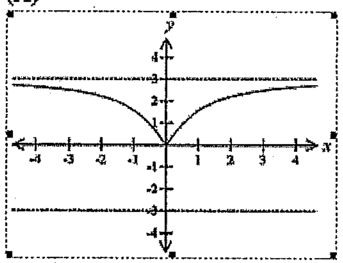
(D) $\frac{1+x}{1-x}$

8 1 The diagram shows the graph of the function $y = f(x)$.

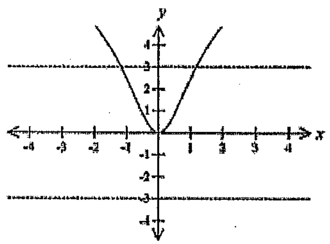


Which of the following is the graph of $y = \sqrt{f(x)}$?

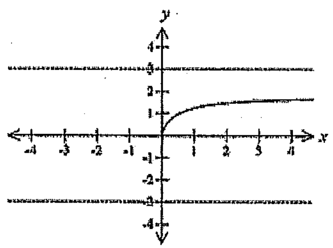
(A)



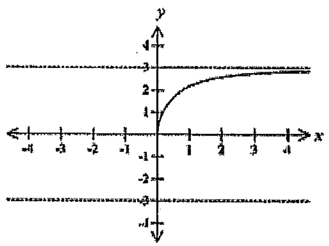
(B)



(C)



(D)



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9 The region enclosed by $y = \sin x$, x=0 and y=1 is rotated around the *y*-axis to produce a solid. The volume of this solid is given by the expression.

A. $\int_0^{\frac{\pi}{2}} 2\pi x \sin x \, dx$

B. $\int_0^{\frac{\pi}{2}} 2\pi x(1 - \sin x) \, dx$

C. $\int_0^1 \pi x^2 \sin^2 x \, dx$

D. $\int_0^1 \pi \sin^{-1} y \, dy$

10 Which of the following is an expression for $\int \frac{2}{x^2+4x+13} dx$?

A. $\frac{1}{3}\tan^{-1}\frac{(x+2)}{3}+c$

B. $\frac{2}{9}\tan^{-1}\frac{(x+2)}{9}+c$

C. $\frac{2}{3}\tan^{-1}\frac{(x+2)}{3}+c$

D. $\frac{1}{9}\tan^{-1}\frac{(x+2)}{9}+c$

Question 11

a) Integrate **2**

$$\int \frac{dx}{\sqrt{5+4x-x^2}}$$

b) Evaluate the definite integral **3**

$$\int_{-1}^{3} x\sqrt{(x+1)}\ dx$$

c) (i) Show that $\int_0^{\pi} \frac{\sin x}{1+cos^2 x} dx = \frac{\pi}{2}$ **2**

(ii) Show that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ **2**

(iii) Using parts (i) and (ii) or otherwise, find the value of the definite integral **2**

$$\int_0^{\pi} \frac{x \sin x}{1+cos^2 x} dx$$

d) Let α, β, γ be the roots of the equation

$x^3 - 3x^2 + 5 = 0$

(i) Find the equations whose roots are $\alpha - 1, \beta - 1, \gamma - 1$ **2**

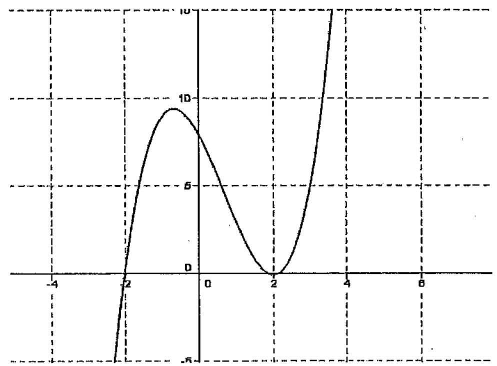
(ii) Find the equation whose roots are α^2, β^2 and γ^2 **1**

(iii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$ **1**

Question 12

a The diagram shows a sketch graph of a function

$y = f(x)$



On separate diagrams, sketch

(i) $y = (f(x))^2$ **1**

(ii) $y = \sqrt{f(x)}$ **1**

(iii) $y^2 = f(x)$ **1**

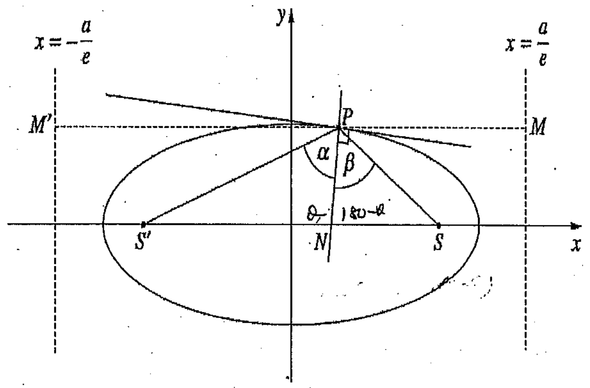
(iv) $y = f|x|$ **2**

(v) $y = \ln(f(x))$ **2**

Question 12 continued onto the next page

b The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ has foci S: $(ae, 0)$ and $S'(-ae,0)$, with corresponding directrices $x=\frac{a}{e}$ and $x=-\frac{a}{e}$.

