TEMPE HS

LABORA FORTITER

2013

HIGHER SCHOOL CERTIFICATE

TRIAL EXAMINATION

Extension 1 Mathematics

General Instructions

- Reading Time - 5 minutes.
- Working Time - 2 hours.
- Write using a blue or black pen.
- Board approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- All necessary working should be shown for every question.
- Begin each question on a fresh sheet of paper.

Total marks (**70**)

- Attempt Questions 1-10. (Multiple Choice - 10 marks)
- Attempt Questions 11-14 (All questions - 15 marks)

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1},\ n\neq -1;\ x\neq 0,\ \textit{if}\ n<0$$

$$\int \frac{1}{x}\,dx = \ln x,\ x>0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax},\ a\neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax,\ a\neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax,\ a\neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax,\ a\neq 0$$

$$\int \sec ax\tan ax\,dx = \frac{1}{a}\sec ax,\ a\neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ a\neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a},\ a>0,\ -a<x<a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right),\ x>a>0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x,\quad x>0$

Section 1- Multiple choice (10 marks)

1. The solution to the inequality $x^3(x^2-4)>0$ is:

a. $x>0$ c. $-2<x<0, x>2$

b. $x\geq 0$ d. $0<x<2$

2. If $y=e^{-x^2}$, then $\frac{d^2y}{dx^2}$ is equal to:

a. $2e^{-x^2}(2x^2-1)$ c. $2e^{-x^2}(e^x+1)$

b. $-2e^{-x^2}(2x^2-1)$ d. $2e^{-x^2}(e^{2x}-1)$

3. Which of the following is an expression for $\int \cos^2 2x\,dx$?

a. $x-\frac{1}{4}\sin 4x+c$ c. $\frac{x}{2}-\frac{1}{8}\sin 4x+c$

b. $x+\frac{1}{4}\sin 4x+c$ d. $\frac{x}{2}+\frac{1}{8}\sin 4x+c$

4. How many distinct permutations of the letters of the word "TEMPE" are possible in a straight line when the word begins and ends with the letter E.

a. 60 c. 24

b. 12 d. 6

5. If $f(x)=e^{x+2}$, what is the inverse function $f^{-1}(x)$?

a. $f^{-1}(x)=e^{y-2}$ c. $f^{-1}(x)=\ln x-2$

b. $f^{-1}(x)=e^{y+2}$ d. $f^{-1}(x)=\ln(x-2)$

6. The graph of $y=\frac{1}{(x+2)(x-1)}$ cuts the $x-$axis:

a. Once c. Three times

b. Twice d. Never

7. The exact value of $\frac{2\tan 15°}{1+\tan^2 15°}$ is:

a. $\frac{1}{\sqrt{3}}$ c. $\frac{1}{2}$

b. $\sqrt{3}$ d. $\frac{1}{\sqrt{2}}$

8. The acute angle between the lines $x-3y=0$ and $3x-2y+1=0$ is:

a. $37°52'$ c. $74°44'$

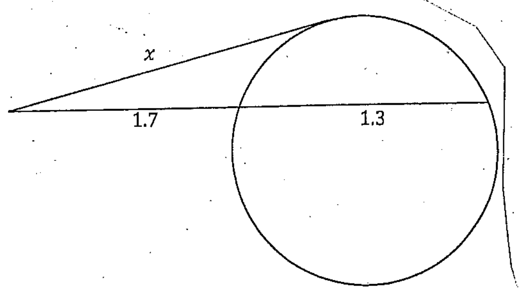
b. $82°15'$ d. $62°52'$

9. The point that divides $(2,-3)$ and $(7,7)$ externally in the ratio of $8:3$ is:

a. $(-1,9)$ c. $(10,13)$

b. $(12,9)$ d. $(13,10)$

10.



The value of x in the diagram above is:

a. 2.21 c. 1.49

b. 2 d. 2.3

End of Section 1

Section 2

Marks

Question 11 (15 Marks) Start a new booklet

a. Evaluate $\displaystyle\int_0^2 \frac{dx}{\sqrt{16-x^2}}$ **2**

b. Evaluate $\displaystyle\lim_{x\to 0}\frac{\sin 3x}{4x}$ **2**

c. Use the substitution $u = \ln x$ to find: **2**

$$\int \frac{dx}{x\sqrt{1-(\ln x)^2}}$$

d. i. Show that the derivative of $x\tan x - \ln(\sec x)$ is $x\sec^2 x$. **3**

ii. Hence, or otherwise, evaluate **2**

$$\int_0^{\frac{\pi}{4}} x\sec^2 x\, dx$$

e. i. Write down the domain and range of $y = \sin^{-1}(\sin x)$ **2**

ii. Draw a neat sketch of $y = \sin^{-1}(\sin x)$ **2**

End of Question 11

Question 12 (15 Marks) Start a new booklet

Marks

a. Prove by mathematical induction that **3**

$$1\times 2^0 + 2\times 2^1 + 3\times 2^2 + \cdots + n\times 2^{n-1} = 1 + (n-1)^{2n}$$