The point P: (x_0, y_0) is on the ellipse. The horizontal line through P meets the directrices at M and M' as shown in the diagram below.



(i) Show that the equation of the normal to the ellipse is

$$y-y_0=\frac{a^2y_0}{b^2x_0}(x-x_0)$$

(ii) The normal at P meets the x axis at the point N. Show that the coordinates of the point N is $(e^2 x_0, 0)$ 2

(iii) Show that $\frac{PS}{PS'}=\frac{NS}{NS'}$ 2

(iv) Let $\alpha=\angle S'PN$ and $\beta=\angle NPS$. 2

Using sine rule to triangles $S'PN$ and NPS, show that

$\alpha=\beta$

Question 13

a Sketch the region in the Complex plane where the inequalities 2

$|z-1-i|<2$ and $\arg(z-1-i)<\frac{\pi}{4}$

b (i) Expand $(cos\theta+isin\theta)^5$ and using De Moivre's theorem show that 2

$\sin 5\theta=16sin^5\theta-20sin^3\theta+5sin\theta$

(ii) Deduce that $x=sin\frac{\pi}{10}$ is a root of $16x^5-20x^3+5x-1=0$ and show that the other three roots are $sin\frac{\pi}{10}$, $\sin\frac{-3\pi}{10}$ and $\sin\frac{-3\pi}{10}$ 3

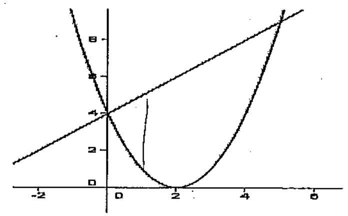
(iii) Find $p(x)$, where $16x^5-20x^3+5x-1=(x-1)p(x)$ 1

(iv) Show that $p(x)=(4x^2+2x-1)^2$ 1

(v) Hence find the exact value for $sin\frac{\pi}{10}$ 2

Question 13 continued onto the next page

c The area bounded by $y = (x-2)^2$ *and* $y = x + 4$ is rotated about the y axis



Using the method of cylindrical shells, find the volume of the solid generated. 4

See next page for question 14

Question 14

a (i) Show that if α *is a* root of multiplicity m *for* $P(x) = 0$, then α is a root of multiplicity *m-1* to $P'(x) = 0$ 1

(ii) $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$, has a root of multiplicity 3. 3

Find this root and factorise $P(x)$

b (i) Show that if $y = mx + k$ is a tangent to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then 2

$m^2a^2 + b^2 = k^2$

(ii) Show that the tangents to the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ from a point $(4,5)$ are at right angles. 2

c Sketch the graph of $y = e^{\cos x}$, by carefully considering the graph of 3

$y = e^x$ *and* $y = \cos x$.

d P $(cp, \frac{c}{p})$ and Q $(cq, \frac{c}{q})$ are points on the rectangular hyperbola $xy = c^2$

(i) Show that the equation of the chord PQ is $x + pqy = cp + cq$ 2

(ii) Find the locus of the mid points of the chords that pass through the point (2,0) 2

Question 15

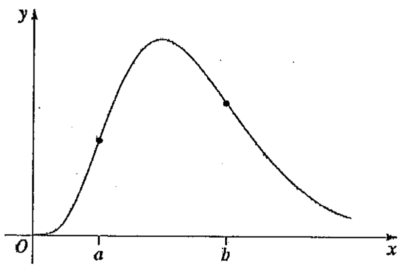
a (i) Show that $a^2 + b^2 \geq 2ab$, *for all real numbers a and b.* 3

And hence deduce that $(a+b+c)^2 \geq 3(ab+bc+ca)$

b Use $\ln t = \int_1^t \frac{1}{x}\,dx$ for $t > 1$ and figures representing suitable areas to deduce that 3

$$1 - \frac{1}{t} \leq \ln t \leq \frac{1}{2}\left(t - \frac{1}{t}\right)$$

c For $x > 0$, *let* $f(x) = x^n e^{-x}$, where n is an integer and $n \geq 2$



(i) Show that $f''(x) = e^{-x}x^{n-2}(x^2 - 2nx + n(n-1))$ 2

(ii) The two points on $y = f(x)$, where $x = a$ *and* $x = b$ shown are points of inflexion. Find *a and b in terms of n* 3

d If $I_n = \int_0^a (a^2 - x^2)^n dx$ for $n \geq 0$. 4

Show that $I_n = \frac{2a^2 n}{2n+1} I_{n-1}$

Question 16

a) Consider the function $f(x) = 2\cos^{-1}\frac{x}{\sqrt{2}} - \sin^{-1}(1 - x^2)$

(i) Show that $f'(x) = 0$ 3

(ii) Hence show that $f(x) = \frac{\pi}{2}$ 2

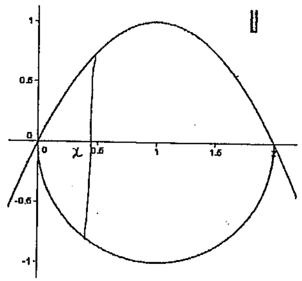
b) (i) Show that $$\frac{1}{\binom{n}{r}} = \frac{r}{r-1}\left[\frac{1}{\binom{n-1}{r-1}} - \frac{1}{\binom{n}{r-1}}\right]$$ 3

(ii) Hence show that if m is an integer with $m \geq r$ 2

$$\frac{1}{\binom{r}{r}} + \frac{1}{\binom{r+1}{r}} + \cdots + \frac{1}{\binom{m}{r}} = \frac{r}{r-1}\left[1 - \frac{1}{\binom{m}{r-1}}\right].$$

Question 16 continued onto the next page

c) The base of a solid is enclosed by the curves $y = x(2-x)$ and $y = -\sqrt{1-(x-1)^2}$ as shown. Cross sections perpendicular to the x *axis*, at a distance x from the origin are triangles whose base is on the base of the solid and whose height is x.



(i) Show that the volume of the solid is given by 2

$$\int_0^2 x^2 - \frac{x^3}{2} + \frac{x}{2}\sqrt{1-(x-1)^2}\,dx$$

(ii) Hence find the volume of the solid 3

END of PAPER

Student Number: Solution.

St Catherine's School

Waverley

2013

HIGHER SCHOOL CERTIFICATE

Extension 2

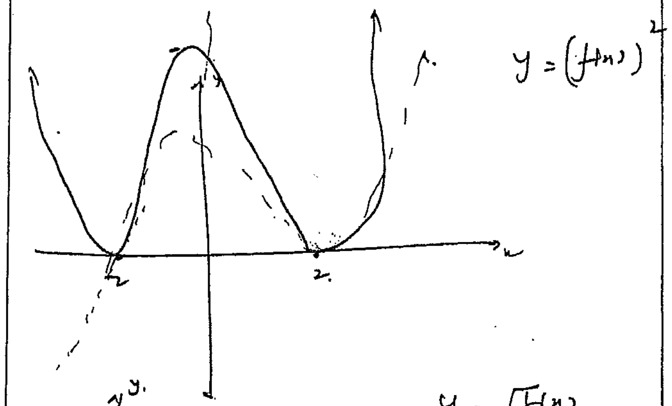
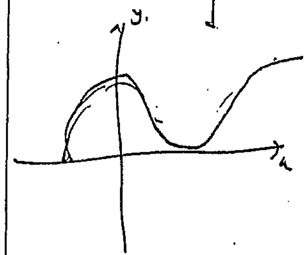
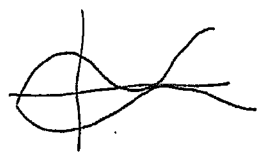
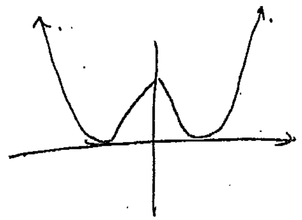
Multiple Choice Answer Sheet

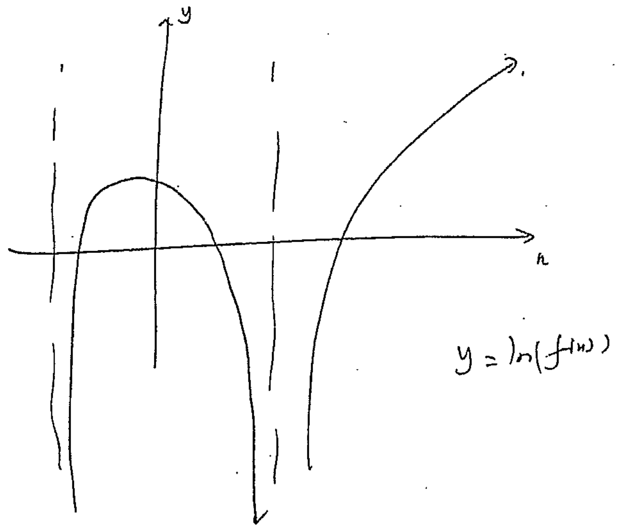
Colour in the correct oval completely

1. A (filled) B C D
2. A B (filled) C D
3. A B (filled) C D
4. A B C (filled) D
5. A B C (filled) D
6. A B C (filled) D
7. A (filled) B C D
8. A B C (filled) D
9. A B (filled) C D
10. A B C (filled) D

Q	Solutions	Marks	Comments
11.	$\int \frac{dx}{\sqrt{5+4x-x^2}}$		
	$= \int \frac{dx}{\sqrt{9-(x-2)^2}} = \sin^{-1}\frac{x-2}{3} + c.$		
b)	$\int_{-1}^{3} x\sqrt{x+1}\,dx$ $u = x+1$, $du = dx$, $x=-1;\ u=0$, $x=3;\ u=4$		
	$\int_0^4 (u-1)\sqrt{u}\,du$		
	$= \frac{2}{5}\left(u^{5/2}\right)_0^4 - \frac{2}{3}\left(u^{3/2}\right)_0^4$		
	$= \frac{2}{5}\times 32 - \frac{2}{3}\times 8$		
	$= \frac{64}{5} - \frac{16}{3} =$		
c)	$\int_0^{\pi} \frac{\sin x}{1+\cos^2 x}\,dx$ if $\cos x = u.$ $-\sin x\,dx = du$, $x=0;\ u=1$, $x=\pi;\ u=-1$		
	$\int_1^{-1} \frac{-du}{1+u^2} = \int_{-1}^{1} \frac{du}{1+u^2}$		
	$= \left(\tan^{-1}u\right)_{-1}^{1}$		
	$= \frac{\pi}{4} - \left(-\frac{\pi}{4}\right)$		
	$= \frac{\pi}{2}$		