b. Consider the function $f(x) = \sin^{-1}(x-1)$

i. Find the domain of the function. **1**

ii. Sketch the graph of the curve $y = f(x)$ showing the endpoints and the x intercept. **2**

iii. The region in the first quadrant bounded by the curve $y = f(x)$ and the y axis between the lines $y = 0$ and $y = \frac{\pi}{2}$ is rotated through one complete revolution about the y axis. Find in simplest exact from the volume of the solid of revolution. **3**

c. Let $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ be two points on the parabola $y = \dfrac{x^2}{2a}$

i. Find the equation of the chord PQ. **2**

ii. If PQ is a focal chord, find the relationship between p and q. **1**

iii. Show that the locus of the midpoint of PQ is a parabola. **3**

End of Question 12

Question 13 (15 Marks) **Start a new booklet** **Marks**

a. If $t = \tan\frac{\theta}{2}$, then $\tan\theta = \frac{2t}{1-t^2}$ **3**

Use the t results, or otherwise, to obtain θ correct to the nearest minute if

$$\frac{7\sin\theta}{2} + 2\cos\theta = 4 \quad \text{for } 0° \leq \theta \leq 360°$$

b. A tower CX is observed at an angle of elevation of 14° from a point A on level ground. The same tower is observed from B, 1km from A, to have an angle of elevation of 17°. $\angle ACB = 120°$ and C is the base of the tower.

i. Draw a diagram showing the information above. **1**

ii. Calculate the height of the tower CX correct to the nearest metre. **2**

c. When a polynomial $P(x)$ is divided by $(x+1)(x-2)$ the result can be written as $P(x) = (x+1)(x-2)Q(x) + R(x)$ where $R(x) = ax + b$.

i. Given that $P(-1) = 3$, find the value of $R(x)$. **1**

ii. Given also that the remainder is -2 when $P(x)$ is divided by $x - 2$, find the values of a and b. **2**

d. Consider the function $f(x) = 2 - \ln x$.

i. Find the equation of the inverse function $f^{-1}(x)$ **1**

ii. Explain why the x coordinate X of the point of intersection P of the graphs of $y = f(x)$ and $y = f^{-1}(x)$ satisfies the equation $e^{2-X} - X = 0$. **2**

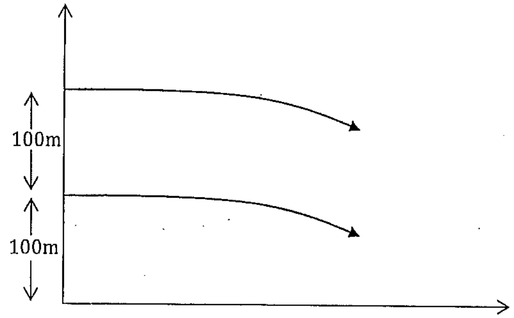
iii. Use two applications of Newton's Method with an initial value of $X = 1.5$ to find the value of X correct to two decimal places. **3**

End of Question 13

Question 14 (15 Marks) **Start a new booklet** **Marks**

a. Find the coefficient of x^4 in the expansion of $\left(3x - \frac{4}{x^2}\right)^7$ **2**

b. A balloon rises from level ground. Two projectiles are fired horizontally from the balloon at a velocity of 80ms^{-1}. The first is fired at a point 100m from the ground and the second when the balloon has risen a further 100m from the ground. How far apart will the projectiles hit the ground? Use $g = 10\text{ms}^{-2}$. **4**



c. 8 dice are rolled, what is the probability (to 3 decimal places), that:

i. 6 appears once. **1**

ii. 6 appears at least once. **2**

d. A particle moves in Simple Harmonic Motion in a straight line. At time t seconds its displacement from a fixed point O on the line is x metres, given by:

$$x = 4\sqrt{2}\cos\left(\frac{\pi t}{4} - \frac{\pi}{4}\right)$$

Its velocity is v ms^{-1} and its acceleration is $\ddot{x}$ ms^{-2}.

i. Find the amplitude and period of motion. **2**

ii. Find the initial position of the particle and determine if it is initially moving towards or away from O. **2**

iii. Find the distance travelled by the particle in its first 3 seconds of motion. **2**

END OF EXAM

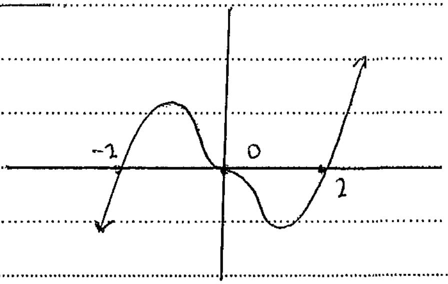
SECTION 1

1) $x^3(x^2-4) > 0$

$x^3(x+2)(x-2) > 0$

$x > 2, \ -2 < x < 0$

$\boxed{C}$



2) $y = e^{-x^2}$

$\frac{dy}{dx} = -2xe^{-x^2}$

$\frac{d^2y}{dx^2} = -2e^{-x^2} + (-2x)(-2xe^{-x^2})$

$= -2e^{-x^2} + 4x^2e^{-x^2}$

$= 2e^{-x^2}(2x^2-1)$

$\boxed{A}$

3) $\int \cos^2 2x \, dx = \frac{1}{2}\int (\cos 4x + 1)\, dx$

$= \frac{1}{2}\left(\frac{\sin 4x}{4} + x\right) + c$

$= \frac{\sin 4x}{8} + \frac{x}{2} + c$

$\boxed{D}$

4) $3! = 6$ $\boxed{D}$

5) $x = e^{y+2}$

$y+2 = \ln x$

$y = \ln x - 2$

$\boxed{C}$

6) x-int $\Rightarrow y = 0$

$\Rightarrow \frac{1}{(x+2)(x-1)} = 0$

$\therefore$ NO SOLN

$\boxed{D}$

7) $\frac{2\tan 15^\circ}{1+\tan^2 15^\circ} = \frac{1}{2}$

$\boxed{C}$

8) $x - 3y = 0$ $\quad 3x - 2y + 1 = 0$

$m = 1/3$ $\quad m = 3/2$

$\tan\theta = \left|\frac{1/3 - 3/2}{1 + 1/3 \times 3/2}\right|$

$\theta = 37°52'$

$\boxed{A}$

9) $P = \left(\frac{8\times 7 - 3\times 2}{8-3}, \frac{8\times 7 - 3\times -3}{8-3}\right) = (10, 13)$

$\boxed{C}$

10) $x^2 = 3 \times 1 \cdot 7$

$x = 2 \cdot 3$

$\boxed{D}$

SECTION 2

11a) $\int_0^2 \frac{dx}{\sqrt{16-x^2}} = \left[\sin^{-1}\frac{x}{4}\right]_0^2$

$= \frac{\pi}{6}$

b) $\lim_{x\to 0} \frac{\sin 3x}{4x} = \frac{3}{4}\lim_{x\to 0}\frac{\sin 3x}{3}$

$= \frac{3}{4}$

c) $\int \frac{dx}{x\sqrt{1-(\ln x)^2}}$ $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$.

$= \int \frac{x\,du}{x\sqrt{1-u^2}}$ $dx = x\,du$

$= \sin^{-1} u + c$

$= \sin^{-1}(\ln x) + c$

d) i) $\frac{d}{dx}\left(x\tan x - \ln(\sec x)\right)$

$= \tan x + x\sec^2 x - \frac{\sec x\tan x}{\sec x}$

$= x\sec^2 x$

ii) $\int_0^{\pi/4} x\sec^2 x\,dx = \left[x\tan x - \ln(\sec x)\right]_0^{\pi/4}$

$= \left(\frac{\pi}{4} - \ln\sqrt{2}\right) - \left(0 - \ln 1\right)$.

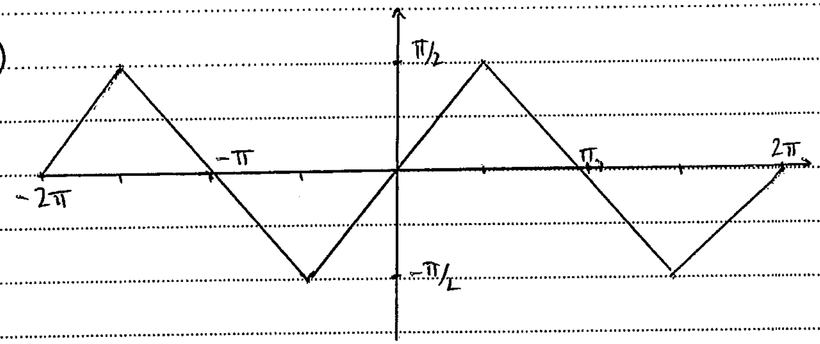
$= \frac{\pi}{4} - \frac{1}{2}\ln 2$

$= \frac{\pi - 2\ln 2}{4}$

e) i) D: $x \in \mathbb{R}$ (all real x)