Q	Solutions	Marks	Comments
(i).	$\int_0^a f(a-x)\,dx$ Let $a-x=u$, $-dx=du$, $x=0;\ u=a$, $x=a;\ u=0$	1	
	$= \int_a^0 f(u)\,(-du)$		
	$= \int_0^a f(u)\,du = \int_0^a f(x)\,dx.$	1	
(ii)	$\int_0^\pi \frac{x\sin x}{1+\cos^2 x} = \int_0^\pi \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}$		
	$= \int_0^\pi \frac{\pi\sin x}{1+\cos^2 x} - \int_0^\pi \frac{x\sin x}{1+\cos^2 x}\,dx$	1	
	$2\int_0^\pi \frac{x\sin x}{1+\cos^2 x} = \pi \times \frac{\pi}{2}.$	1	
	$\therefore \int_0^\pi \frac{x\sin x}{1+\cos^2 x} = \frac{\pi^2}{4}.$		
d)	(i). α is a root of $P(x)=0$ $\therefore P(\alpha)=0$		
	Note $P(\alpha-1+1)=0$ $\therefore \alpha-1$ is a root of $P(x+1)=0$		
	$\therefore$ The eqn is $(x+1)^3 - 3(x+1)^2 + 5 = 0$.		
	(ii) $P(\sqrt{x})=0$; $(x\sqrt{x}) - 3x + 5 = 0$		
	$(x\sqrt{x})^2 = (3x-5)^2$		
	$x^3 = (3x-5)^2$		

Q	Solutions	Marks	Comments
	$\alpha^3+\beta^3+\gamma^3 = 3(\alpha^2+\beta^2+\gamma^2) - 15$		
	$= 3\left((\alpha+\beta+\gamma)^2 - 2\sum\alpha\beta\right) - 15$		
	$= 3(3^2 - 0) - 15$		
	$= 27 - 15$	1	
	$= 12.$		
a)	 $y = (f(x))^2$		
b)	 $y = \sqrt{f(x)}$		
c)	 $y^2 = f(x)$		
d)	 $\lvert x\rvert = x,\ x>0$; $= -x;\ x<0$		

Q	Solutions	Marks	Comments
			
b)	$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$		
	$\frac{2x}{a^2}+\frac{2y}{b^2}y'=0$		
	$y'=\frac{-x}{a^2}\times\frac{b^2}{y}=\frac{-b^2}{a^2}\cdot\frac{x}{y}$		
	$y'_{at\,P}=\frac{-b^2}{a^2}\frac{x_0}{y_0}$	1	
	Grad. of normal at P is $\frac{a^2y_0}{b^2x_0}$		
	Eq^n of normal		
	$y-y_0=\frac{a^2y_0}{b^2x_0}(x-x_0)$	1	

Q	Solutions	Marks	Comments
Q.12 b	This normal meets x axis at $y=0$		
	$-y_0=\frac{a^2y_0}{b^2x_0}(x-x_0)$		
	$-b^2x_0y_0=a^2xy_0-a^2x_0y_0$	1	
	$x=\frac{1}{a^2y_0}\cdot x_0y_0(a^2-b^2)$		
	$=x_0\left(1-\frac{b^2}{a^2}\right)$ $b^2=a^2(1-e^2)$, $\frac{b^2}{a^2}=1-e^2$, $e^2=1-\frac{b^2}{a^2}$		
	$=x_0e^2$		
	$\therefore N:(e^2x_0,0)$		
	$\frac{PS}{PS'}=\frac{e\,PM}{e\,PM'}=\frac{\frac{a}{e}-x_0}{\frac{a}{e}+x_0}$		
	$=\frac{a-ex_0}{a+ex_0}$	1	
	$\frac{NS}{NS'}=\frac{ae-e^2x_0}{ae+e^2x_0}=\frac{\cancel{e}(a-ex_0)}{\cancel{e}(a+ex_0)}$		
	Hence $\frac{PS}{PS'}=\frac{NS}{NS'}$		
	In $\Delta PS'N$		
	$\frac{\sin\angle PNS'}{PS'}=\frac{\sin\alpha}{NS'}$		
	or $\frac{PS'}{NS'}=\frac{\sin\angle PNS'}{\sin\alpha}$		

Q	Solutions	Marks	Comments

In Δ NPS

$$\frac{\sin \angle PNS}{PS} = \frac{\sin \beta}{NS}$$

$$\therefore \frac{PS}{NS} = \frac{\sin \angle PNS}{\sin \beta}$$

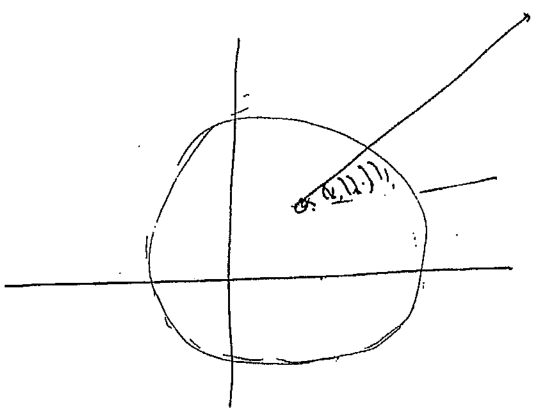
Using $\frac{PS}{PS'} = \frac{NS}{NS'}$, i.e. $\frac{PS}{NS} = \frac{PS'}{NS'}$

$$\frac{\sin \angle PNS'}{\sin \alpha} = \frac{\sin \angle PNS}{\sin \beta}$$

but $\sin \angle PNS' = \sin \angle PNS$ (the angles being supplementary)

$$\therefore \sin \beta = \sin \alpha \text{ or } \alpha = \beta.$$

Q.13



b) $(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(i\sin\theta)^2 + 10\cos^2\theta(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5$

$\therefore \sin 5\theta = 5\cos^4\theta\sin\theta - 10\cos^2\theta\sin^3\theta + \sin^5\theta$

$= 5(1-s^2)^2 s - 10(1-s^2)s^3 + s^5 \quad (s = \sin\theta)$

Q	Solutions	Marks	Comments

$= 5(1+s^4-2s^2).s - 10(s^3 - s^5) + s^5$

$= 5s + 5s^5 - 10s^3 - 10s^3 + 10s^5 + s^5$

$= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta \quad (s : \sin\theta)$

if $x = \sin\theta$

Roots of $16x^5 - 20x^3 + 5x - 1 = 0$

or $16x^5 - 20x^3 + 5x = 1$

are roots of $\sin 5\theta = 1$

$5\theta = \frac{\pi}{2}, \; 2\pi + \frac{\pi}{2}, \; 4\pi + \frac{\pi}{2}, \; 6\pi + \frac{\pi}{2}, \; 8\pi + \frac{\pi}{2}$

$\theta = \frac{\pi}{10}, \frac{\pi}{2}, \frac{9\pi}{10}, \frac{13\pi}{10}, \frac{17\pi}{10}$

$\therefore \sin\frac{\pi}{10}, \; \sin\frac{\pi}{2} = 1; \; \sin\frac{9\pi}{10} = \sin\frac{\pi}{10}; \; \sin\frac{17\pi}{10} = -\sin\frac{3\pi}{10}$ are

$\sin\frac{13\pi}{10} = -\sin\frac{3\pi}{10}$

The roots.

(iii)

$$\begin{array}{r l}
 & 16x^4 + 16x^3 - 4x^2 - 4x + 1 \\
x-1 \,) & 16x^5 - 20x^3 + 5x - 1 \\
 & 16x^5 - 16x^4 \\
 & 16x^4 - 20x^3 + 5x - 1 \\
 & 16x^4 - 16x^3 \\
 & -4x^3 + 5x - 1 \\
 & -4x^3 + 4x^2 \\
 & -4x^2 + 5x - 1 \\
 & -4x^2 + 4x \\
 & x - 1 \\
 & x - 1 \\
 & 0
\end{array}$$

$\therefore p(x) = 16x^4 + 16x^3 - 4x^2 - 4x + 1$

or $p(x)$ can be written using observation.

Q	Solutions	Marks	Comments

$(4x^2+2x-1)^2$

$= 16x^4 + 4x^2 + 1 + 16x^3 - 4x - 8x^2$

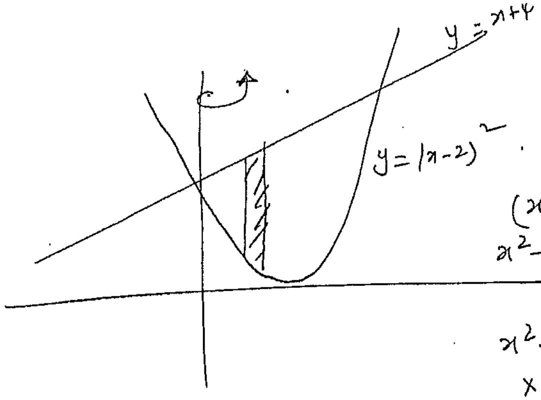
$= 16x^4 + 16x^3 - 4x^2 - 4x + 1$

$\sin\frac{\pi}{10}$; $\sin\frac{-3\pi}{10}$ are roots of $4x^2+2x-1=0$ — 1M.

$x = \dfrac{-2 \pm \sqrt{4+16}}{8}$

$= \dfrac{-2 \pm 2\sqrt{5}}{8} = \dfrac{-1 \pm \sqrt{5}}{4}$

$\sin\frac{\pi}{10} > 0 \quad \therefore \sin\frac{\pi}{10} = \dfrac{-1+\sqrt{5}}{4}$



$(x-2)^2 = (x+4)$

$x^2 - 4x + 4 = x + 4$

$x^2 - 5x = 0$

$x = 0; x = 5$ — (1M)

$\Delta V = 2\pi \cdot x \left((x+4) - (x-2)^2\right) \Delta x$

$= 2\pi \int_0^5 x\left(x + \cancel{4} - x^2 + 4x - \cancel{4}\right) dx$

$= 2\pi \int_0^5 \left(5x^2 - x^3\right) dx = 2\pi \left(\frac{5x^3}{3} - \frac{x^4}{4}\right)_0^5$