R: $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$.

ii)

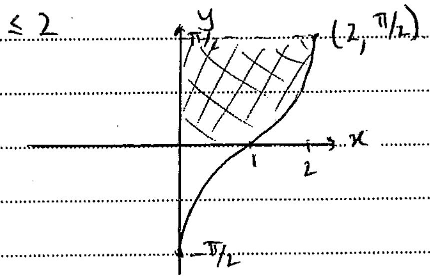


12 a)

b) i) $-1 \leq x-1 \leq 1$

$0 \leq x \leq 2$

ii)



iii) $y = \sin^{-1}(x-1)$

$x - 1 = \sin y$

$x = 1 + \sin y$

$x^2 = 1 + \sin^2 y + 2\sin y$

$= 1 + \frac{1}{2}\left(1 - \cos 2y\right) + 2\sin y$

$= 1 + \frac{1}{2} - \frac{\cos 2y}{2} + 2\sin y$

$$x^2 = \frac{3}{2} - \frac{\cos 2y}{2} + 2\sin y$$

$$V = \pi \int_0^{\pi/2} \left(\frac{3}{2} - \frac{\cos 2y}{2} + 2\sin y \right) dy$$

$$= \pi \left[\frac{3x}{2} - \frac{\sin 2y}{4} - 2\cos y \right]_0^{\pi/2}$$

$$= \pi \left(\left[\frac{3\pi}{4} - 0 - 0 \right] - \left[0 - 0 - 2 \right] \right)$$

$$= \pi \left[\frac{3\pi}{4} + 2 \right] u^3$$

c) i) $m_{PQ} = \dfrac{ap^2 - aq^2}{2ap - 2aq}$

$$= \frac{a(p+q)(p-q)}{2a(p-q)}$$

$$= \frac{p+q}{2}.$$

chord PQ $\rightarrow$ $y - ap^2 = \dfrac{p+q}{2}(x - 2ap)$.

$$= \left(\frac{p+q}{2}\right) x - ap^2 - apq$$

$$y = \left(\frac{p+q}{2}\right) x - apq$$

ii) Focal $\rightarrow$ $x = 0$, $y = a/2$.

$\rightarrow$ $a/2 = -apq$

$pq = -\dfrac{1}{2}.$

iii) $M\left[\dfrac{2ap + 2aq}{2}, \dfrac{ap^2 + aq^2}{2}\right]$

$x = a(p+q)$

$$\frac{x^2}{a^2} = p^2 + 2pq + q^2$$

$$p^2 + q^2 = \frac{x^2}{a^2} - 2pq$$

$$= \frac{x^2}{a^2} + 1.$$

$$y = \frac{a}{2}(p^2 + q^2).$$

$$= \frac{a}{2}\left[\frac{x^2}{a^2} + 1\right]$$

$$y = \frac{x^2}{2a} + \frac{a}{2}.$$

13 a) $\dfrac{7}{2}\left[\dfrac{2t}{1+t^2}\right] + 2\left[\dfrac{1-t^2}{1+t^2}\right] = 4$

$$7t + 2(1-t^2) = 4(1+t^2)$$

$$7t + 2 - 2t^2 = 4 + 4t^2$$

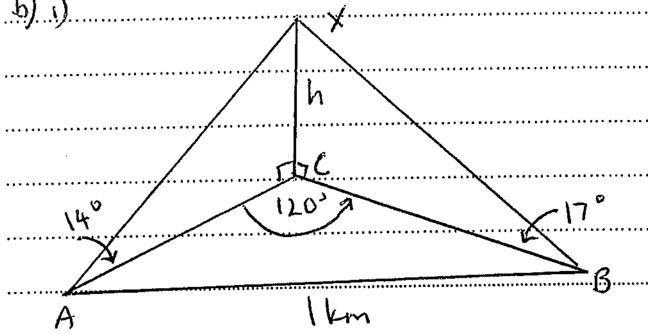
$$6t^2 - 7t + 2 = 0$$

$$(3t-2)(2t-1) = 0$$

$$\therefore \tan \theta/2 = 2/3, \ 1/2$$

$$\theta = 67°23', \ 53°8'$$

b) i)



ii) $\tan 14° = \dfrac{h}{AC}$

$AC = h \cot 14°$

$BC = h \cot 17°$

$h^2\cot^2 14° + h^2\cot^2 17° + 2(h\cot 14°)(h\cot 17°)\cos 120° = 1$

$h^2\cot^2 14° + h^2\cot^2 17° + h^2\cot 14° \cot 17° = 1$

$\therefore h = \sqrt{\dfrac{1}{\cot^2 14° + \cot^2 17° + \cot 14° \cot 17°}}$

$= 0.158 \text{ km}$

c) $P(x) = (x+1)(x-2)Q(x) + R(x)$

i) $P(x) = (x+1)(x-2)Q(x) + R(x)$

$P(-1) = R(-1) = 3$

ii) $P(2) = R(2) = -2$

$\therefore -a + b = 3$ ①

$2a + b = -2$ ②

① - ② $\Rightarrow -3a = 5$

$a = -5/3$

$b = 4/3$

d) i) $x = 2 - \ln y$

$\ln y = 2 - x$

$y = e^{2-x}$

ii) let $x = X$ be x-value of P.O.I.

Since $f(x)$ and $f^{-1}(x)$ intersect on $y = x$,

$e^{2-x} - x = 0$

$\therefore e^{2-X} - X = 0$ gives P.O.I.

iii) $x_0 = 1.5$

$x_1 = 1.5 - \dfrac{e^{2-1.5} - 1.5}{-e^{2-1.5} - 1} \doteq 1.5561$

$x_2 = x_1 - \dfrac{e^{2-x_1} - x_1}{-e^{2-x_1} - 1}$

$\doteq 1.557\ldots$

$\doteq 1.56$

14) a) $(3x - 4x^{-2})^7 \Rightarrow T_{r+1} = {}^7C_r (3x)^{7-r}(-4x^{-2})^r$

$= {}^7C_r\, 3^{7-r} x^{7-r} (-4)^r (x^{-2})^r$

$= {}^7C_r\, 3^{7-r} (-4)^r x^{7-3r}$

$x^4 \Rightarrow 7 - 3r = 4$

$r = 1$

$\therefore$ coeff of x^4 is ${}^7C_1\, 3^6 (-4)^1 = -20412$.

b) Projectile 1 | Projectile 2.

HOR $\rightarrow$ $\ddot{x}_1 = 0$ | $\ddot{x}_2 = 0$

$\dot{x}_1 = 80$ | $\dot{x}_2 = 80$

$x_1 = 80t$ | $x_2 = 80t$

VERT $\rightarrow$ $\ddot{y}_1 = -10$ | $\ddot{y}_2 = -10$

$\dot{y}_1 = -10t$ | $\dot{y}_2 = -10t$

$y_1 = -5t^2 + 100$ $\qquad$ $y_2 = -5t^2 + 200$

Hits ground $\rightarrow$ $y = 0$

$y_1 = 0 \Rightarrow 5t^2 = 100$ $\qquad$ $y_2 = 0 \rightarrow 5t^2 = 200$

$t^2 = 20$ $\qquad$ $t^2 = 40$

$t = 2\sqrt{5}$ $\qquad$ $t = 2\sqrt{10}$

$\therefore x_1 = 80 \times 2\sqrt{5}$ $\qquad$ $x_2 = 80 \times 2\sqrt{10}$

$x_2 - x_1 = 160(\sqrt{10} - \sqrt{5})$

$\doteqdot 148\text{m.}$

c) i) $P(1) = {}^8C_1 \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^7$

$\doteqdot 0.372$

ii) $P(\text{at least one}) = 1 - P(0)$

$= 1 - \left(\frac{5}{6}\right)^8$

$= 0.767$

d) $x = 4\sqrt{2}\cos\left(\frac{\pi t}{4} - \frac{\pi}{4}\right)$

i) amp $= 4\sqrt{2}$

$T = \dfrac{2\pi}{\pi/4} = 8$

ii) $\dot{x} = -\frac{\pi}{4} \times 4\sqrt{2}\sin\left(\frac{\pi t}{4} - \frac{\pi}{4}\right)$

$= -\pi\sqrt{2}\sin\left(\frac{\pi t}{4} - \frac{\pi}{4}\right)$

$t = 0$, $x = 4$, $\dot{x} = \pi$

$\therefore$ particle is moving away from O

iii) $t = 3$, $x = 4\sqrt{2}\cos\left(\frac{3\pi}{4} - \frac{\pi}{4}\right)$

$= 0$

$\therefore$ at $t = 3$ particle is at O.

$\therefore$ particle starts at $x = 4$, moves to $x = 4\sqrt{2}$ then returns to $x = 0$

$\therefore d = 4\sqrt{2} + (4\sqrt{2} - 4)$

$= (8\sqrt{2} - 4)\text{ m.}$