$= \dfrac{2\pi \times 625}{12} = \dfrac{625\pi}{6}$

8

Q	Solutions	Marks	Comments

14. $P(x) = (x-\alpha)^m Q(x)$

α being a root of multiplicity m for $P(x)$

$P'(x) = (x-\alpha)^m \cdot Q'(x) + m(x-\alpha)^{m-1} \cdot Q(x)$

$= (x-\alpha)^{m-1}\left(\underbrace{(x-\alpha) \cdot Q'(x) + m \cdot Q(x)}_{\text{This is a polynomial}}\right)$

$\therefore \alpha$ is a root of multiplicity $(m-1)$ for $P'(x)$

b) if α is the root of multiplicity 3 for the given $P(x) = x^4 + x^3 - 3x^2 - 5x - 2$

α is a root of $P'(x) = 4x^3 + 3x^2 - 6x - 5$

and $P''(x) = 12x^2 + 6x - 6$

$= 6(2x^2 + x - 1)$

$= 6(x+1)(2x-1)$

Note $P'(-1) = 4(-1)^3 + 3(-1)^2 - 6(-1) - 5$

$= 0$

and $P(-1) = (-1)^4 + (-1)^3 - 3(-1)^2 - 5(-1) - 2$

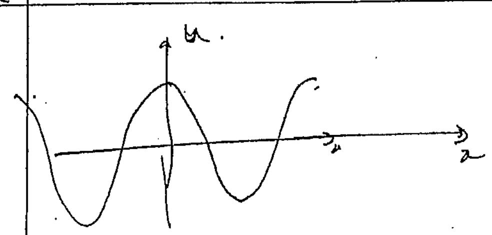
$= 0$

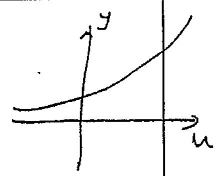
$\therefore -1$ is the root of multiplicity 3

$P(x) = (x+1)^3 \cdot Q(x)$

$= (x+1)^3 (x-2)$ by observation.

Q	Solutions	Marks	Comments





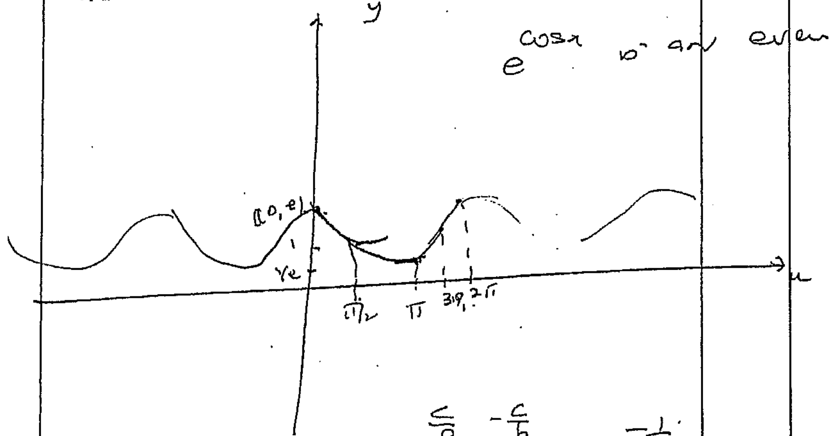
$$y = e^{\cos x}$$

$x = 0, = 2\pi$; $\cos 0 = 1$; $e^1 = e$

$x = \pi/2, = 3\pi/2$; $\cos \pi/2 = 0$; $e^0 = 1$.

$x = \pi$; $\cos \pi = -1$; $e^{-1} = \frac{1}{e}$

$e^{\cos x}$ is an even fn.



d) grad. of $PQ = \dfrac{\frac{c}{q} - \frac{c}{p}}{q - p} = -\dfrac{1}{pq}$

Eqn. of PQ: $y - \dfrac{c}{p} = -\dfrac{1}{pq}(x - cp)$

$pqy - cq = -x + cp$.

$x + pqy = cp + cq$.

Q	Solutions	Marks	Comments

$y = mx + k$ is a tgt. to $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$

if the descriminant of

$$\frac{x^2}{a^2} + \frac{(mx+k)^2}{b^2} = 1 \text{ is } 0.$$

$$b^2x^2 + a^2(m^2x^2 + 2mxk + k^2) = a^2b^2$$

$$x^2(b^2 + a^2m^2) + 2mka^2\,x + a^2(k^2 - b^2) = 0.$$

$$\Delta = 4m^2k^2a^4 - 4(b^2 + a^2m^2).a^2(k^2 - b^2) = 0.$$

$$m^2k^2a^2 - (b^2 + a^2m^2)(k^2 - b^2) = 0$$

$$m^2k^2a^2 - (b^2k^2 - b^4 + a^2m^2k^2 - a^2m^2b^2) = 0$$

$$b^4 - b^2k^2 + a^2m^2b^2 = 0$$

$$b^2 - k^2 + a^2m^2 = 0$$

$$k^2 = b^2 + a^2m^2.$$

(ii) $y = mx + k$ passes through $(4, 5)$
($a\ b^2 = 16$; $a^2 = 25$)

$\therefore\ 5 = 4m + k$.

also $k^2 = 16 + 25m^2$

$(5 - 4m)^2 = 16 + 25m^2$

$25 + 16m^2 - 40m = 16 + 25m^2$

$9m^2 + 40m - 9 = 0$ —— (*)

$m_1 \times m_2 = -1$ $\therefore$ The tangent $y = m_1x + k_1$; $y = m_2x + k_2$ are $\perp$; m_1 & m_2 being the roots of —(*)

Q	Solutions	Marks	Comments

$x + pqy = cp + cq$ passes through $(2,0)$

$\therefore \; 2 = c(p+q)$

Mid pt. of $PQ = \left(\frac{cp+cq}{2}, \frac{c/p + c/q}{2} \right)$

$\therefore \; X = c\left(\frac{p+q}{2}\right)$

$Y = c\left(\frac{q+p}{2pq}\right)$

and $p+q = \frac{2}{c}$

$\therefore \; X = c \times \frac{1}{c} = 1$ is the locus.

Q.14

$(a-b)^2 \geq 0$

$a^2+b^2 \geq 2ab$

$b^2+c^2 \geq 2bc$

$c^2+a^2 \geq 2ca$

$2(a^2+b^2+c^2) \geq 2(ab+bc+ca)$

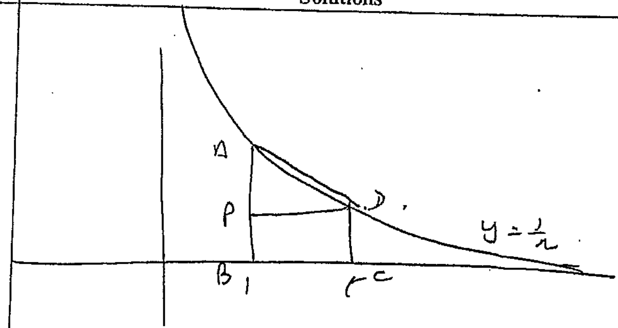
$a^2+b^2+c^2 \geq ab+bc+ca$

$(a+b+c)^2 = a^2+b^2+c^2 + 2(ab+bc+ca)$

$\geq 3(ab+bc+ca)$

b). P.T.O.

Q	Solutions	Marks	Comments



$\int_1^t \frac{1}{x}\,dx$ is the area bounded by $y = \frac{1}{x}$; $x=1$; $x=t$.

Area of BPDC $< \int_1^t \frac{1}{x}\,dx <$ Area of trapez. ABCD

$(t-1) \times \frac{1}{t} < (\ln x)_1^t < \frac{1}{2}\left(1+\frac{1}{t}\right)(t-1)$

$1 - \frac{1}{t} < \ln t < \frac{1}{2}\left(t + \cancel{1} - \cancel{1} - \frac{1}{t}\right)$

c)

$f(x) = x^n e^{-x}$

$f'(x) = x^n(-e^{-x}) + e^{-x}(nx^{n-1})$

$= e^{-x}(nx^{n-1} - x^n)$

$f''(x) = e^{-x}(n(n-1)x^{n-2} - nx^{n-1}) + (nx^{n-1} - x^n)(-e^{-x})$

$= e^{-x}\left[n(n-1)x^{n-2} - nx^{n-1} - nx^{n-1} + x^n\right]$

$= e^{-x}x^{n-2}\left[n(n-1) - 2nx + x^2\right]$.

Q	Solutions	Marks	Comments
	$f''(a) = 0$; $f''(b) = 0$ being points		of inflexion
	$\therefore$ a & b are roots of		
	$f''(x) = 0$		
	a & b are roots of		
	$x^2 - 2nx + n(n-1) = 0$		
	$x = \dfrac{2n \pm \sqrt{4n^2 - 4n(n-1)}}{2}$		
	$= \dfrac{2n \pm 2\sqrt{n}}{2}$		
	$n+\sqrt{n}$; $n-\sqrt{n}$.		
	$\therefore a = n-\sqrt{n}$ & $b = n+\sqrt{n}$	($b > a$)	
d).	$I_n = \int_0^a (a^2-x^2)^n \, dx$.		
	$u = (a^2-x^2)^n$		$v' = 1$
	$u' = n(a^2-x^2)^{n-1}(-2x)$		$v = x$.
	$\int uv' =$	$uv - \int u'v$	
	$\therefore I_n = \left(x(a^2-x^2)^n\right)_0^a + 2n\int_0^a x^2(a^2-x^2)^{n-1}$		dx.
	$= 0 - 2n\int_0^a (a^2 - x^2 - a^2)(a^2-x^2)^{n-1}$		dx.
	$= -2n\int_0^a \left[(a^2-x^2)^n - a^2(a^2-x^2)^{n-1} \right]$		dx.

Q	Solutions	Marks	Comments
	$I_n = -2n\, I_n + 2na^2\, I_{n-1}$		
	$(1+2n) I_n = +2na^2\, I_{n-1}$		
	$I_n = \dfrac{+2na^2}{1+2n} I_{n-1}$.		
Q.1b	$f(x) = 2\cos^{-1}\dfrac{x}{\sqrt{2}} - \sin^{-1}(1-x^2)$		
	$f'(x) = \dfrac{-2}{\sqrt{1-\frac{x^2}{2}}} \cdot \dfrac{1}{\sqrt{2}} + \dfrac{2x}{\sqrt{1-(1-x^2)^2}}$		
	$= \dfrac{1}{\sqrt{2}}\left(\dfrac{-2\sqrt{2}}{\sqrt{2-x^2}}\right) + \dfrac{2x}{\sqrt{-x^4+2x^2}}$.		
	$= \dfrac{-2}{\sqrt{2-x^2}} + \dfrac{2\cancel{x}}{\cancel{x}\sqrt{2-x^2}} = 0$		
	$f(x) =$ Constant.		
	Consider $f(0) = 2\cos^{-1}0 - \sin^{-1}1$		
	$= 2\times\frac{\pi}{2} - \frac{\pi}{2}$		
	$= \frac{\pi}{2}$.		

Q	Solutions	Marks	Comments

4(i)

$$\frac{1}{^{n}C_r} = \frac{r!\,(n-r)!}{n!}$$

$$\frac{r}{r-1}\left[\frac{1}{^{n-1}C_{r-1}} - \frac{1}{^{n}C_{r-1}}\right]$$

$$= \frac{r}{r-1}\left[\frac{(r-1)!\,(n-r)!}{(n-1)!} - \frac{(r-1)!\,(n-r+1)!}{n!}\right]$$

$$= \frac{r}{r-1}\cdot (r-1)!\left[\frac{n(n-r)! - (n-r+1)!}{n!}\right]$$

$$= \frac{r!}{\cancel{r-1}}\cdot (n-r)!\left[\frac{\cancel{n-n+r-1}}{n!}\right]$$

$$= \frac{1}{^{n}C_r}.$$

$$\frac{1}{^{r}C_r} = \frac{r}{r-1}\left[\frac{1}{^{r-1}C_{r-1}} - \cancel{\frac{1}{^{r}C_{r-1}}}\right]$$

$$\frac{1}{^{r+1}C_r} = \frac{r}{r-1}\left[\cancel{\frac{1}{^{r}C_{r-1}}} - \frac{1}{^{r+1}C_{r-1}}\right]$$

$$\vdots$$

$$\frac{1}{^{m}C_r} = \frac{r}{r-1}\left[\frac{1}{^{m-1}C_{r-1}} - \frac{1}{^{m}C_{r-1}}\right].$$

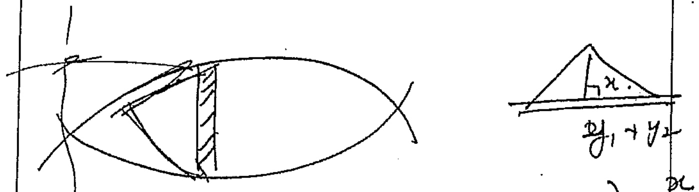
Q	Solutions	Marks	Comments

Adding.

$$\frac{1}{^{r}C_r} + \frac{1}{^{r+1}C_r} + \cdots + \frac{1}{^{m}C_r}$$

$$= \frac{r}{r-1}\left[\frac{1}{^{r-1}C_{r-1}} - \frac{1}{^{m}C_{r-1}}\right]$$

$$= \frac{r}{r-1}\left[1 - \frac{1}{^{m}C_{r-1}}\right].$$



$$\Delta V = \frac{1}{2}\left((2x - x^2) + \sqrt{1-(x-1)^2}\right)\cdot x$$

$$= \frac{1}{2}(2x^2 - x^3) + \frac{1}{2}x\sqrt{1-(x-1)^2}$$

$$\therefore V = \int_0^2 x^2 - \frac{x^3}{2} + \frac{x}{2}\sqrt{1-(x-1)^2}\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^2 - \frac{1}{2}\left[\frac{x^4}{4}\right]_0^2 + \frac{1}{2}\int_0^2 x\sqrt{1-(x-1)^2}\,dx$$

$$= \frac{8}{3} - 2 + \frac{1}{2}I$$

Q	Solutions	Marks	Comments
	$I = \int_0^2 x\sqrt{1-(x-1)^2}\,dx$		
	Let $x-1 = \sin\theta.$		
	$dx = \cos\theta\, d\theta$		
	$x=0;\ \theta = -\pi/2$		
	$x=2;\ \theta = \pi/2$		
	$\therefore I = \int_{-\pi/2}^{\pi/2} (\sin\theta + 1)\cos^2\theta\, d\theta.$		
	$= +\underbrace{\int_{-\pi/2}^{\pi/2} \sin\theta.\cos^2\theta\, d\theta}_{\text{odd fn.}} \cdot + \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta$		
	$= 0. + 2\int_0^{\pi/2} \cos^2\theta\, d\theta$		
	$= 0 + \int_0^{\pi/2} (1+\cos 2\theta)\, d\theta$		
	$= \left(\theta + \frac{\sin 2\theta}{2}\right)_0^{\pi/2}$		
	$= \frac{\pi}{2}.$		
	$\therefore \text{Vol} = \frac{8}{3} - 2 + \frac{\pi}{4} = \frac{2}{3} + \frac{\pi}{4}\, u^3.$		